普通高等院校规划教材

An Introduction to Complex Analysis

复变函数引论（第2版）

曹怀信　主编

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An Introduction to Complex Analysis

复变函数引论

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Preface

For several years, I have been conducting courses in Complex Analysis, Real Analysis and Functional Analysis in a so-called "bilingual" way. That is, the lessons are given with Chinese textbooks, but mainly teached in English. The main purpose of teaching in this way is to improve the undergraduate students' ability to read and write English. Using a Chinese textbook in such "bilingual" courses is not, however, useful for training students' ability of English-thinking. Consequently, although there are a number of books on complex analysis in Chinese, in order to meet the requirements of bilingual teaching, it is necessary to write a textbook on complex analysis in English for Chinese undergraduate students. This is just the main aim of compiling the present book.

Roughly, analysis may be characterized as the study of functions and their various generalizations by using limits. In Mathematical Analysis, or Calculus, real-valued continuous functions of real variables were mainly discussed. Complex Analysis, or theory of functions of one complex variable, is devoted to the study of analytic complex-valued functions of one complex variable. The main tool used in complex analysis may be the theory of integrals.

Starting with the real number field $\mathbf{R}$, the complex number field $\mathbf{C}$ is introduced in Chapter 1, which is defined as the set of all pairs (a,b) of real numbers a and b with the addition and multiplication:

$$(a,b)+(x,y)=(a+x,b+y), \quad (a,b)(x,y)=(ax-by,ay+bx).$$

Moreover, the algebraic and geometric structures of the complex number system are surveyed there.

In Chapter 2, functions of a complex variable are discussed and a theory of differentiation for them is developed. The main goal of this chapter is to introduce analytic functions, which play a center role in complex analysis.

In Chapter 3, various elementary functions of a complex variable are considered, including exponential function $\exp z$, sine function $\sin z$, cosine function $\cos z$, etc..

In Chapter 4, the theory of integration of complex-valued functions of a complex variable is studied, which is very important in complex analysis and becomes a powerful tool for dealing with complex functions.

In Chapter 5, series of complex numbers are considered. With power series, representations of analytic functions are obtained, which are called Taylor expansions and Laurent expansions of analytic functions. Continuity and differentiability of the sums of power series and Laurent series are studied.

In Chapter 6, residues and poles of analytic functions are introduced in terms of Laurent expansions of the functions. Zeros of analytic functions are also discussed and the behavior of a function near isolated singular points is studied as well.

In Chapter 7, some important applications of the theory of residues are discussed, including evaluation of certain types of definite and improper integrals occurring in real analysis and applied mathematics.

In Chapter 8, the concept of a conformal mapping is introduced and discussed, which is a continuation of the geometric interpretation of a function of a complex variable as a mapping, or transformation introduced in Sections 2.2 and 2.3 of Chapter 2.

Most of the basic results are stated as propositions, theorems, or corollaries, followed by examples and exercises illustrating those results.

Some of materials of this book are from the book "Complex Variable and Applications" written by James Ward Brown and Ruel V. Churchill (7^{th} edition) and the book "Functions of one Complex Variable" by John B. Conway. My heart-felt thanks should be extended to the authors of these books.

In the preparations of this book, continual interest and support have been provided by a number of people, including my colleagues, students and my family members. The following deserve special thanks: Dr. Zhang Jianhua and Chen Zhengli, co-authors of this book, without whose cooperation this book could not have been completed. I would like to thank Professor Wu Jianhua, Professor Ji Guoxing, Professor Wu Baowei and Professor Liu Xinping for their kindly encouragement. Especially, I should like to thank Professor Yang Ming for correcting some errors in English writing.

I am also indebted to the support from the Bilingual Teaching Found of Shaanxi Normal University.

Any comments and suggestions from the readers will be gratefully appreciated.

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Chapter I

Complex Number Field

In this chapter, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known. The positive integer number system, integer number system, rational number system and real number system are denoted by $\mathbf{N},\mathbf{Z},\mathbf{Q}$ and $\mathbf{R}$, respectively.

§1.1. Sums and Products

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the *complex plane*, with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line. When real numbers x are displayed as points $(x,0)$ on the *real axis,* it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called *pure imaginary numbers* if $y \neq 0$. The y axis is then referred to as the *imaginary axis*. The set of all complex numbers is always denoted by $\mathbf{C}$ and called the *complex number system.*

It is customary to denote a complex number (x, y) by z, so that

$$z = (x, y). \tag{1.1.1}$$

The real numbers x and y are called the *real and imaginary parts* of z, respectively; and we write

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y. \tag{1.1.2}$$

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal whenever they have the same real parts and the same imaginary parts. Thus, the statement $z_1 = z_2$ means that z_1 and z_2 correspond to the same point in the complex plane, or z plane.

The *sum* $z_1 + z_2$ and the *product* $z_1 z_2$ of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{1.1.3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2). \tag{1.1.4}$$

Note that the operations defined by equations (1.1.3) and (1.1.4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1,0)+(x_2,0)=(x_1+x_2,0),\quad (x_1,0)(x_2,0)=(x_1x_2,0).$$

The complex number system is, therefore, a natural extension of the real number system.

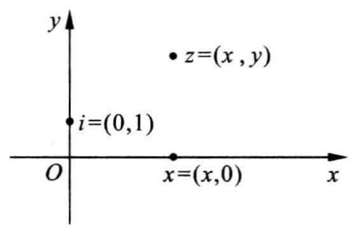

Fig. 1-1

Any complex number $z=(x,y)$ can be written as $z=(x,0)+(0,y)$. Since $(0,1)(y,0)=(0,y)$, we have $z=(x,0)+(0,1)(y,0)$. If we think of a real number x as the complex number $(x,0)$, that is, we identify a real number x with a corresponding complex number $(x,0)$, and let i denote the imaginary number $(0,1)$ (Fig. 1-1) , it is clear that

$$z=x+iy, \tag{1.1.5}$$

which is called the *rectangular form* of the number z. Thus, the complex number system can be written as

$$\mathbf{C}=\{(x,y):x,y\in\mathbf{R}\}=\{x+iy:x,y\in\mathbf{R}\}.$$

Also, with the convention $z^2=zz, z^3=zz^2$, etc., we find that

$$i^2=(0,1)(0,1)=(-1,0)=-1. \tag{1.1.6}$$

Thus, the equation $z^2+1=0$ has a root $z=i$ in $\mathbf{C}$.

In view of expression (1.1.5), definitions (1.1.3) and (1.1.4) become

$$(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2), \tag{1.1.7}$$

$$(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2). \tag{1.1.8}$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left replacing i^2 by -1 when it occurs.

§1.2. Basic Algebraic Properties

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

The *commutative laws*:

$$z_1+z_2=z_2+z_1, \quad z_1z_2=z_2z_1 \tag{1.2.1}$$

and the *associative laws*:

$$(z_1+z_2)+z_3=z_1+(z_2+z_3), \quad (z_1z_2)z_3=z_1(z_2z_3) \tag{1.2.2}$$

follow easily from the definitions (1.1.3) and (1.1.4) and the fact that real numbers obey these laws. For example, if $z_1=(x_1,y_1)$, and $z_2=(x_2,y_2)$, then

$$z_1+z_2=(x_1+x_2,y_1+y_2)=(x_2+x_1,y_2+y_1)=z_2+z_1.$$

Verification of the rest of the above laws, as well as the *distributive law*:

$$z(z_1+z_2)=zz_1+zz_2, \tag{1.2.3}$$

are similar.

According to the commutative law for multiplication, $iy=yi$. Hence one can write $z=x+yi$ instead of $z=x+iy$. Also, because of the associative laws, a sum $z_1+z_2+z_3$ and a product $z_1z_2z_3$ are well defined without parentheses, as is the case with real numbers. Especially, for every positive integer n and every complex number z, we make the following conventions

$$nz=\overbrace{z+z+\cdots+z}^{n} \text{ and } z^n=\overbrace{zz\cdots z}^{n}.$$

The *additive identity* $0=(0,0)$ and the *multiplicative identity* $1=(1,0)$ for real numbers carry over to the entire complex number system. That is,

$$z+0=z \text{ and } z\cdot 1=z \tag{1.2.4}$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 9).

For each complex number $z=(x,y)$, there is an *additive inverse*

$$-z=(-x,-y), \tag{1.2.5}$$

satisfying the equation $z+(-z)=0$. Moreover, there is only one additive inverse for any given z, since the equation $(x,y)+(u,v)=(0,0)$ implies that $u=-x$ and $v=-y$. Expression (1.2.5) can also be written $-z=-x-iy$ without ambiguity since (Exercise 8)

$$-(iy)=(-i)y=i(-y).$$

Additive inverses are used to define *subtraction*:

$$z_1-z_2=z_1+(-z_2), \forall z_1,z_2\in\mathbf{C}. \tag{1.2.6}$$

So if $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$, then

$$z_1-z_2=(x_1-x_2,y_1-y_2)=(x_1-x_2)+i(y_1-y_2). \tag{1.2.7}$$

For any *nonzero* complex number $z=(x,y)=x+iy$, there is a number z^{-1} such that $zz^{-1}=1$, called the *multiplicative inverse* of z. To find it, we observe that

$$(x,y)(u,v)=(1,0) \text{ if and only if } \begin{cases} xu-yv=1, \\ yu+xv=0. \end{cases}$$

The last system of *linear simultaneous equations* has a unique solution

$$u=\frac{x}{x^2+y^2}, \quad v=\frac{-y}{x^2+y^2}.$$

So the multiplicative inverse of $z=(x,y)=x+iy$ is

$$z^{-1}=\left(\frac{x}{x^2+y^2},\frac{-y}{x^2+y^2}\right)=\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2} \ (z\neq 0). \qquad (1.2.8)$$

The inverse z^{-1} is not defined when $z=0$. In fact, $z=0$ means that

$$x^2+y^2=0;$$

and this is not permitted in expression (1.2.8).

From the discussion above, we conclude that the set $\mathbf{C}$ of all complex numbers becomes a *field*, called the *field of complex numbers*, or the *complex number field.*

Exercises

1. Verify that
   (*a*) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$;
   (*b*) $(2,-3)(-2,1)=(-1,8)$;
   (*c*) $(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right)=(2,1)$.
2. Show that
   (*a*) $\operatorname{Re}(iz)=-\operatorname{Im} z$; (*b*) $\operatorname{Im}(iz)=\operatorname{Re} z$.
3. Show that $(1+z)^2=1+2z+z^2$.
4. Verify that each of the two numbers $z=1\pm i$ satisfies the equation
   $$z^2-2z+2=0.$$
5. Prove that multiplication is commutative, as stated in the second of equations (1.2.1), Sec. 1.2.
6. Verify that
   (*a*) the associative law for addition, stated in the first of equations (1.2.2), Sec.1.2;
   (*b*) the distributive law (1.2.3), Sec. 1.2.
7. Use the associative law for addition and the distributive law to show that
   $$z(z_1+z_2+z_3)=zz_1+zz_2+zz_3.$$
8. By writing $i=(0,1)$ and $y=(y,0)$, show that $-(iy)=(-i)y=i(-y)$.

9. (*a*) Show that the complex number $0=(0,0)$ is the unique additive identity.
   (*b*) Show that the number $1=(1,0)$ is the unique multiplicative identity.

10. Solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing

$$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y\neq 0$.

Ans. $z=\left(-\frac{1}{2},\pm\frac{\sqrt{3}}{3}\right)$.

11. For any nonempty sets $E,F\subset \mathbf{C}$, define the following sets

$$E\pm F=\{z\pm w: z\in E, w\in F\},\quad EF=\{zw: z\in E, w\in F\},$$
$$E/F=\{z/w: z\in E, w\in F\}\ (\text{if } 0\notin F),\quad wE=\{wz: z\in E\}(w\in \mathbf{C}),$$
$$w\pm E=\{w\pm z: z\in E\}(w\in \mathbf{C}),\quad E^n=\{z^n: z\in E\}(n\in \mathbf{N}).$$

Find four examples of E that satisfy the following conditions, respectively.

(*a*) $E+E\neq 2E$;
(*b*) $E-E\neq\{0\}$;
(*c*) $EE\neq E^2$;
(*d*) $E/E\neq\{1\}$.

12. Show that $\mathbf{C}=\mathbf{R}+i\mathbf{R}$, $\mathbf{Z}=\mathbf{N}-\mathbf{N}$ and $\mathbf{Q}=\mathbf{Z}/\mathbf{N}$.

§1.3. Further Properties

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described in Sec.1.2. Because such properties continue to be anticipated, the reader can easily pass to Sec.1.4 without serious disruption.

We begin with the observation that the existence of multiplicative inverses enables us to show that if a product z_1z_2 is zero, then so is at least one of the factors z_1 and z_2. For suppose that $z_1z_2=0$ and $z_1\neq 0$. The inverse z_1^{-1} exists; and, according to the definition of multiplication, any complex number times zero is zero. Hence

$$z_2=1\cdot z_2=(z_1^{-1}z_1)z_2=z_1^{-1}(z_1z_2)=z_1^{-1}\cdot 0=0.$$

Thus, if $z_1z_2=0$, then either $z_1=0$ or $z_2=0$; or possibly both z_1 and z_2 equal zero. Another way to state this result is that *if two complex numbers* z_1 *and*

z_2 *are nonzero, then so is their product* $z_1 z_2$.

Division by a nonzero complex number is defined as follows:

$$\frac{z_1}{z_2} = z_1 z_2^{-1} \ (z_2 \neq 0). \tag{1.3.1}$$

If $z_1 = (x_1, y_1) = x_1 + iy_1$ and $z_2 = (x_2, y_2) = x_2 + iy_2$, then equation (1.3.1) and expression (1.2.8) in Sec.1.2 tell us that

$$\frac{z_1}{z_2} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right).$$

Thus,

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \ (z_2 \neq 0). \tag{1.3.2}$$

Although expression (1.3.2) is not easy to remember, it can be obtained by writing (see Exercise 4)

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}, \tag{1.3.3}$$

multiplying out the products in the numerator and denominator on the right, and then using the property

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \ (z_3 \neq 0). \tag{1.3.4}$$

The motivation for starting with equation (1.3.3) appears in Sec. 1.5.

There are some expected identities, involving quotients, that follow from the relation

$$\frac{1}{z_2} = z_2^{-1} \ (z_2 \neq 0), \tag{1.3.5}$$

which is equation (1.3.1) when $z_1 = 1$. Relation (1.3.5) enables us, for example, to write equation (1.3.1) in the form

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \quad (z_2 \neq 0). \tag{1.3.6}$$

Also, by observing that (see Exercise 3)

$$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1 \quad (z_2 \neq 0),$$

and hence that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$, one can use relation (1.3.5) to show that

$$\frac{1}{z_1 z_2} = (z_1 z_2)^{-1} = z_1^{-1} z_2^{-1} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right)(z_1 \neq 0, z_2 \neq 0). \tag{1.3.7}$$

Another useful identity, to be derived in the exercises, is

$$\frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)(z_3 \neq 0, z_4 \neq 0). \tag{1.3.8}$$

Example. Computations such as the following are now justified:

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right) = \frac{1}{(2-3i)(1+i)} = \frac{1}{5-i}\cdot\frac{5+i}{5+i} = \frac{5+i}{(5-i)(5+i)}$$

$$= \frac{5+i}{26} = \frac{5}{26} + \frac{i}{26} = \frac{5}{26} + \frac{1}{26}i.$$

Finally, we note that the *binomial formula* involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \ (n = 1, 2, \ldots) \ (\textit{Binomial Formula}) \tag{1.3.9}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$. The proof, by *mathematical induction*, is left as an exercise.

Exercises

1. Reduce each of these quantities to a real number:

   $(a)\ \dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$; $(b)\ \dfrac{5i}{(1-i)(2-i)(3-i)}$; $(c)\ (1-i)^4$.

2. Show that

   $(a)\ (-1)z = -z$; $(b)\ \dfrac{1}{1/z} = z(z \neq 0)$.

3. Use the associative and commutative laws for multiplication to show that

   $$(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4).$$

4. Use identity (1.3.8) in Sec.1.3 to derive the cancellation law:

   $$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2}(z_2 \neq 0, z \neq 0).$$

5. Use mathematical induction to verify the binomial formula (1.3.9) in Sec.1.3. More precisely, note first that the formula is true when $n = 1$. Then, assuming that it is valid when $n = m$ where m denotes any positive integer, show that in must hold when $n = m + 1$.

§1.4. Moduli

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z (Sec. 1.1) in the complex plane. In fact, we often refer to z as the point z or the vector z . In Fig. 1-2, the number $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radial vectors.

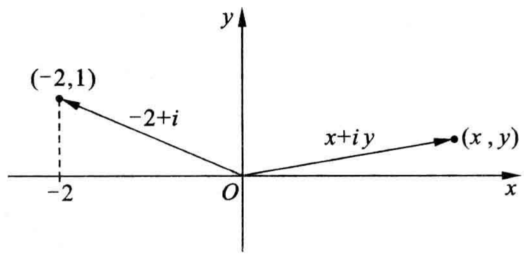

Fig. 1-2

According to the definition of the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $z_1 + z_2$ may be obtained vectorially as shown in Fig. 1-3.

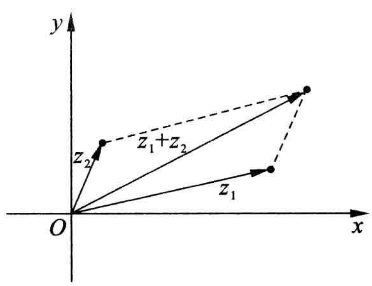

Fig. 1-3

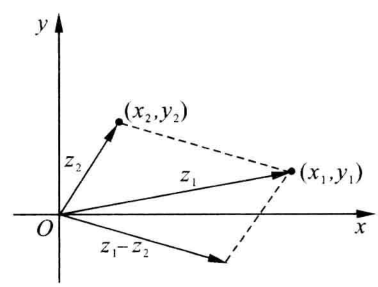

Fig. 1-4

The difference $z_1 - z_2 = z_1 + (-z_2)$ corresponds to the sum of the vectors z_1 and $-z_2$ (Fig. 1-4).

Although the product of two complex number z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The *modulus*, or *absolute value*, of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$|z| = \sqrt{x^2 + y^2}. \tag{1.4.1}$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the vector representing z. It reduces to the usual absolute value in the real number *system* when $y = 0$.

Note that, while the inequality $z_1 < z_2$ is meaningless unless both z_1 and z_2 are real, the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

Example 1. Since $|-3+2i| = \sqrt{13}$ and $|1+4i| = \sqrt{17}$, the point $-3+2i$ is closer to the origin than $1+4i$ is.

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is $|z_1 - z_2|$. This is clear from Fig.1-4, since $|z_1 - z_2|$ is the length of the vector $z_1 - z_2$; and, by translating the radius vector $z_1 - z_2$, one can interpret $z_1 - z_2$ as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1). Alternatively, it follows from the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

and definition (1.4.1) that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of these points simply as the circle $|z - z_0| = R$, denoted by $C(z_0, R)$.

Example 2. The equation $|z - 1 + 3i| = 2$ represents the circle whose center is the point $z_0 = (1, -3)$ and whose radius is $R = 2$.

It also follows from definition (1.4.1) that the real numbers $|z|$, $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$ are related by the equation

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2. \tag{1.4.2}$$

Thus

$$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \text{ and } \operatorname{Im} z \le |\operatorname{Im} z| \le |z|. \tag{1.4.3}$$