周毓麟论文集（三）

Selected Papers of Zhou Yulin (3)

世界图书出版公司

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序

1992 年 6 月，在庆祝周毓麟院士七十华诞之际，出版了他自选的一本论文集，共 17 篇论文， 352 页，当时由于篇幅有限，且考虑各方面合作者，远远未能反映作者的全面研究工作和成果，这次欣逢周毓麟院士八十诞辰之际，我们在他发表的论文与著作中，继续出版他的论文选集（二）和（三），其中论文选集（二）共有论文 26 篇，论文选集（三）共有论文 28 篇，由于周先生在数学这块科学园地上辛勤耕耘了六十多年，所取得的科学成果是极其丰富的，我们以后将继续出版周先生的论文集。

在论文选集 (二) 中，主要收录了周毓麟教授在拓扑学、非线性抛物型方程，非线性波动色散方程等方面的论文，其中包括 KdV 方程、非线性 Schrödinger 方程、铁磁链方程、Benjamin – Ono 型方程、有限深度流体方程等，我们可以看到他以独特的方法建立了这些方程（方程组）完整的系统的数学理论，得到了一系列深刻而重要的结果。在论文选集（三）中，主要收录了周毓麟教授有关离散泛函分析的数学理论，其中包括离散的 Sobolev 插值不等式，某些非线性发展方程差分格式的迭代收敛性、稳定性。此外，他还提出了新型的并行格式并作了理论分析和实际计算等。出版这些论文集的目的除了有重要的纪念意义外，更重要的是为了使更多的学者，特别是年轻的偏微分方程、计算数学家，能够从这些论文中学习周毓麟教授严谨的治学态度，敏锐的分析问题的方法和深厚的数学功底，促进我国偏微分方程和计算数学的迅猛发展。

在此，我们向为“周毓麟选集”出版作出辛勤贡献的所有人员表示衷心的感谢，并衷心祝愿周毓麟教授健康长寿，在科学事业上取得更大的成就。

郭柏灵

2002 年 10 月

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GENERAL INTERPOLATION FORMULAS FOR SPACES OF DISCRETE FUNCTIONS WITH NONUNIFORM MESHES*

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Abstract

The unequal meshsteps are unavoidable in general for scientific and engineering computations especially in large scale computations. The analysis of difference schemes with nonuniform meshes is very rare even by use of fully heuristic methods. For the purpose of the systematic and theoretical study of the finite differnce method with nonuniform meshes for the problems of partial differential equations, the general interpolation formulas for the spaces of discrete functions of one index with unequal meshsteps are established in the present work. These formulas give the connected relationships among the norms of various types, such as the sum of powers of discrete values, the discrete maximum modulo, the discrete Hölder and Lipschitz coefficients.

1. Introduction

The great number of problems for the large scale scientific and engineering computations concern the numerical solutions of various problems for the partial differential equations and systems in mathematical physics. The finite difference method is the most commonly used in these computations. So the theoretical and numerical studies of the finite difference schmes for the problems of the partial differential equations and systems naturally call people's great attentions.

The imbedding theorems and the interpolation formulas for the functions of Sobolev's spaces are very useful in the linear and nonlinear theory of the partial differential equations [1-4]. It is natural that the analogous extensions of the interpolation formulas for the discrete functional spaces must play the extremely important role in the study of the finite difference approximations to the problems of linear and nonlinear partial differential equations and systems. The discrete interpolatin formulas and their consequences can be used in the study of the convergence and stability of the finite difference schemes for the various problems of linear and nonlinear systems of partial differential equations of different types. And they can also be used to construct the weak, generalized and classical, local and global solution for the problems of partial differential equations and systems [5-11].

The finite difference schemes with unequal meshsteps for the problems of partial differential equations are much more complicated than the schemes with equal meshstep. There are only very few simple results concerning this topic. Establishment of the general interpolation formulas for the spaces of discrete functions with unequal meshsteps obviously gives the possibility and strong apparatus for the systematic studies of the finite difference schemes

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with unequal meshsteps for the problems of partial differential equations.

The purpose of the present work is to establish a series of general interpolation formulas for the discrete functional spaces of discrete functions with equal and unequal meshsteps. These general interpolation formulas give the connected relationship among the discrete norms as the summations of powers, the maximum modulo and the Lipschitz and Hölder quotients for different discrete functional spaces. Also a series of consequences, derivations and applications for these interpolation formulas are justified. They are very commonly used in the further study for the finite difference approximations to the theory of partial differential equations.

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Let us divide the finite interval $[0, l]$ into the small segments by the grid points $\{x_j, | j = 0, 1, \cdots, J\}$, where $0 = x_0 < x_1 < \cdots < x_{J-1} < x_J = l$, J is an integer and $h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 \, (j = 0, 1, \cdots, J-1)$ are the equal and unequal meshsteps. The discrete function $u_h = \{u_j \,|\, j = 0, 1, \cdots, J\}$ is defined on the grid points $\{x_j \,|\, j = 0, 1, \cdots, J\}$ with unequal in general meshsteps $h = \{h_{j+\frac{1}{2}} \,|\, j = 0, 1, \cdots, J-1\}$. Let us denote $\Delta_+ u_j = u_{j+1} - u_j$ or simply $\Delta u_j = u_{j+1} - u_j \, (j = 0, 1, \cdots, J\}$ and $\Delta_- u_j = u_j - u_{j-1} \, (j = 0, 1, \cdots, J)$.

Now let us introduce some notations of the difference quotients for the discrete function $u_h = \{u_j \,|\, j = 0, 1, \cdots, J\}$. As the discrete functions we take the notation for the difference quotient of first order

$$\delta u_h = \left\{ \delta u_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}} \right| \left. j = 0, 1, \cdots, J-1 \right\}, \tag{1}$$

which can be regarded as a discrete function defined on the grid points

$$\left\{ x^{(1)}_{j+\frac{1}{2}} = \frac{1}{2}(x_{j+1} + x_j) \right| \left. j = 0, 1, \cdots, J-1 \right\}$$

of the interval $[x^{(1)}_{\frac{1}{2}}, x^{(1)}_{J-\frac{1}{2}}]$ of length $x^{(1)}_{J-\frac{1}{2}} - x^{(1)}_{\frac{1}{2}} = l - \frac{1}{2}(h_{\frac{1}{2}} + h_{J-\frac{1}{2}})$ with the unequal in general meshsteps

$$\left\{ h^{(1)}_{j+\frac{1}{2}} = h_{j+\frac{1}{2}} \right| \left. j = 0, 1, \cdots, J-1 \right\}.$$

The difference quotient of second order for the discrete function $u_h = \{u_j \,|\, j = 0, 1, \cdots, J\}$ is a discrete function

$$\delta^2 u_h = \left\{ \delta^2 u_j = \frac{\delta u_{j+\frac{1}{2}} - \delta u_{j-\frac{1}{2}}}{h^{(2)}_j} \right| \left. j = 1, \cdots J-1 \right\}. \tag{2}$$

The grid points of this discrete function are

$$\left\{ x^{(2)}_j = \frac{1}{2} \left(x^{(1)}_{j+\frac{1}{2}} + x^{(1)}_{j-\frac{1}{2}} \right) \right| \left. j = 0, 1, \cdots, J-1 \right\}$$

of the interval $[x_1^{(2)}, x_{J-1}^{(2)}]$ with length $x_{J-1}^{(2)} - x_1^{(2)}$ and the corresponding unequal meshsteps are

$$\left\{ h_j^{(2)} = \frac{1}{2}\left(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}}\right) \middle| j = 0, 1, \cdots, J-1 \right\}.$$

For the difference quotients of higher order, we have

$$\begin{aligned} \delta^3 u_h &= \left\{ \delta^3 u_{j+\frac{1}{2}} = \frac{\delta^2 u_{j+1} - \delta^2 u_j}{h_{j+\frac{1}{2}}^{(3)}} \middle| j = 1, \cdots, J-2 \right\}, \\ \delta^4 u_h &= \left\{ \delta^4 u_j = \frac{\delta^3 u_{j+\frac{1}{2}} - \delta^3 u_{j-\frac{1}{2}}}{h_j^{(4)}} \middle| j = 2, \cdots, J-2 \right\}, \end{aligned} \tag{3}$$

$$\begin{aligned} \delta^{2k+1} u_h &= \left\{ \delta^{2k+1} u_{j+\frac{1}{2}} = \frac{\delta^{2k} u_{j+1} - \delta^{2k} u_j}{h_{j+\frac{1}{2}}^{(2k+1)}} \middle| j = k, k+1, \cdots, J-(k+1) \right\}, \\ \delta^{2k+2} u_h &= \left\{ \delta^{2k+2} u_j = \frac{\delta^{2k+1} u_{j+\frac{1}{2}} - \delta^{2k+1} u_{j-\frac{1}{2}}}{h_j^{(2k+2)}} \middle| j = k+1, \cdots, J-(k+1) \right\}, \\ & k = 0, 1, \cdots, \end{aligned}$$

where

$$\begin{aligned} h_{j+\frac{1}{2}}^{(1)} &= h_{j+\frac{1}{2}}; \\ h_j^{(2)} &= \frac{1}{2}(h_{j+\frac{1}{2}}^{(1)} + h_{j-\frac{1}{2}}^{(1)}) = \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}}); \\ h_{j+\frac{1}{2}}^{(3)} &= \frac{1}{2}(h_{j+1}^{(2)} + h_j^{(2)}) = \frac{1}{4}(h_{j+\frac{1}{2}} + 2h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}}); \\ h_j^{(4)} &= \frac{1}{2}\left(h_{j+\frac{1}{2}}^{(3)} + h_{j-\frac{1}{2}}^{(3)}\right) = \frac{1}{8}\left(h_{j+\frac{3}{2}} + 3h_{j+\frac{1}{2}} + 3h_{j-\frac{1}{2}} + h_{j-\frac{3}{2}}\right); \\ & \cdots \\ h_{j+\frac{1}{2}}^{(2k+1)} &= \frac{1}{2}\left(h_{j+1}^{(2k)} + h_j^{(2k)}\right) = \frac{1}{2^{2k}} \sum_{i=0}^{2k} \binom{2k}{i} h_{j+k+\frac{1}{2}-i}; \\ h_j^{(2k+2)} &= \frac{1}{2}\left(h_{j+\frac{1}{2}}^{(2k+1)} + u_{j-\frac{1}{2}}^{(2k+1)}\right) = \frac{1}{2^{2k+1}} \sum_{i=0}^{2k+1} \binom{2k+1}{i} h_{j+k+\frac{1}{2}-i}, \\ & k = 0, 1, \cdots. \end{aligned} \tag{4}$$

The discrete difference quotients $\delta^{2k+1} u_h$ and $\delta^{2k+2} u_h$ $(k \geq 0)$ can be regarded as the discrete functions defined on the grid points $\{x_{j+\frac{1}{2}}^{(2k+1)} \,|\, j = k, \cdots, J-(k+1)\}$ and $\{x_j^{(2k+2)} | j = k+1, \cdots, J_1 - (k+1)\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}}^{(2k+1)} j = k, \cdots, J-(k+1)\}$ and $\{h_j^{(2k+2)} | j = k+1, \cdots, J-(k+1)\}$ of the intervals $[x_{k+\frac{1}{2}}^{(2k+1)}, x_{J-(k+\frac{1}{2})}^{(2k+1)}]$ and $[x_{k+1}^{(2k+1)}, x_{J-(k+1)}^{(2k+2)}]$ with the lengths $\bar{l}_{2k+1} = x^{(2k+1)}_{J-(k+\frac{1}{2})} - x^{(2k+1)}_{k+\frac{1}{2}}$ and $\bar{l}_{2k+2} = x^{(2k+2)}_{J-(k+1)} - x^{(2k+2)}_{k+1}$ repectively, where

$$x^{(2k+1)}_{j+\frac{1}{2}} = \frac{1}{2^{2k}} \sum_{i=0}^{2k+1} \binom{2k+1}{i} x_{j+i-k}, \qquad (j = k, \cdots, J-(k+1));$$
$$x^{(2k+2)}_{j} = \frac{1}{2^{2k+1}} \sum_{i=0}^{2k+2} \binom{2k+2}{i} x_{j+i-(k+1)}, \qquad (j = k+1, \cdots, J-(k+1)) \tag{5}$$

and

$$\begin{aligned} h^{(2k+1)}_{j+\frac{1}{2}} &= x^{(2k)}_{j+1} - x^{(2k)}_{j}; \\ x^{(2k+1)}_{j+\frac{1}{2}} &= \frac{1}{2}\left(x^{(2k)}_{j+1} + x^{(2k)}_{j}\right), \qquad j = k, \cdots, J-(k+1); \\ h^{(2k+2)}_{j} &= x^{(2k+1)}_{j+\frac{1}{2}} - x^{(2k+1)}_{j-\frac{1}{2}}, \\ x^{(2k+2)}_{j} &= \frac{1}{2}\left(x^{(2k+1)}_{j+\frac{1}{2}} + x^{(2k+1)}_{j-\frac{1}{2}}\right), \qquad j = k+1, \cdots, J-(k+1) \end{aligned} \tag{6}$$

with $x^0_j = x_j (j = 0, 1, \cdots, J)$.

Let us denote

$$h^* = \max_{j=0,1,\cdots,J-1} h_{j+\frac{1}{2}}, \qquad h_i = \min_{j=0,1,\cdots,J-1} h_{j+\frac{1}{2}}. \tag{7}$$

It is clear that

$$h^* \geq \max_{j=k,\cdots,J-(k+1)} h^{(2k+1)}_{j+\frac{1}{2}}, \qquad h^* \geq \max_{j=k+1,\cdots,J-(k+1)} h^{(2k+2)}_{j},$$

and

$$h_* \leq \min_{j=k,\cdots,J-(k+1)} h^{(2k+1)}_{j+\frac{1}{2}}, \qquad h_* \leq \min_{j=k+1,\cdots,J-(k+1)} h^{(2k+2)}_{j}$$

for $k = 0, 1, \cdots$. And it can be verified that

$$l \geq \bar{l}_{2k+1} \geq l - 2kh^* \geq \frac{1}{2}l, \qquad l > \bar{l}_{2k+2} \geq l - (2k+1)h^* \geq \frac{1}{2}l$$

for $k = 0, 1, \cdots$.

3

The norms of the disctrte function $u_h = \{u_j \,|\, j = 0, 1, \cdots, J\}$ with unequal meshsteps are defined as

$$\|u_h\|_p = \left(\frac{1}{2}|u_0|^p h_{\frac{1}{2}} + \sum_{j=1}^{J-1} |u_j|^p \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}}) + \frac{1}{2}|u_J|^p h_{J-\frac{1}{2}}\right)^{\frac{1}{p}} \tag{8}$$

or

$$\|u_h\|_p = \left(\sum_{j=0}^{J} \frac{1}{2}(h_{j+\frac{1}{2}} + h_{j-\frac{1}{2}})|u_j|^p\right)^{\frac{1}{p}} \tag{9}$$

or

$$\|u_h\|_p = \left(\sum_{j=0}^{J-1} \frac{1}{2}(|u_j|^p + |u_{j+1}|^p) h_{j+\frac{1}{2}} \right)^{\frac{1}{p}} \tag{10}$$

and

$$\|u_h\|_\infty = \max_{j=0,1,\cdots,J} |u_j|, \tag{11}$$

where $h_{-\frac{1}{2}} = h_{J+\frac{1}{2}} = 0$ and $1 \leq p < \infty$.

The difference quotient of first order is the discrete function δu_h has the norm as

$$\|\delta u_h\|_\infty = \max_{j=0,1,\cdots,J-1} |\delta u_{j+\frac{1}{2}}| \tag{12}$$

and

$$\|\delta u_h\|_p = \left(\sum_{j=0}^{J-1} |\delta u_{j+\frac{1}{2}}|^p h_{j+\frac{1}{2}} \right)^{\frac{1}{p}} \tag{13}$$

where $1 \leq p < \infty$ is a real number. The norms of the difference quotients $\delta^2 u_h$ of second order for the discrete function u_h have the expressions as

$$\|\delta^2 u_h\|_\infty = \max_{j=1,\cdots,J-1} |\delta^2 u_j| \tag{14}$$

and

$$\|\delta^2 u_h\|_p = \left(\sum_{j=1}^{J-1} |\delta^2 u_j|^p h_j^{(2)} \right)^{\frac{1}{p}}, \tag{15}$$

where $1 \leq p < \infty$.

Then for the norms of the difference quotients $\delta^k u_h$ of order $k \geq 1$, we take the notations as follows:

$$\begin{aligned} \|\delta^{2k+1} u_h\|_p &= \left(\sum_{j=k}^{J-(k+1)} |\delta^{2k+1} u_{j+\frac{1}{2}}|^p h_{j+\frac{1}{2}}^{(2k+1)} \right)^{\frac{1}{p}}, \\ \|\delta^{2k+2} u_h\|_p &= \left(\sum_{j=k+1}^{J-(k+1)} |\delta^{2k+2} u_j|^p h_j^{(2k+2)} \right)^{\frac{1}{p}}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \|\delta^{2k+1} u_h\|_\infty &= \max_{j=k,\cdots,J-(k+1)} |\delta^{2k+1} u_{j+\frac{1}{2}}|, \\ \|\delta^{2k+2} u_h\|_\infty &= \max_{j=k+1,\cdots,J-(k+1)} |\delta^{2k+2} u_j|, \end{aligned} \tag{17}$$

where $1 \leq p < \infty$ and $k = 0, 1, \cdots$.

Denote by

$$M = \max_{j=0,1,\cdots,J-1} \left\{ \frac{h_{j-\frac{1}{2}}}{h_{j+\frac{1}{2}}}, \frac{h_{j+\frac{1}{2}}}{h_{j-\frac{1}{2}}} \right\}$$

the maximum ratio constant of two consecutive unequal meshsteps or simply the ratio constant of meshsteps.

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Lemma 1. For any diserete functions $u_h = \{u_j \mid j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j \mid j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 \mid j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for any constants $1 \leq q, r \leq \infty$ and $q \leq p \leq \infty$, there is

$$\|u_h\|_p \leq C(\|u_h\|_q^{1-\alpha}\|\delta u_h\|_r^{\alpha} + l^{\frac{1}{p}-\frac{1}{q}}\|u_h\|_q) \tag{18}$$

with

$$\frac{1}{p} = \frac{1-\alpha}{q} + \alpha\left(\frac{1}{r} - 1\right) \tag{19}$$

and

$$0 \leq \alpha \leq \frac{\frac{1}{q}}{1 - \frac{1}{r} + \frac{1}{q}} \leq 1, \tag{20}$$

where C is a constant independent of the constants p, q, r, the finite length $1 < \infty$, the meshsteps $\{h_{j+\frac{1}{2}} > 0 \mid j = 0, 1, \cdots, J-1\}$ and the discrete function u_h.

Proof. For any $u_h = \{u_j \mid j = 0, 1, \cdots, J\}$, we have

$$|u_m|^d - |u_s|^d \leq |u_m^d - u_s^d| \leq d\sum_{j=s}^{m-1}(|u_{j+1}|^{d-1} + |u_j|^{d-1})|u_{j+1} - u_j|,$$

where $d > 1$ and $0 \leq s < m \leq J$. Let $1 \leq g, r < \infty$ and

$$\frac{1}{g} + \frac{1}{r} = 1.$$

Here we then have

$$\begin{aligned}
|u_m|^d \leq & d\left[\sum_{j=s}^{m-1}(|u_{j+1}|^{d-1} + |u_j|^{d-1})^g h_{j+\frac{1}{2}}\right]^{\frac{1}{g}}\left[\sum_{j=s}^{m-1}\left|\frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}}\right|^r h_{j+\frac{1}{2}}\right]^{\frac{1}{r}} + |u_s|^d \\
\leq & 2d\left[\sum_{j=s}^{m-1}(|u_{j+1}|^{(d-1)g} + |u_j|^{(d-1)g})h_{j+\frac{1}{2}}\right]^{\frac{1}{g}}\|\delta u_h\|_r + |u_s|^d \\
\leq & 2d2^{\frac{1}{g}}\|u_h\|_{(d-1)g}^{d-1}\|\delta u_h\|_r + \|u_s\|^d.
\end{aligned}$$

Take $(d-1) = q(q \geq 1)$, then

$$\frac{1}{d} = \frac{\frac{1}{q}}{1 - \frac{1}{r} + \frac{1}{q}}.$$

If for $j = 0, 1, \cdots, J$, $|u_j| \geq a > 0$, then

$$\|u_h\|_q \geq a l^{\frac{1}{q}}.$$

Thus there always exists such a u_s, that

$$\|u_h\|_q \geq |u_s| l^{\frac{1}{q}}.$$

Taking this special u_s, we get for any $m = 0, 1, \cdots, J$,

$$|u_m|^d \leq 4d\|u_h\|_q^{d-1}\|\delta u_h\|_r + l^{-\frac{d}{q}}\|u_h\|_q^d.$$

Hence we have

$$\|u_h\|_\infty \leq 4\|u_h\|_q^{1-\frac{1}{d}}\|\delta u_h\|_r^{\frac{1}{d}} + l^{-\frac{1}{q}}\|u_h\|_q,$$

where $(4d)^{\frac{1}{d}} \leq 4$ for $d \geq 1$.

For any $1 \leq q \leq p < \infty$, there is

$$\|u_h\|_p^p \leq \|u_h\|_\infty^{p-q}\|u_h\|_q^q.$$

Therefore, we have

$$\|u_h\|_p \leq 4\|u_h\|_q^{1-\alpha}\|\delta u_h\|_r^\alpha + l^{\frac{1}{p}-\frac{1}{q}}\|u_h\|_q,$$

where

$$\alpha = \frac{\frac{1}{q} - \frac{1}{p}}{1 - \frac{1}{r} + \frac{1}{q}}$$

for any $1 \leq q \leq p < \infty$.

By means of Hölder inequality, we have

$$\frac{\|u_h\|_q}{l^{\frac{1}{q}}} \leq \frac{\|u_h\|_p}{l^{\frac{1}{p}}} \leq \|u_h\|_\infty, \qquad p \geq q$$

and then also

$$\frac{\|\delta u_h\|_r}{l^{\frac{1}{r}}} \leq \|\delta u_h\|_\infty.$$

These show that

$$\lim_{q\to\infty} \|u_h\|_q = \|u_h\|_\infty, \qquad \lim_{r\to\infty} \|\delta u_h\|_r = \|\delta u_h\|_\infty. \tag{21}$$

Hence the obtained estimate is valid also for $r = \infty$ and $q = \infty$. Then the lemma is proved.

Lemma 2. For every discrete function $u_h = \{u_j \,|\, j = 0, 1, \cdots, J\}$ defined on the grid points $\{x_j \,|\, j = 0, 1, \cdots, J\}$ with unequal meshsteps $\{h_{j+\frac{1}{2}} = x_{j+1} - x_j > 0 \,|\, j = 0, 1, \cdots, J-1\}$ of the interval $[0, l]$ of finite length $l < \infty$ and for any constants $1 \leq q, r \leq \infty$ and $q \leq p \leq \infty$, there is

$$\|\delta^k u_h\|_p \leq C(\|\delta^k u_h\|_q^{1-\alpha}\|\delta^{k+1} u_h\|_r^\alpha + l^{\frac{1}{p}-\frac{1}{q}}\|\delta^k u_h\|_q) \tag{22}$$

with

$$\frac{1}{p}=\frac{1-\alpha}{q}+\alpha\left(\frac{1}{r}-1\right) \tag{23}$$

and

$$0\le\alpha\le\frac{\frac{1}{q}}{1-\frac{1}{r}+\frac{1}{q}}\le 1, \tag{24}$$

where $k\ge 1$ and C is a constant independent of the constants p,q,r, the finite length $l<\infty$, the meshsteps $\{h_{j+\frac{1}{2}}>0\,|\,j=0,1,\cdots,J-1\}$ and the discrete function u_h and dependent on the ratio constant M of meshsteps.

Proof. For the sake of brevity, let us consider the case of k being odd integer, $k=2k'+1, k'=0,1,\cdots$. Then let $v_h=\delta^k u_h$ or

$$v_h=\{v_j=\delta^k u_{j+\frac{1}{2}}\,|\,j=k',\cdots,J-(k'+1)\}.$$

This discrete function $v_h=\delta^k u_h$ is defined on the grid points

$$\left\{y_j=x^{(k)}_{j+\frac{1}{2}}\,|\,j=k',\cdots,J-(k'+1)\right\}$$

with the meshsteps

$$\left\{\tau_{j+\frac{1}{2}}=y_{j+1}-y_j=x^{(k)}_{j+\frac{3}{2}}-x^{(k)}_{j+\frac{1}{2}}=h^{(k+1)}_{j+1}\,|\,j=k',\cdots,J-k'-2\right\}$$

on the interval $[y_{k'},y_{J-(k'+1)}]\equiv[x^{(k)}_{k'+\frac{1}{2}},x^{(k)}_{J-k'-\frac{1}{2}}]$ of length $\bar{l}_k=y_{J-k'-1}-y_{k'}=x^{(k)}_{J-k'-\frac{1}{2}}-x^{(k)}_{k'+\frac{1}{2}}\ge\frac{1}{2}l$. Here we also have $\delta v_h=\delta^{k+1}u_h$, in fact

$$\delta v_{j+\frac{1}{2}}=\frac{v_{j+1}-v_j}{\tau_{j+\frac{1}{2}}}=\frac{\delta^k u_{j+\frac{3}{2}}-\delta^k u_{j+\frac{1}{2}}}{h^{(k+1)}_{j+1}}=\delta^{k+1}u_{j+1}$$

for $j=k'\cdots,J-k'-2$.

By the same way as the begin of the proof of Lemma 1, we have for $d>0, k'\le s<m\le J-k'-1$ and $1\le q,r<\infty$ with $\frac{1}{g}+\frac{1}{r}=1$, the estimate

$$|v_m|^d\le 2d\left[\sum_{j=s}^{m-1}(|v_{j+1}|^q+|v_j|^q)\tau_{j+\frac{1}{2}}\right]^{\frac{1}{g}}\left[\sum_{j=s}^{m-1}\left|\frac{v_{j+1}-v_j}{\tau_{j+\frac{1}{2}}}\right|^r\tau_{j+\frac{1}{2}}\right]^{\frac{1}{r}}+|v_s|^d$$

or

$$\begin{aligned}|\delta^k u_{m+\frac{1}{2}}|^d\le&2d\left[\sum_{j=s}^{m-1}(|\delta^k u_{j+\frac{3}{2}}|^q+|\delta^k u_{j+\frac{1}{2}}|^q)h^{(k+1)}_{j+1}\right]^{\frac{1}{g}}\\&\times\left[\sum_{j=s}^{m-1}|\delta^{k+1}u_{j+1}|^r h^{(k+1)}_{j+1}\right]^{\frac{1}{r}}+|\delta^k u_{s+\frac{1}{2}}|^d,\end{aligned}$$

where $q = (d-1)g \geq 1$. Then we have

$$|\delta^k u_{m+\frac{1}{2}}|^d \leq 2dW^{\frac{1}{g}} \|\delta^{k+1} u_h\|_r + |\delta^k u_{s+\frac{1}{2}}|^d,$$

where

$$W = \sum_{j=k'}^{J-k'-2} (|\delta^k u_{j+\frac{3}{2}}|^q + |\delta^k u_{j+\frac{1}{2}}|^q) h_{j+1}^{(k+1)}.$$

Here we have

$$\begin{aligned}
W =& |\delta^k u_{k'+\frac{1}{2}}|^q h_{k'+1}^{(k+1)} + \sum_{j=k'+1}^{(J-k'-2)} |\delta^k u_{j+\frac{1}{2}}|^q (h_j^{(k+1)} + h_{j+1}^{(k+1)}) \\
&+ |\delta^k u_{J-k'-\frac{1}{2}}|^q h_{J-k'-1}^{(k+1)} \\
=& |\delta^k u_{k'+\frac{1}{2}}|^q h_{k'+\frac{1}{2}}^{(k)} \left(\frac{h_{k'+1}^{(k+1)}}{h_{k'+\frac{1}{2}}^{(k)}} \right) + \sum_{j=k'+1}^{J-k'-2} |\delta^k u_{j+\frac{1}{2}}|^q h_{j+\frac{1}{2}}^{(k)} \left(\frac{h_j^{(k+1)} + h_{j+1}^{(k+1)}}{h_{j+\frac{1}{2}}^{(k)}} \right) \\
&+ |\delta^k u_{J-k'-\frac{1}{2}}|^q h_{J-k'-\frac{1}{2}}^{(k)} \left(\frac{h_{J-k'-1}^{(k+1)}}{h_{J-k'-\frac{1}{2}}^{(k)}} \right) \\
\leq& 2\|\delta^k u_h\|_q^q \left\{ \max_{j=k'+1,\cdots,J-k'-1} \frac{h_j^{(k+1)}}{h_{j+\frac{1}{2}}^{(k)}}, \max_{j=k',\cdots,J-k'-2} \frac{h_{j+1}^{(k+1)}}{h_{j+\frac{1}{2}}^{(k)}} \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{h_j^{(k+1)}}{h_{j+\frac{1}{2}}^{(k)}} =& \frac{h_j^{(2k'+2)}}{h_{j+\frac{1}{2}}^{(2k'+1)}} = \frac{\frac{1}{2^{2k'+1}} \sum_{i=0}^{2k'+1} \binom{2k'+1}{i} h_{j+k'+\frac{1}{2}-i}}{\frac{1}{2^{2k'}} \sum_{i=0}^{2k'} \binom{2k'}{i} h_{j+k'+\frac{1}{2}-i}} \\
\leq& \frac{1}{2}(1+M)M^{k-1}, \qquad j = k'+1, \cdots, J-k'-1
\end{aligned}$$

and

$$\frac{h_{j+1}^{(k+1)}}{h_{j+\frac{1}{2}}^{(k)}} \leq \frac{1}{2}(1+M)M^{k-1}, \qquad j = k', \cdots, J-k'-2,$$

then we have

$$W \leq (1+M)M^{k-1} \|\delta^k u_h\|_q^q.$$

This shows that

$$|\delta^k u_{m+\frac{1}{2}}|^d \leq 2d(1+M)^{\frac{1}{g}} M^{\frac{k-1}{g}} \|\delta^k u_h\|_q^{d-1} \|\delta^{k+1} u_h\|_r + |\delta^k u_{s+\frac{1}{2}}|^d.$$

Similarly to the proof of Lemma 1, we get the estimate

$$\|\delta^k u_h\|_p \leq C(M)(\|\delta^k u_h\|_q^{1-\alpha} \|\delta^{k+1} u_h\|_r^\alpha + l^{\frac{1}{p}-\frac{1}{q}} \|\delta^k u_h\|_q)$$

with

$$\frac{1}{p} = \frac{1-\alpha}{q} + \alpha\left(\frac{1}{r} - 1\right),$$

where $C(M)$ depends on the maximum ratio constant M of two consecutive unequal meshsteps. This gives the result of the lemma for the case $k = 2k' + 1$ being odd.

For the case of k being even, $k = 2k', k' = 1, 2, \cdots$, let us denote $v_h = \delta^k u_h$ or

$$v_h = \{v_j = \delta^k u_j \,|\, j = k' \cdots, J - k'\}.$$

This discrete function v_h is defined on the grid points

$$\{y_j = x_j^{(k)} \,|\, j = k' \cdots, J - k'\}$$

with the meshsteps

$$\left\{\tau_{j+\frac{1}{2}} = y_{j+1} - y_j = x_{j+1}^{(k)} - x_j^{(k)} = h_{j+\frac{1}{2}}^{(k+1)} \,|\, j = k', \cdots, J - k' - 1\right\}$$

on the interval $[y_{k'}, y_{J-k'}] \equiv [x_{k'}^{(k)}, x_{J-k'}^{(k)}]$ of length $\bar{l}_k = y_{J-k'} - y_{k'} = x_{J-k'}^{(k)} - x_{k'}^{(k)} \geq \frac{1}{2}l$. Here we have $\delta v_h = \delta^{k+1} u_h$, in fact

$$\delta v_{j+\frac{1}{2}} = \frac{v_{j+1} - v_j}{\tau_{j+\frac{1}{2}}} = \frac{\delta^k u_{j+1} - \delta^k u_j}{h_{j+\frac{1}{2}}^{(k+1)}} = \delta^{k+1} u_{j+\frac{1}{2}}$$

for $j = k', \cdots, J - k' - 1$.

Similarly we have for $d > 0, k' \leq s < m \leq J - k'$ and $1 \leq q, r < \infty$ with $\frac{1}{g} + \frac{1}{r} = 1$, the estimate

$$|v_m|^d \leq 2d \left[\sum_{j=s}^{m-1} (|v_{j+1}|^q + |v_j|^q)\tau_{j+\frac{1}{2}}\right]^{\frac{1}{g}} \left[\sum_{j=s}^{m-1} \left|\frac{v_{j+1} - v_j}{\tau_{j+\frac{1}{2}}}\right|^r \tau_{j+\frac{1}{2}}\right]^{\frac{1}{r}} + |\delta^k v_s|^d$$

or

$$\begin{aligned}|\delta^k u_m|^d \leq & 2d \left[\sum_{j=s}^{m-1} (|\delta^k u_{j+1}|^q + |\delta^k u_j|^q) h_{j+\frac{1}{2}}^{(k+1)}\right]^{\frac{1}{g}} \\ & \left[\sum_{j=s}^{m-1} \left|\delta^{k+1} u_{j+\frac{1}{2}}\right|^r h_{j+\frac{1}{2}}^{(k+1)}\right]^{\frac{1}{r}} + |\delta^k u_s|^d,\end{aligned}$$

where $q = (d-1)g \geq 1$. Then we have also

$$|\delta^k u_m|^d \leq 2dW^{\frac{1}{g}} \|\delta^{k+1} u_h\|_r + |\delta^k u_s|^d,$$

where

$$W = \sum_{j=k'}^{J-k'-1} (|\delta^k u_{j+1}|^q + |\delta^k u_j|^q) h_{j+\frac{1}{2}}^{(k+1)}.$$

Similarly, we can prove that

$$W \leq (1 + M) M^{k-1} \|\delta^k u_h\|_q^q.$$