

国外数学名著系列

(影印版) 26

Christian Bonatti Lorenzo J. Díaz Marcelo Viana

Dynamics Beyond Uniform Hyperbolicity

A Global Geometric and Probabilistic Perspective

一致双曲性之外的动力学 一种整体的几何学的与概率论的观点

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来,需要数学家淡泊名利并付出更艰苦地努力。另一方面,我们也要从客观上为数学家创造更有利的发展数学事业的外部环境,这主要是加强对数学事业的支持与投资力度,使数学家有较好的工作与生活条件,其中也包括改善与加强数学的出版工作。

从出版方面来讲,除了较好较快地出版我们自己的成果外,引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说,施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书,使我国广大数学家能以较低的价格购买,特别是在边远地区工作的数学家能普遍见到这些书,无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权,一次影印了23本施普林格出版社出版的数学书,就是一件好事,也是值得继续做下去的事情。大体上分一下,这23本书中,包括基础数学书5本,应用数学书6本与计算数学书12本,其中有些书也具有交叉性质。这些书都是很新的,2000年以后出版的占绝大部分,共计16本,其余的也是1990年以后出版的。这些书可以使读者较快地了解数学某方面的前沿,例如基础数学中的数论、代数与拓扑三本,都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点,基础数学类的书以“经典”为主,应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家,例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士,曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然,23本书只能涵盖数学的一部分,所以,这项工作还应该继续做下去。更进一步,有些读者面较广的好书还应该翻译成中文出版,使之有更大的读者群。

总之,我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持,并盼望这一工作取得更大的成绩。

王 元

2005年12月3日

Preface

What is Dynamics about?

In broad terms, the goal of Dynamics is to describe the long term evolution of systems for which an “infinitesimal” evolution rule is known. Examples and applications arise from all branches of science and technology, like physics, chemistry, economics, ecology, communications, biology, computer science, or meteorology, to mention just a few.

These systems have in common the fact that each possible state may be described by a finite (or infinite) number of observable quantities, like position, velocity, temperature, concentration, population density, and the like. Thus, the space of states (*phase space*) is a subset M of an Euclidean space \mathbb{R}^m . Usually, there are some constraints between these quantities: for instance, for ideal gases pressure times volume must be proportional to temperature. Then the space M is often a manifold, an n -dimensional surface for some $n < m$.

For *continuous time* systems, the evolution rule may be a differential equation: to each state $x \in M$ one associates the speed and direction in which the system is going to evolve from that state. This corresponds to a vector field $X(x)$ in the phase space. Assuming the vector field is sufficiently regular, for instance continuously differentiable, there exists a unique curve tangent to X at every point and passing through x : we call it the *orbit* of x .

Even when the real phenomenon is supposed to evolve in continuous time, it may be convenient to consider a *discrete time* model, for instance, if observations of the system take place at fixed intervals of time only. In this case the evolution rule is a transformation $f : M \rightarrow M$, assigning to the present state $x \in M$ the one $f(x)$ the system will be in after one unit of time. Then the *orbit* of x is the sequence x_n obtained by iteration of the transformation: $x_{n+1} = f(x_n)$ with $x_0 = x$.

In both cases, one main problem is *to describe the behavior as time goes to infinity for the majority of orbits*, for instance, for a full probability set of initial states. Another problem, equally important, is *to understand whether that limit behavior is stable under small changes of the evolution law*, that is,

whether it remains essentially the same if the vector field X or the transformation f are slightly modified. It is easy to see why this is such a crucial question, both conceptually and for the practical applications: mathematical models are always simplifications of the real system (a model of a chemical reaction, say, taking into account the whole universe would be obviously unpractical ...) and, in the absence of stability, conclusions drawn from the model might be specific to it and not have much to do with the actual phenomenon.

It is tempting to try to address these problems by “solving” the dynamical system, that is, by looking for analytic expressions for the trajectories, and indeed that was the prevailing point of view in differential equations until little more than a century ago. However, that turns out to be impossible in most cases, both theoretically and in practice. Moreover, even when such an analytic expressions can be found, it is usually difficult to deduce from them useful conclusions about the global dynamics.

Then, by the end of the 19th century, Poincaré proposed to bring in methods from other disciplines, such as topology or ergodic theory, *to find qualitative information on the dynamics without actually finding the solutions*. A beautiful example, among many others, is the Poincaré-Birkhoff theorem stating that an area preserving homeomorphism of the annulus which rotates the two boundary circles in opposite directions must have some fixed point. This proposal, which was already present in Poincaré’s early works and attained full maturity in his revolutionary contribution to Celestial Mechanics, is usually considered to mark the birth of Dynamics as a mathematical discipline.

Hyperbolicity and stability.

This direction was then pursued by Birkhoff in the thirties. In particular, he was much interested in the phenomenon of transverse *homoclinic points*, that is, points where the stable manifold and the unstable manifold of the same fixed or periodic saddle point intersect transversely. This phenomenon had been discovered in the context of the N -body problem by Poincaré, who immediately recognized it as a major source of dynamical complexity. Birkhoff made this intuition much more precise by proving that any transverse homoclinic orbit is accumulated by periodic points. A definitive understanding of this phenomenon unfolded at the beginning of the sixties, when Smale introduced the *horseshoe*, a simple geometric model whose dynamics can be understood rather completely, and whose presence in the system is equivalent to the existence of transverse homoclinic points.

The horseshoe, and other robust models containing infinitely many periodic orbits, such as Thom’s cat map (hyperbolic toral automorphism), were unified by Smale’s notion of uniformly *hyperbolic set*: a subset of the phase space invariant under the dynamical system and such that the tangent space at each point splits into two complementary subspaces that are uniformly contracted under, respectively, forward and backward iterations. Then Smale also introduced the notion of *uniformly hyperbolic dynamical system* (Axiom A)

which essentially means that the limit set, consisting of all forward or backward accumulation points of orbits, is a hyperbolic set. These ideas much influenced contemporary remarkable work of Anosov where it was shown that the geodesic flow on any manifold with negative curvature is ergodic.

Another major achievement of uniform hyperbolicity was to provide a characterization of structurally stable dynamical systems. The notion of *structural stability*, introduced in the thirties by Andronov, Pontrjagin, means that the whole orbit structure remains the same when the system is slightly modified: there exists a homeomorphism of the ambient manifold mapping orbits of the initial system into orbits of the modified one, and preserving the time arrow. Indeed, uniform hyperbolicity proved to be the key ingredient of structurally stable systems, together with a transversality condition, as conjectured by Palis, Smale.

In the process, a theory of uniformly hyperbolic systems was developed, mostly from the sixties to the mid eighties, whose importance extended much beyond the original objectives. It was part of a revolution in our vision of determinism, strongly driven by observations originating from experimental sciences, which shattered the classical opposition between deterministic evolutions and random evolutions. The uniformly hyperbolic theory provided a mathematical foundation for the fact that deterministic systems, even with a small number of degrees of freedom, often present chaotic behavior in a robust fashion. Thus, it led to the almost paradoxical conclusion that “chaos” may be stable.

On the other hand, structural stability and uniform hyperbolicity were soon realized to be less universal properties than was initially thought: there exist many classes of systems that are robustly unstable and non-hyperbolic and, in fact, that is often the case for specific models coming from concrete applications. The dream of a general paradigm in Dynamics had to be postponed.

Beyond uniform hyperbolicity.

The next years saw the theory being extended in several distinct directions:

- The study of specific classes of systems, such as quadratic maps, Lorenz flows, and Hénon attractors, which introduced a host of new methods and ideas.
- Bifurcation theory including, in particular, the study of the boundary of uniformly hyperbolic systems, and of the local and global mechanisms leading to chaotic behavior, especially homoclinic bifurcations.
- New developments in the ergodic theory of smooth systems and, especially, the theory of non-uniformly hyperbolic systems (Pesin theory).
- Weaker formulations of hyperbolicity, still with a uniform flavor but where one allows for invariant “neutral” directions (partial hyperbolicity, projective hyperbolicity or existence of a dominated splitting).

- The converse implication in the stability conjecture (hyperbolicity is necessary for stability), which led to the introduction of new perturbation lemmas (ergodic closing lemma, connecting lemma).

Building on remarkable progress obtained in these directions, especially in the eighties and early nineties, several ideas have been put forward and a new point of view has emerged recently, which again allow us to dream of a global understanding of “most” dynamical systems. Initiated as a survey paper requested to us by David Ruelle, the present work is an attempt to put such recent developments in a unified perspective, and to point open problems and likely directions of further progress.

Two semi-local mechanisms, very different in nature but certainly not mutually exclusive, have been identified as the main sources of persistently non-hyperbolic dynamics:

- What we call here “critical behavior”, corresponding to critical points in one-dimensional dynamics and, more generally, to homoclinic tangencies, and which is at the heart of Hénon-like dynamics. This is now reasonably well understood, in terms of non-uniformly hyperbolic behavior. Moreover, recent results show that this type of behavior is always present in connection to non-hyperbolic dynamics in low dimensions.
- In higher dimensions, dynamical robustness (robust transitivity, stable ergodicity) extends well outside the uniformly hyperbolic domain, roughly speaking associated to coexistence of uniformly hyperbolic behavior with different unstable dimensions. It requires some uniform geometric structure (transverse invariant bundles: partial hyperbolicity, dominated decomposition) that we refer to as “non-critical behavior”.

On the other hand, new perturbation lemmas permitted to organize the global dynamics of generic dynamical systems, by breaking it into elementary pieces separated by a filtration. A great challenge is to understand the dynamics on (the neighborhood of) these elementary dynamical pieces, which should involve a deeper analysis of the two mechanisms mentioned previously. Indeed, a good understanding has already been possible in several cases, especially at the statistical level.

What is this book, and what is it not?

The text is aimed at researchers, both young and senior, willing to get a quick yet broad view of this part of Dynamics. Main ideas, methods, and results are discussed, at variable degrees of depth, with references to the original works for details and complementary information.

We assume the reader is familiar with the fundamental objects of smooth Dynamics, like manifolds or C^r diffeomorphisms and vector fields, as well as with the basic facts in the local theory of dynamical systems close to a hyperbolic periodic point, such as the Hartman-Grobman linearization theorem and the stable manifold theorem. This material is covered by several

books, like Bowen [86], Irwin [225], Palis, de Melo [342], Ruelle [394], Katok, Hasselblatt [232], or Robinson [382].

Familiarity with the classical theory of uniformly hyperbolic systems is also desirable, of course. This is also covered by a number of books, including Bowen [86], Shub [411], Mañé [281], Palis, Takens [345], and Katok, Hasselblatt [232]. For the reader's convenience, in Chapter 1 we review the main conclusions of the theory that are relevant for our purposes. In that chapter we also give an introductory discussion of robust mechanisms of non-hyperbolicity, and other key issues outside the hyperbolic set-up. This is to be much expanded afterwards, so at that point our presentation is sketchier than elsewhere.

Apart from these pre-requisites, we have tried to keep the text self-contained, giving the precise definitions of all relevant non-elementary notions. Occasionally, this is done in an informal fashion at places where the notion is first needed in a non-crucial way, with the formal definition appearing at some later section where it really is at the heart of the subject. This is especially true about Chapter 1, as explained in the previous paragraph.

Although we have used parts of this book as a basis for graduate courses, it is certainly not designed as a text book that could be used for that purpose all alone. The properties of the main notions are often only stated, and most results are presented with just an outline of the proof.

The book is also not meant to be an exhaustive presentation of the recent results in Dynamics. We are only too conscious of the many fundamental topics we left outside, or touched only briefly. Deciding where to stop could be one of the most difficult and most important problems in this kind of project, and no answer is entirely satisfactory.

How should this book be used and what does it contain?

The 12 chapters are organized so as to convey a global perspective of dynamical systems. The 5 appendices include several other important results, older and new, which we feel should not be omitted, either because they are used in the text or because they provide complementary views of some aspects of the theory.

Although there is, naturally, a global coherence in the text, we have tried to keep the various chapters rather independent, so that the reader may choose to read one chapter without really needing to go through the previous ones. This means that we often recall main notions and statements introduced elsewhere, or else give precise references to where they can be found. On the other hand, the chapters often rely on ideas and results from the appendices.

The main text may be, loosely, split into the following blocks:

- Chapter 1 contains a brief review of uniformly hyperbolic theory and an introduction to main themes to be developed throughout the text.
- Chapters 2 to 4 are devoted to critical behavior in various aspects: one-dimensional dynamics, homoclinic tangencies, Hénon-like dynamics.

- Chapter 5 shows that, for low dimensional systems, far from critical behavior the dynamical behavior is hyperbolic.
- Chapters 6 to 9 treat non-critical behavior, especially the relation between robustness and existence of invariant splittings. While most of the text focusses on dissipative discrete time systems, Chapter 8 deals with conservative diffeomorphisms and Chapter 9 is devoted to flows.
- In Chapter 10 we try to give a global framework for the dynamics of generic maps, where critical and non-critical behavior could fit together.
- Chapter 11 presents some of the progress attained in describing the dynamics in ergodic terms, both in critical and in non-critical situations (either separate or coexisting). Lyapunov exponents are an important tool in this analysis, and Chapter 12 is devoted to their study and control.

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Christian Bonatti
Lorenzo J. Díaz
Marcelo Viana

Contents

1	Hyperbolicity and Beyond	1
1.1	Spectral decomposition	1
1.2	Structural stability	3
1.3	Sinai-Ruelle-Bowen theory	4
1.4	Heterodimensional cycles	6
1.5	Homoclinic tangencies	6
1.6	Attractors and physical measures	7
1.7	A conjecture on finitude of attractors	9
2	One-Dimensional Dynamics	13
2.1	Hyperbolicity	13
2.2	Non-critical behavior	16
2.3	Density of hyperbolicity	18
2.4	Chaotic behavior	18
2.5	The renormalization theorem	20
2.6	Statistical properties of unimodal maps	21
3	Homoclinic Tangencies	25
3.1	Homoclinic tangencies and Cantor sets	26
3.2	Persistent tangencies, coexistence of attractors	27
3.2.1	Open sets with persistent tangencies	32
3.3	Hyperbolicity and fractal dimensions	34
3.4	Stable intersections of regular Cantor sets	38
3.4.1	Renormalization and pattern recurrence	39
3.4.2	The scale recurrence lemma	41
3.4.3	The probabilistic argument	43
3.5	Homoclinic tangencies in higher dimensions	44
3.5.1	Intrinsic differentiability of foliations	45
3.5.2	Frequency of hyperbolicity	47
3.6	On the boundary of hyperbolic systems	50

4	Hénon-like Dynamics	55
4.1	Hénon-like families	56
4.1.1	Identifying the attractor	58
4.1.2	Hyperbolicity outside the critical regions	59
4.2	Abundance of strange attractors	61
4.2.1	The theorem of Benedicks-Carleson	61
4.2.2	Critical points of dissipative diffeomorphisms	62
4.2.3	Some conjectures and open questions	65
4.3	Sinai-Ruelle-Bowen measures	69
4.3.1	Existence and uniqueness	69
4.3.2	Solution of the basin problem	74
4.4	Decay of correlations and central limit theorem	79
4.5	Stochastic stability	83
4.6	Chaotic dynamics near homoclinic tangencies	87
4.6.1	Tangencies and strange attractors	87
4.6.2	Saddle-node cycles and strange attractors	90
4.6.3	Tangencies and non-uniform hyperbolicity	92
5	Non-Critical Dynamics and Hyperbolicity	97
5.1	Non-critical surface dynamics	97
5.2	Domination implies almost hyperbolicity	99
5.3	Homoclinic tangencies <i>vs.</i> Axiom A	100
5.4	Entropy and homoclinic points on surfaces	102
5.5	Non-critical behavior in higher dimensions	104
6	Heterodimensional Cycles and Blenders	107
6.1	Heterodimensional cycles	108
6.1.1	Explosion of homoclinic classes	108
6.1.2	A simplified example	109
6.1.3	Unfolding heterodimensional cycles	113
6.2	Blenders	114
6.2.1	A simplified model	115
6.2.2	Relaxing the construction	117
6.3	Partially hyperbolic cycles	120
7	Robust Transitivity	123
7.1	Examples of robust transitivity	124
7.1.1	An example of Shub	125
7.1.2	An example of Mañé	125
7.1.3	A local criterium for robust transitivity	126
7.1.4	Robust transitivity without hyperbolic directions	127
7.2	Consequences of robust transitivity	128
7.2.1	Lack of domination and creation of sinks or sources	130
7.2.2	Dominated splittings <i>vs.</i> homothetic transformations	132
7.2.3	On the dynamics of robustly transitive sets	134

7.2.4	Manifolds supporting robustly transitive maps	136
7.3	Invariant foliations	138
7.3.1	Pathological central foliations	138
7.3.2	Density of accessibility	140
7.3.3	Minimality of the strong invariant foliations	142
7.3.4	Compact central leaves	144
8	Stable Ergodicity	147
8.1	Examples of stably ergodic systems	148
8.1.1	Perturbations of time-1 maps of geodesic flows	148
8.1.2	Perturbations of skew-products	148
8.1.3	Stable ergodicity without partial hyperbolicity	149
8.2	Accessibility and ergodicity	150
8.3	The theorem of Pugh-Shub	151
8.4	Stable ergodicity of torus automorphisms	152
8.5	Stable ergodicity and robust transitivity	153
8.6	Lyapunov exponents and stable ergodicity	154
9	Robust Singular Dynamics	157
9.1	Singular invariant sets	158
9.1.1	Geometric Lorenz attractors	158
9.1.2	Singular horseshoes	161
9.1.3	Multidimensional Lorenz attractors	163
9.2	Singular cycles	164
9.2.1	Explosions of singular cycles	165
9.2.2	Expanding and contracting singular cycles	166
9.2.3	Singular attractors arising from singular cycles	168
9.3	Robust transitivity and singular hyperbolicity	169
9.3.1	Robust globally transitive flows	170
9.3.2	Robustness and singular hyperbolicity	173
9.4	Consequences of singular hyperbolicity	178
9.4.1	Singularities attached to regular orbits	178
9.4.2	Ergodic properties of singular hyperbolic attractors	179
9.4.3	From singular hyperbolicity back to robustness	180
9.5	Singular Axiom A flows	183
9.6	Persistent singular attractors	186
10	Generic Diffeomorphisms	189
10.1	A quick overview	189
10.2	Notions of recurrence	192
10.3	Decomposing the dynamics to elementary pieces	193
10.3.1	Chain recurrence classes and filtrations	195
10.3.2	Maximal weakly transitive sets	196
10.3.3	A generic dynamical decomposition theorem	197
10.4	Homoclinic classes and elementary pieces	199

10.4.1	Homoclinic classes and maximal transitive sets	199
10.4.2	Homoclinic classes and chain recurrence classes	202
10.4.3	Isolated homoclinic classes	202
10.5	Wild behavior <i>vs.</i> tame behavior	204
10.5.1	Finiteness of homoclinic classes	204
10.5.2	Dynamics of tame diffeomorphisms	205
10.6	A sample of wild dynamics	207
10.6.1	Coexistence of infinitely many periodic attractors	207
10.6.2	C^1 coexistence phenomenon in higher dimensions	208
10.6.3	Generic coexistence of aperiodic pieces	208
11	SRB Measures and Gibbs States	213
11.1	SRB measures for certain non-hyperbolic maps	214
11.1.1	Intermingled basins of attraction	214
11.1.2	A transitive map with two SRB measures	216
11.1.3	Robust multidimensional attractors	217
11.1.4	Open sets of non-uniformly hyperbolic maps	219
11.2	Gibbs u -states for $E^u \oplus E^{cs}$ systems	221
11.2.1	Existence of Gibbs u -states	221
11.2.2	Structure of Gibbs u -states	223
11.2.3	Every SRB measure is a Gibbs u -state	225
11.2.4	Mostly contracting central direction	231
11.2.5	Differentiability of Gibbs u -states	232
11.3	SRB measures for dominated dynamics	233
11.3.1	Non-uniformly expanding maps	234
11.3.2	Existence of Gibbs cu -states	236
11.3.3	Simultaneous hyperbolic times	237
11.3.4	Stability of cu -Gibbs states	239
11.4	Generic existence of SRB measures	240
11.4.1	A piecewise affine model	241
11.4.2	Transfer operators	243
11.4.3	Absolutely continuous invariant measure	245
11.5	Extensions and related results	247
11.5.1	Zero-noise limit and the entropy formula	247
11.5.2	Equilibrium states of non-hyperbolic maps	249
12	Lyapunov Exponents	253
12.1	Continuity of Lyapunov exponents	254
12.2	A dichotomy for conservative systems	258
12.3	Deterministic products of matrices	261
12.4	Abundance of non-zero exponents	264
12.4.1	Bundle-free cocycles	266
12.4.2	A geometric criterium for non-zero exponents	267
12.4.3	Conclusion and an application	267
12.5	Looking for non-zero Lyapunov exponents	269

12.5.1	Removing zero Lyapunov exponents	269
12.5.2	Lower bounds for Lyapunov exponents	270
12.5.3	Genericity of non-uniform hyperbolicity	272
12.6	Hyperbolic measures are exact dimensional	274
A	Perturbation Lemmas	277
A.1	Closing lemmas	278
A.2	Ergodic closing lemma	279
A.3	Connecting lemmas	279
A.4	Some ideas of the proofs	281
A.5	A connecting lemma for pseudo-orbits	284
A.6	Realizing perturbations of the derivative	285
B	Normal Hyperbolicity and Foliations	287
B.1	Dominated splittings	287
B.1.1	Definition and elementary properties	287
B.1.2	Proofs of the elementary properties:	290
B.2	Invariant foliations	293
B.3	Linear Poincaré flows	295
C	Non-Uniformly Hyperbolic Theory	299
C.1	The linear theory	299
C.2	Stable manifold theorem	301
C.3	Absolute continuity of foliations	302
C.4	Conditional measures along invariant foliations	303
C.5	Local product structure	304
C.6	The disintegration theorem	305
D	Random Perturbations	311
D.1	Markov chain model	311
D.2	Iterations of random maps	313
D.3	Stochastic stability	314
D.4	Realizing Markov chains by random maps	317
D.5	Shadowing versus stochastic stability	319
D.6	Random perturbations of flows	320
E	Decay of Correlations	323
E.1	Transfer operators: spectral gap property	324
E.2	Expanding and piecewise expanding maps	325
E.3	Invariant cones and projective metrics	326
E.4	Uniformly hyperbolic diffeomorphisms	328
E.5	Uniformly hyperbolic flows	329
E.6	Non-uniformly hyperbolic systems	331
E.7	Non-exponential convergence	336
E.8	Maps with neutral fixed points	342

E.9 Central limit theorem	344
Conclusion	349
References	353
Index	375