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Christian Bonatti　Lorenzo J. Díaz　Marcelo Viana

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A Global Geometric and Probabilistic Perspective

一致双曲性之外的动力学

一种整体的几何学的与概率论的观点

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Preface

What is Dynamics about?

In broad terms, the goal of Dynamics is to describe the long term evolution of systems for which an "infinitesimal" evolution rule is known. Examples and applications arise from all branches of science and technology, like physics, chemistry, economics, ecology, communications, biology, computer science, or meteorology, to mention just a few.

These systems have in common the fact that each possible state may be described by a finite (or infinite) number of observable quantities, like position, velocity, temperature, concentration, population density, and the like. Thus, the space of states (*phase space*) is a subset M of an Euclidean space $\mathbb{R}^m$. Usually, there are some constraints between these quantities: for instance, for ideal gases pressure times volume must be proportional to temperature. Then the space M is often a manifold, an n-dimensional surface for some $n < m$.

For *continuous time* systems, the evolution rule may be a differential equation: to each state $x \in M$ one associates the speed and direction in which the system is going to evolve from that state. This corresponds to a vector field $X(x)$ in the phase space. Assuming the vector field is sufficiently regular, for instance continuously differentiable, there exists a unique curve tangent to X at every point and passing through x: we call it the *orbit of* x.

Even when the real phenomenon is supposed to evolve in continuous time, it may be convenient to consider a *discrete time* model, for instance, if observations of the system take place at fixed intervals of time only. In this case the evolution rule is a transformation $f : M \to M$, assigning to the present state $x \in M$ the one $f(x)$ the system will be in after one unit of time. Then the *orbit of* x is the sequence x_n obtained by iteration of the transformation: $x_{n+1} = f(x_n)$ with $x_0 = x$.

In both cases, one main problem is *to describe the behavior as time goes to infinity for the majority of orbits*, for instance, for a full probability set of initial states. Another problem, equally important, is *to understand whether that limit behavior is stable under small changes of the evolution law*, that is, whether it remains essentially the same if the vector field X or the transformation f are slightly modified. It is easy to see why this is such a crucial question, both conceptually and for the practical applications: mathematical models are always simplifications of the real system (a model of a chemical reaction, say, taking into account the whole universe would be obviously unpractical ...) and, in the absence of stability, conclusions drawn from the model might be specific to it and not have much to do with the actual phenomenon.

It is tempting to try to address these problems by "solving" the dynamical system, that is, by looking for analytic expressions for the trajectories, and indeed that was the prevailing point of view in differential equations until little more than a century ago. However, that turns out to be impossible in most cases, both theoretically and in practice. Moreover, even when such an analytic expressions can be found, it is usually difficult to deduce from them useful conclusions about the global dynamics.

Then, by the end of the 19th century, Poincaré proposed to bring in methods from other disciplines, such as topology or ergodic theory, *to find qualitative information on the dynamics without actually finding the solutions.* A beautiful example, among many others, is the Poincaré-Birkhoff theorem stating that an area preserving homeomorphism of the annulus which rotates the two boundary circles in opposite directions must have some fixed point. This proposal, which was already present in Poincaré's early works and attained full maturity in his revolutionary contribution to Celestial Mechanics, is usually considered to mark the birth of Dynamics as a mathematical discipline.

Hyperbolicity and stability.

This direction was then pursued by Birkhoff in the thirties. In particular, he was much interested in the phenomenon of transverse *homoclinic points*, that is, points where the stable manifold and the unstable manifold of the same fixed or periodic saddle point intersect transversely. This phenomenon had been discovered in the context of the N-body problem by Poincaré, who immediately recognized it as a major source of dynamical complexity. Birkhoff made this intuition much more precise by proving that any transverse homoclinic orbit is accumulated by periodic points. A definitive understanding of this phenomenon unfolded at the beginning of the sixties, when Smale introduced the *horseshoe*, a simple geometric model whose dynamics can be understood rather completely, and whose presence in the system is equivalent to the existence of transverse homoclinic points.

The horseshoe, and other robust models containing infinitely many periodic orbits, such as Thom's cat map (hyperbolic toral automorphism), were unified by Smale's notion of uniformly *hyperbolic set*: a subset of the phase space invariant under the dynamical system and such that the tangent space at each point splits into two complementary subspaces that are uniformly contracted under, respectively, forward and backward iterations. Then Smale also introduced the notion of *uniformly hyperbolic dynamical system* (Axiom A) which essentially means that the limit set, consisting of all forward or backward accumulation points of orbits, is a hyperbolic set. These ideas much influenced contemporary remarkable work of Anosov where it was shown that the geodesic flow on any manifold with negative curvature is ergodic.

Another major achievement of uniform hyperbolicity was to provide a characterization of structurally stable dynamical systems. The notion of *structural stability*, introduced in the thirties by Andronov, Pontrjagin, means that the whole orbit structure remains the same when the system is slightly modified: there exists a homeomorphism of the ambient manifold mapping orbits of the initial system into orbits of the modified one, and preserving the time arrow. Indeed, uniform hyperbolicity proved to be the key ingredient of structurally stable systems, together with a transversality condition, as conjectured by Palis, Smale.

In the process, a theory of uniformly hyperbolic systems was developed, mostly from the sixties to the mid eighties, whose importance extended much beyond the original objectives. It was part of a revolution in our vision of determinism, strongly driven by observations originating from experimental sciences, which shattered the classical opposition between deterministic evolutions and random evolutions. The uniformly hyperbolic theory provided a mathematical foundation for the fact that deterministic systems, even with a small number of degrees of freedom, often present chaotic behavior in a robust fashion. Thus, it led to the almost paradoxical conclusion that "chaos" may be stable.

On the other hand, structural stability and uniform hyperbolicity were soon realized to be less universal properties than was initially thought: there exist many classes of systems that are robustly unstable and non-hyperbolic and, in fact, that is often the case for specific models coming from concrete applications. The dream of a general paradigm in Dynamics had to be postponed.

Beyond uniform hyperbolicity.

The next years saw the theory being extended in several distinct directions:

- The study of specific classes of systems, such as quadratic maps, Lorenz flows, and Hénon attractors, which introduced a host of new methods and ideas.
- Bifurcation theory including, in particular, the study of the boundary of uniformly hyperbolic systems, and of the local and global mechanisms leading to chaotic behavior, especially homoclinic bifurcations.
- New developments in the ergodic theory of smooth systems and, especially, the theory of non-uniformly hyperbolic systems (Pesin theory).
- Weaker formulations of hyperbolicity, still with a uniform flavor but where one allows for invariant "neutral" directions (partial hyperbolicity, projective hyperbolicity or existence of a dominated splitting).

- The converse implication in the stability conjecture (hyperbolicity is necessary for stability), which led to the introduction of new perturbation lemmas (ergodic closing lemma, connecting lemma).

Building on remarkable progress obtained in these directions, especially in the eighties and early nineties, several ideas have been put forward and a new point of view has emerged recently, which again allow us to dream of a global understanding of "most" dynamical systems. Initiated as a survey paper requested to us by David Ruelle, the present work is an attempt to put such recent developments in a unified perspective, and to point open problems and likely directions of further progress.

Two semi-local mechanisms, very different in nature but certainly not mutually exclusive, have been identified as the main sources of persistently non-hyperbolic dynamics:

- What we call here "critical behavior", corresponding to critical points in one-dimensional dynamics and, more generally, to homoclinic tangencies, and which is at the heart of Hénon-like dynamics. This is now reasonably well understood, in terms of non-uniformly hyperbolic behavior. Moreover, recent results show that this type of behavior is always present in connection to non-hyperbolic dynamics in low dimensions.
- In higher dimensions, dynamical robustness (robust transitivity, stable ergodicity) extends well outside the uniformly hyperbolic domain, roughly speaking associated to coexistence of uniformly hyperbolic behavior with different unstable dimensions. It requires some uniform geometric structure (transverse invariant bundles: partial hyperbolicity, dominated decomposition) that we refer to as "non-critical behavior".

On the other hand, new perturbation lemmas permitted to organize the global dynamics of generic dynamical systems, by breaking it into elementary pieces separated by a filtration. A great challenge is to understand the dynamics on (the neighborhood of) these elementary dynamical pieces, which should involve a deeper analysis of the two mechanisms mentioned previously. Indeed, a good understanding has already been possible in several cases, especially at the statistical level.

What is this book, and what is it not?

The text is aimed at researchers, both young and senior, willing to get a quick yet broad view of this part of Dynamics. Main ideas, methods, and results are discussed, at variable degrees of depth, with references to the original works for details and complementary information.

We assume the reader is familiar with the fundamental objects of smooth Dynamics, like manifolds or C^r diffeomorphisms and vector fields, as well as with the basic facts in the local theory of dynamical systems close to a hyperbolic periodic point, such as the Hartman-Grobman linearization theorem and the stable manifold theorem. This material is covered by several books, like Bowen [86], Irwin [225], Palis, de Melo [342], Ruelle [394], Katok, Hasselblatt [232], or Robinson [382].

Familiarity with the classical theory of uniformly hyperbolic systems is also desirable, of course. This is also covered by a number of books, including Bowen [86], Shub [411], Mañé [281], Palis, Takens [345], and Katok, Hasselblat [232]. For the reader's convenience, in Chapter 1 we review the main conclusions of the theory that are relevant for our purposes. In that chapter we also give an introductory discussion of robust mechanisms of non-hyperbolicity, and other key issues outside the hyperbolic set-up. This is to be much expanded afterwards, so at that point our presentation is sketchier than elsewhere.

Apart from these pre-requisites, we have tried to keep the text self-contained, giving the precise definitions of all relevant non-elementary notions. Occasionally, this is done in an informal fashion at places where the notion is first needed in a non-crucial way, with the formal definition appearing at some later section where it really is at the heart of the subject. This is especially true about Chapter 1, as explained in the previous paragraph.

Although we have used parts of this book as a basis for graduate courses, it is certainly not designed as a text book that could be used for that purpose all alone. The properties of the main notions are often only stated, and most results are presented with just an outline of the proof.

The book is also not meant to be an exhaustive presentation of the recent results in Dynamics. We are only too conscious of the many fundamental topics we left outside, or touched only briefly. Deciding where to stop could be one of the most difficult and most important problems in this kind of project, and no answer is entirely satisfactory.

How should this book be used and what does it contain?

The 12 chapters are organized so as to convey a global perspective of dynamical systems. The 5 appendices include several other important results, older and new, which we feel should not be omitted, either because they are used in the text or because they provide complementary views of some aspects of the theory.

Although there is, naturally, a global coherence in the text, we have tried to keep the various chapters rather independent, so that the reader may choose to read one chapter without really needing to go through the previous ones. This means that we often recall main notions and statements introduced elsewhere, or else give precise references to where they can be found. On the other hand, the chapters often rely on ideas and results from the appendices.

The main text may be, loosely, split into the following blocks:

- Chapter 1 contains a brief review of uniformly hyperbolic theory and an introduction to main themes to be developed throughout the text.
- Chapters 2 to 4 are devoted to critical behavior in various aspects: one-dimensional dynamics, homoclinic tangencies, Hénon-like dynamics.

- Chapter 5 shows that, for low dimensional systems, far from critical behavior the dynamical behavior is hyperbolic.
- Chapters 6 to 9 treat non-critical behavior, especially the relation between robustness and existence of invariant splittings. While most of the text focusses on dissipative discrete time systems, Chapter 8 deals with conservative diffeomorphisms and Chapter 9 is devoted to flows.
- In Chapter 10 we try to give a global framework for the dynamics of generic maps, where critical and non-critical behavior could fit together.
- Chapter 11 presents some of the progress attained in describing the dynamics in ergodic terms, both in critical and in non-critical situations (either separate or coexisting). Lyapunov exponents are an important tool in this analysis, and Chapter 12 is devoted to their study and control.

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