PRINCETON LECTURES IN ANALYSIS II

COMPLEX ANALYSIS

复分析

ELIAS M. STEIN & RAMI SHAKARCHI

PRINCETON UNIVERSITY PRESS

世界图书出版公司
www.wpcbj.com.cn

PRINCETON LECTURES IN ANALYSIS

II

COMPLEX ANALYSIS

Elias M. Stein

&

Rami Shakarchi

PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD

图书在版编目（CIP）数据

复分析 = Complex Analysis：英文/（美）斯坦恩（Stein，E. M.）著．—影印本．—北京：世界图书出版公司北京公司，2011.12

ISBN 978 - 7 - 5100 - 4054 - 2

Ⅰ.①复…　Ⅱ.①斯…　Ⅲ.①复分析—高等学校—教材—英文　Ⅳ.①O174.5

中国版本图书馆 CIP 数据核字（2011）第 217040 号

书　　名： Complex Analysis
作　　者： Elias M. Stein & Rami Shakarchi
中 译 名： 复分析
责任编辑： 高蓉　刘慧

出 版 者： 世界图书出版公司北京公司
印 刷 者： 三河市国英印务有限公司
发　　行： 世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话： 010 - 64021602，010 - 64015659
电子信箱： kjb@ wpcbj. com. cn

开　　本： 24 开
印　　张： 17
版　　次： 2013 年 1 月
版权登记： 图字：01 - 2006 - 6620

书　　号： 978 - 7 - 5100 - 4054 - 2　　**定　　价：** 59.00 元

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540
In the United Kingdom: Princeton University Press,
3 Market Place, Woodstock, Oxfordshire OX20 1SY

ISBN 0-691-11385-8
Library of Congress Cataloging-in-Publication data has been applied for

British Library Cataloging-in-Publication Data is available

The publisher would like to acknowledge the authors of this volume for
providing the camera-ready copy from which this book was printed

影印版前言

本套丛书是数学大师给本科生写的分析学系列教材。第一作者 E. M. Stein 是调和分析大师（1999 年 Wolf 奖获得者），也是一位卓越的教师。他的学生，和学生的学生，加起来超过两百多人，其中有两位已经获得过 Fields 奖，2006 年 Fields 奖的获奖者之一即为他的学生陶哲轩。

这本教材在 Princeton 大学使用，同时在其它学校，比如 UCLA 等名校也在本科生教学中得到使用。其教学目的是，用统一的、联系的观点来把现代分析的“核心”内容教给本科生，力图使本科生的分析学课程能接上现代数学研究的脉络。共四本书，顺序是：

Ⅰ. 傅立叶分析

Ⅱ. 复分析

Ⅲ. 实分析

Ⅳ. 泛函分析

这些课程仅仅假定读者读过大一微积分和线性代数，所以可看作是本科生高年级（大二到大三共四个学期）的必修课程，每学期一门。

非常值得注意的是，作者把傅立叶分析作为学完大一微积分后的第一门高级分析课。同时，在后续课程中，螺旋式上升，将其贯穿下去。我本人是极为赞同这种做法的，一者，现代数学中傅立叶分析无处不在，既在纯数学，如数论的各个方面都有深入的应用，又在应用数学中是绝对的基础工具。二者，傅立叶分析不光有用，其本身的内容，可以说，就能够把数学中的几大主要思想都体现出来。这样，学生们先学这门课，对数学就能有鲜活的了解，既知道它的用处，又能够“连续”地欣赏到数学中的各种大思想、大美妙。接着，是学同样具有深刻应用和理论优美性于一体的复分析。学完这两门课，学生已经有了相当多的例子和感觉，既懂得其用又懂得其妙。这样，再学后面比较抽象的实分析和泛函分析时，就自然得多、动机充分得多。

这种教法，国内还很欠缺，也缺乏相应的教材。这主要是因为我们的教育体制还存在一些问题，比如数学系研究生入学考试，以往最关键的是初试，但初试只考数学分析和高等代数，也就是本科生低年级的课程。长此以往，中国的大多数本科生，只用功在这两门低年级课程上，而在高年级后续课程，以及现代数学的眼界上有很大的欠缺。这样，导致他们在研究生阶段后劲不足，需要补的东西过多，而疲于奔命。

那么，为弥补这种不足，国内的教材显然是不够的。列举几个原因如下：

1. 比如复变函数这门课，即使国内最好的本科教材，其覆盖的主要内容也仅是这套书中《复分析》的1/3，也就是前一百页。其后面的内容，我们很多研究生也未必学到，但那些知识，在以后做数学研究时，却往往用到。

2. 国内的教材，往往只教授其知识本身，对这个知识的来龙去脉，后续应用，均有很大的欠缺。比如实变函数（实分析），为什么要学这么抽象的东西呢，从书本上是不太能看到的，但是Stein却以Fourier分析为线索，将这些知识串起来，说明了其中的因果。

因此在目前情况下，这种大学数学教育有很大的欠缺。尤其是有些偏远学校的本科生，他们可能很用功，已经很好地掌握了数学分析、高等代数这两门低年级课程，研究生初试成绩很高。但对于高年级课程掌握不够，有些甚至未学过，所以在入学考试的第二阶段——面试过程中，就捉襟见肘，显露出不足。所以，最近几年，各高校亦开始重视研究生考试的面试阶段。那些知识面和理解度不够的同学，往往会在面试时被刷下来。如果他们能够读完Stein这套本科生教材，相信他们的知识面足以在分析学领域，应付得了国内任何一所高校的研究生面试，也会更加明白，学了数学以后，要干什么，怎么样去干。

本套丛书由世界图书出版公司北京公司引进出版。影印版的发行，将使得这些本科生有可能买得起这套丛书，形成讨论班，互相研讨，琢磨清楚。这对大学数学教育质量的提升，乃至对中国数学研究梯队的壮大，都将是非常有益的。

首都师范大学数学系　王永晖

2006－10－8

Foreword

Beginning in the spring of 2000, a series of four one-semester courses were taught at Princeton University whose purpose was to present, in an integrated manner, the core areas of analysis. The objective was to make plain the organic unity that exists between the various parts of the subject, and to illustrate the wide applicability of ideas of analysis to other fields of mathematics and science. The present series of books is an elaboration of the lectures that were given.

While there are a number of excellent texts dealing with individual parts of what we cover, our exposition aims at a different goal: presenting the various sub-areas of analysis not as separate disciplines, but rather as highly interconnected. It is our view that seeing these relations and their resulting synergies will motivate the reader to attain a better understanding of the subject as a whole. With this outcome in mind, we have concentrated on the main ideas and theorems that have shaped the field (sometimes sacrificing a more systematic approach), and we have been sensitive to the historical order in which the logic of the subject developed.

We have organized our exposition into four volumes, each reflecting the material covered in a semester. Their contents may be broadly summarized as follows:

I. Fourier series and integrals.

II. Complex analysis.

III. Measure theory, Lebesgue integration, and Hilbert spaces.

IV. A selection of further topics, including functional analysis, distributions, and elements of probability theory.

However, this listing does not by itself give a complete picture of the many interconnections that are presented, nor of the applications to other branches that are highlighted. To give a few examples: the elements of (finite) Fourier series studied in Book I, which lead to Dirichlet characters, and from there to the infinitude of primes in an arithmetic progression; the X-ray and Radon transforms, which arise in a number of problems in Book I, and reappear in Book III to play an important role in understanding Besicovitch-like sets in two and three dimensions; Fatou's theorem, which guarantees the existence of boundary values of bounded holomorphic functions in the disc, and whose proof relies on ideas developed in each of the first three books; and the theta function, which first occurs in Book I in the solution of the heat equation, and is then used in Book II to find the number of ways an integer can be represented as the sum of two or four squares, and in the analytic continuation of the zeta function.

A few further words about the books and the courses on which they were based. These courses where given at a rather intensive pace, with 48 lecture-hours a semester. The weekly problem sets played an indispensable part, and as a result exercises and problems have a similarly important role in our books. Each chapter has a series of "Exercises" that are tied directly to the text, and while some are easy, others may require more effort. However, the substantial number of hints that are given should enable the reader to attack most exercises. There are also more involved and challenging "Problems"; the ones that are most difficult, or go beyond the scope of the text, are marked with an asterisk.

Despite the substantial connections that exist between the different volumes, enough overlapping material has been provided so that each of the first three books requires only minimal prerequisites: acquaintance with elementary topics in analysis such as limits, series, differentiable functions, and Riemann integration, together with some exposure to linear algebra. This makes these books accessible to students interested in such diverse disciplines as mathematics, physics, engineering, and finance, at both the undergraduate and graduate level.

It is with great pleasure that we express our appreciation to all who have aided in this enterprise. We are particularly grateful to the students who participated in the four courses. Their continuing interest, enthusiasm, and dedication provided the encouragement that made this project possible. We also wish to thank Adrian Banner and Jose Luis Rodrigo for their special help in running the courses, and their efforts to see that the students got the most from each class. In addition, Adrian Banner also made valuable suggestions that are incorporated in the text.

We wish also to record a note of special thanks for the following individuals: Charles Fefferman, who taught the first week, (successfully launching the whole project!); Paul Hagelstein, who in addition to reading part of the manuscript taught several weeks of one of the courses, and has since taken over the teaching of the second round of the series; Daniel Levine who gave valuable help in proof-reading. Last but not least, our thanks go to Gerree Pecht, for her consummate skill in typesetting and for the time and energy she spent in the preparation of all aspects of the lectures, such as transparencies, notes, and the manuscript.

We are also happy to acknowledge our indebtedness for the support we received from the 250th Anniversary Fund of Princeton University, and the National Science Foundation's VIGRE program.

Elias M. Stein
Rami Shakarchi

Princeton, New Jersey
August 2002

Introduction

> ... In effect, if one extends these functions by allowing complex values for the arguments, then there arises a harmony and regularity which without it would remain hidden.
>
> *B. Riemann,* 1851

When we begin the study of complex analysis we enter a marvelous world, full of wonderful insights. We are tempted to use the adjectives magical, or even miraculous when describing the first theorems we learn; and in pursuing the subject, we continue to be astonished by the elegance and sweep of the results.

The starting point of our study is the idea of extending a function initially given for real values of the argument to one that is defined when the argument is complex. Thus, here the central objects are functions from the complex plane to itself

$$f : \mathbb{C} \to \mathbb{C},$$

or more generally, complex-valued functions defined on open subsets of $\mathbb{C}$. At first, one might object that nothing new is gained from this extension, since any complex number z can be written as $z = x + iy$ where $x, y \in \mathbb{R}$ and z is identified with the point (x, y) in $\mathbb{R}^2$.

However, everything changes drastically if we make a natural, but misleadingly simple-looking assumption on f: that it is differentiable in the complex sense. This condition is called **holomorphicity**, and it shapes most of the theory discussed in this book.

A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at the point $z \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad (h \in \mathbb{C})$$

exists. This is similar to the definition of differentiability in the case of a real argument, except that we allow h to take *complex* values. The reason this assumption is so far-reaching is that, in fact, it encompasses a multiplicity of conditions: so to speak, one for each angle that h can approach zero.

Although one might now be tempted to prove theorems about holomorphic functions in terms of real variables, the reader will soon discover that complex analysis is a new subject, one which supplies proofs to the theorems that are proper to its own nature. In fact, the proofs of the main properties of holomorphic functions which we discuss in the next chapters are generally very short and quite illuminating.

The study of complex analysis proceeds along two paths that often intersect. In following the first way, we seek to understand the universal characteristics of holomorphic functions, without special regard for specific examples. The second approach is the analysis of some particular functions that have proved to be of great interest in other areas of mathematics. Of course, we cannot go too far along either path without having traveled some way along the other. We shall start our study with some general characteristic properties of holomorphic functions, which are subsumed by three rather miraculous facts:

1. CONTOUR INTEGRATION: If f is holomorphic in Ω, then for appropriate closed paths in Ω

$$\int_\gamma f(z)dz = 0.$$

2. REGULARITY: If f is holomorphic, then f is indefinitely differentiable.

3. ANALYTIC CONTINUATION: If f and g are holomorphic functions in Ω which are equal in an arbitrarily small disc in Ω, then $f = g$ everywhere in Ω.

These three phenomena and other general properties of holomorphic functions are treated in the beginning chapters of this book. Instead of trying to summarize the contents of the rest of this volume, we mention briefly several other highlights of the subject.

- The zeta function, which is expressed as an infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

  and is initially defined and holomorphic in the half-plane $\text{Re}(s) > 1$, where the convergence of the sum is guaranteed. This function and its variants (the L-series) are central in the theory of prime numbers, and have already appeared in Chapter 8 of Book I, where we proved Dirichlet's theorem. Here we will prove that ζ extends to a meromorphic function with a pole at $s = 1$. We shall see that the behavior of $\zeta(s)$ for $\text{Re}(s) = 1$ (and in particular that ζ does not vanish on that line) leads to a proof of the prime number theorem.

- The theta function

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z},$$

which in fact is a function of the two complex variables z and τ, holomorphic for all z, but only for τ in the half-plane $\text{Im}(\tau) > 0$. On the one hand, when we fix τ, and think of Θ as a function of z, it is closely related to the theory of elliptic (doubly-periodic) functions. On the other hand, when z is fixed, Θ displays features of a modular function in the upper half-plane. The function $\Theta(z|\tau)$ arose in Book I as a fundamental solution of the heat equation on the circle. It will be used again in the study of the zeta function, as well as in the proof of certain results in combinatorics and number theory given in Chapters 6 and 10.

Two additional noteworthy topics that we treat are: the Fourier transform with its elegant connection to complex analysis via contour integration, and the resulting applications of the Poisson summation formula; also conformal mappings, with the mappings of polygons whose inverses are realized by the Schwarz-Christoffel formula, and the particular case of the rectangle, which leads to elliptic integrals and elliptic functions.

Contents

1 Preliminaries to Complex Analysis

> The sweeping development of mathematics during the last two centuries is due in large part to the introduction of complex numbers; paradoxically, this is based on the seemingly absurd notion that there are numbers whose squares are negative.
>
> *E. Borel,* 1952

This chapter is devoted to the exposition of basic preliminary material which we use extensively throughout of this book.

We begin with a quick review of the algebraic and analytic properties of complex numbers followed by some topological notions of sets in the complex plane. (See also the exercises at the end of Chapter 1 in Book I.)

Then, we define precisely the key notion of holomorphicity, which is the complex analytic version of differentiability. This allows us to discuss the Cauchy-Riemann equations, and power series.

Finally, we define the notion of a curve and the integral of a function along it. In particular, we shall prove an important result, which we state loosely as follows: if a function f has a primitive, in the sense that there exists a function F that is holomorphic and whose derivative is precisely f, then for any closed curve γ

$$\int_\gamma f(z)\,dz = 0.$$

This is the first step towards Cauchy's theorem, which plays a central role in complex function theory.

1 Complex numbers and the complex plane

Many of the facts covered in this section were already used in Book I.

1.1 Basic properties

A complex number takes the form $z = x + iy$ where x and y are real, and i is an imaginary number that satisfies $i^2 = -1$. We call x and y the