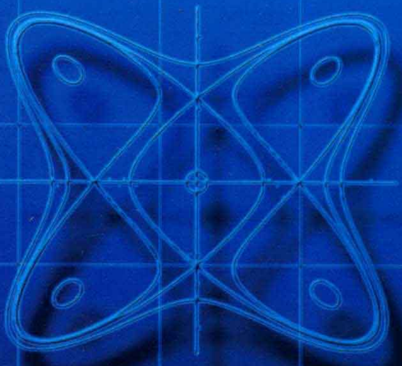

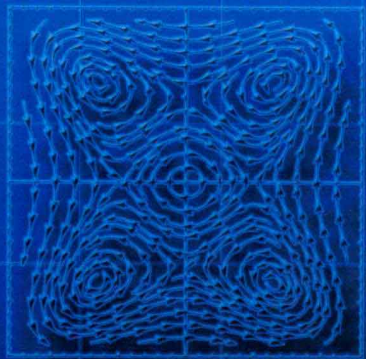

The Qualitative Methods and Numerical Simulations of Differential Equations

微分方程定性方法和数值模拟

◎刘正荣　编著

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内容提要

本教材包含线性系统的相图，非线性系统的线性近似，具有零特征值奇点的性质，高阶奇点，极限环和它们的分支，无穷远奇点及奇点指数，关于相图应用的例子等内容。本书可作为高等院校数学类、自动化控制、信息处理等专业的本科生和研究生的选修课教材，也可作为对微分方程及数值模拟感兴趣的朋友的自学读本。

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前　言

这本教材是本科生的《常微分方程》教材的后续教材，是为高年级本科生或研究生的选修课程编写的英语教材。

众所周知，很多实际问题中的数学模型都是微分方程，因此微分方程是数学联系实际的重要桥梁之一。不幸的是，大多数这样的方程是找不到精确解的。定量的研究方法对它们是行不通的，只能采取定性研究或数值研究的方法。

本教材处理的对象主要是由两个自治微分方程构成的方程组，这样的方程组被称为平面自治系统。针对这样的系统，我们介绍定性分析方法和数值模拟方法，并利用这两种方法的结果相互验证其正确性。我们介绍的数值模拟工具是数学软件Mathematica，我们的愿望是想通过该课程的学习，使学生在掌握一些微分方程定性分析方法的同时，也开始接触一些专业外语，并学会把数学软件应用到该课程学习中，但由于编者水平有限，以上愿望不一定能圆满实现。

本教材的出版获得了“华南理工大学创新人才培养计划资助项目”(项目编号：yjzk2011005)、“数学与应用数学”国家级特色专业建设和华南理工大学出版基金的资助，深表感谢。由于编者水平有限，恳切希望同行专家及读者对本书的不足与疏漏给予批评指正。

编　者

2012 年 10 月

Contents

Chapter 1

Phase Portraits of Linear Systems

1.1 Standard Forms of Linear Systems

First of all, consider system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x. \end{cases} \tag{1.1}$$

Clearly,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c\cos t \\ c\sin t \end{pmatrix}$$

is a solution of (1.1). For given $c > 0$, in the t–x–y space, the solution $x = c\cos t$, $y = c\sin t$ determines a curve Γ_c which is called an integral curve of (1.1) (see Figure 1.1).

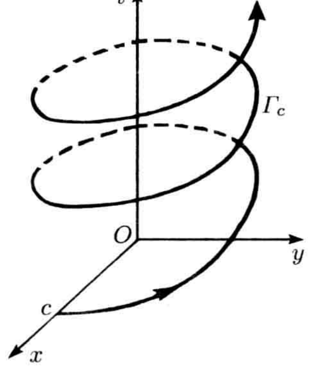


Figure 1.1 The integral curve $x = c\cos t$, $y = c\sin t$ of system (1.1)

On x–y plane, the project of the integral curve Γ_c is l_c: $x^2+y^2=c^2$.

The x–y plane is called phase plane, and l_c is called orbit (see Figure 1.2). When t increases, the direction of Γ_c is also called the direction of l_c.

Generally, we consider autonomous system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = p(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = q(x,y). \end{cases} \tag{1.2}$$

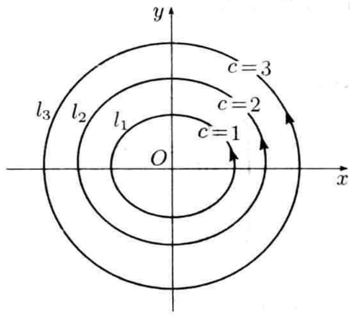


Figure 1.2 The phase plane and the orbits of system (1.1)

Definition 1.1 In the t–x–y space, the curve defined by a solution $x=x(t)$, $y=y(t)$ of (1.2) is called an integral curve. On x–y plane the project of an integral curve is called an orbit and the x–y plane is called phase plane. The combination of orbits are called phase portraits. If (x^*,y^*) satisfies $p(x^*,y^*)=q(x^*,y^*)=0$, then (x^*,y^*) is called a singular point of (1.2).

Now we consider linear system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = ax+by, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = cx+dy, \end{cases} \tag{1.3}$$

where a, b, c, d are constants and satisfy

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0. \tag{1.4}$$

Obviously, $(0, 0)$ is unique singular point of (1.3). The equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0, \tag{1.5}$$

that is,

$$\lambda^2 - (a+d)\lambda + ad - bc = 0, \tag{1.6}$$

is called characteristic equation of (1.3).

The two roots of (1.6) are denoted by

$$\lambda_1 = \frac{1}{2}\left[a + d + \sqrt{(a+d)^2 - 4(ad-bc)}\right], \tag{1.7}$$

and

$$\lambda_2 = \frac{1}{2}\left[a + d - \sqrt{(a+d)^2 - 4(ad-bc)}\right], \tag{1.8}$$

which are called eigenvalues.

Via the eigenvalues, system (1.3) can be changed into some standard forms. We use the following proposition to state them.

Proposition 1.1 Assume that $\lambda_1\lambda_2 \neq 0$. System (1.3) can be changed into some standard forms as follows:

(i) If λ_1 and λ_2 are real and $\lambda_1 \neq \lambda_2$, then under the following transformations

$$\begin{cases} \xi = (d-\lambda_1)x - by, \\ \eta = (d-\lambda_2)x - by, \end{cases} \quad \text{for } b \neq 0, \tag{1.9}$$

or

$$\begin{cases} \xi = -cx + (a-\lambda_1)y, \\ \eta = -cx + (a-\lambda_2)y, \end{cases} \quad \text{for } c \neq 0, \tag{1.10}$$

system (1.3) is changed into

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = \lambda_1\xi, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = \lambda_2\eta. \end{cases} \tag{1.11}$$

(ii) If $\lambda_1 = \lambda_2 = \lambda \neq 0$, then under the transformations

$$\begin{cases} \xi = x, \\ \eta = (\lambda - d)x + by, \end{cases} \quad \text{for } b \neq 0, \tag{1.12}$$

or

$$\begin{cases} \xi = y, \\ \eta = cx + (\lambda - a)y, \end{cases} \quad \text{for} \quad c \neq 0, \tag{1.13}$$

system (1.3) is changed into

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = \lambda\xi + \eta, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = \lambda\eta. \end{cases} \tag{1.14}$$

(iii) If $\lambda_1 = \alpha + \mathrm{i}\beta$ and $\lambda_2 = \alpha - \mathrm{i}\beta$ ($\beta > 0$), then under the transformations

$$\begin{cases} \xi = (d - \alpha)x - by, \\ \eta = \beta x, \end{cases} \quad \text{for} \quad b \neq 0, \tag{1.15}$$

or

$$\begin{cases} \xi = -cx + (a - \alpha)y, \\ \eta = \beta y, \end{cases} \quad \text{for} \quad c \neq 0, \tag{1.16}$$

system (1.3) is changed into

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = \alpha\xi + \beta\eta, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = -\beta\xi + \alpha\eta. \end{cases} \tag{1.17}$$

Remark 1.1 When $b = c = 0$, system (1.3) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = ax, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = dy, \end{cases} \tag{1.18}$$

which is of the same form with system (1.11). If $a = d = \mu$, then (1.18) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \mu x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \mu y. \end{cases} \tag{1.19}$$

The systems (1.11), (1.14), (1.17) and (1.19) are called standard forms of linear systems.

1.2 Classification of Singular Points for Linear Systems

Now we consider the four standard forms above.

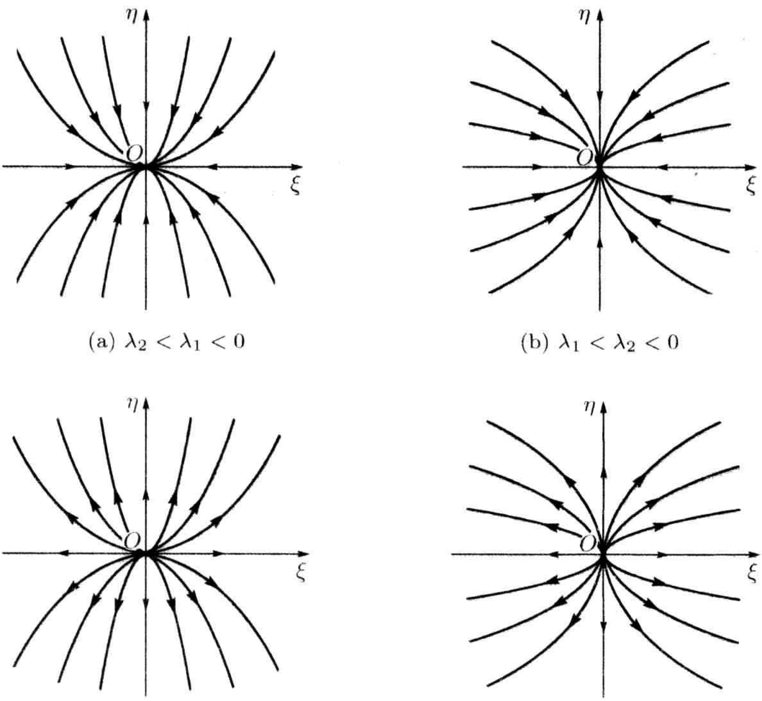
Case 1 When λ_1 and λ_2 are real and $\lambda_1 \neq \lambda_2$, from (1.11) we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\lambda_2 \eta}{\lambda_1 \xi}. \tag{1.20}$$

Obviously, equation (1.20) has solution

$$\eta = c\xi^{\lambda_2/\lambda_1}. \tag{1.21}$$

From (1.21) we draw the phase portraits of system (1.11) as Figure 1.3 and Figure 1.4.



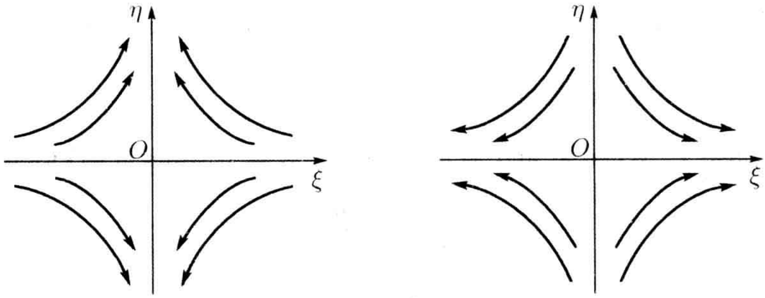
(a) $\lambda_2 < \lambda_1 < 0$

(b) $\lambda_1 < \lambda_2 < 0$

(c) $0 < \lambda_1 < \lambda_2$

(d) $0 < \lambda_2 < \lambda_1$

Figure 1.3 The phase portraits of system (1.11) when $\lambda_1\lambda_2 > 0$



(a) $\lambda_1 < 0 < \lambda_2$ (b) $\lambda_2 < 0 < \lambda_1$

Figure 1.4 The phase portraits of system (1.11) when $\lambda_1\lambda_2 < 0$

Definition 1.2 In Figure 1.3, the singular point $(0,0)$ is called node, that is, when $\lambda_1\lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$, $(0,0)$ is called node. When $\lambda_2 < \lambda_1 < 0$ or $\lambda_1 < \lambda_2 < 0$, $(0,0)$ is called stable node (see Figure 1.3(a), Figure 1.3(b)). When $\lambda_2 > \lambda_1 > 0$ or $\lambda_1 > \lambda_2 > 0$, $(0,0)$ is called unstable node (see Figure 1.3(c), Figure 1.3(d)).

Definition 1.3 In Figure 1.4, the singular point $(0,0)$ is called saddle, that is, when $\lambda_1\lambda_2 < 0$, $(0,0)$ is called saddle.

Case 2 When $\lambda_1 = \lambda_2 = \lambda$, from (1.14) we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\lambda\eta}{\lambda\xi + \eta}, \tag{1.22}$$

that is,

$$\lambda(\eta\mathrm{d}\xi - \xi\mathrm{d}\eta) = \eta\mathrm{d}\eta. \tag{1.23}$$

Multiplying the two sides of equation (1.23) by $1/\eta^2$, it follows that

$$\frac{\lambda(\eta\mathrm{d}\xi - \xi\mathrm{d}\eta)}{\eta^2} = \frac{\mathrm{d}\eta}{\eta}. \tag{1.24}$$

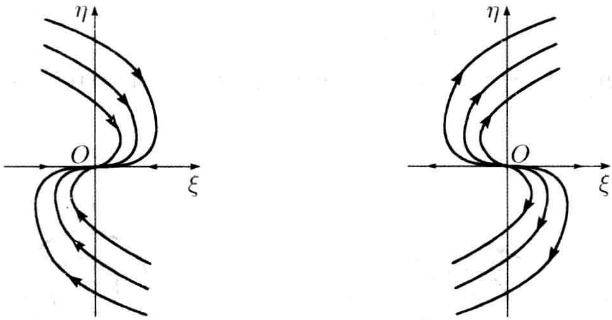
Via (1.24) we get

$$\frac{\lambda\xi}{\eta} = \ln|\eta| + c. \tag{1.25}$$

This implies that equation (1.22) has solution

$$\xi = \frac{\eta}{\lambda}(\ln|\eta| + c). \tag{1.26}$$

From (1.26) we draw the phase portraits of system (1.14) as Figure 1.5.



(a) $\lambda < 0$ (b) $\lambda > 0$

Figure 1.5 The phase portraits of system (1.14)

Definition 1.4 In Figure 1.5, the singular point $(0, 0)$ is called degenerate node, that is, when $\lambda_1 = \lambda_2 = \lambda \neq 0$, $(0, 0)$ is called degenerate node. When $\lambda < 0$, $(0, 0)$ is called stable degenerate node. When $\lambda > 0$, $(0, 0)$ is called unstable degenerate node.

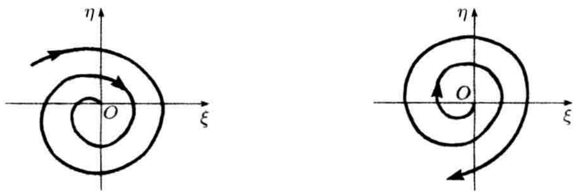
Case 3 When $\lambda_1 = \alpha + \mathrm{i}\beta$ and $\lambda_2 = \alpha - \mathrm{i}\beta$ ($\beta > 0$), consider system (1.17). Substituting $\xi = r\cos\theta$ and $\eta = r\sin\theta$ into (1.17), it follows that

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t}\cos\theta - r\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\sin\theta = \alpha r\cos\theta + \beta r\sin\theta, \\ \dfrac{\mathrm{d}r}{\mathrm{d}t}\sin\theta + r\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\cos\theta = -\beta r\cos\theta + \alpha r\sin\theta. \end{cases} \tag{1.27}$$

From (1.27) we get

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = \alpha r, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\beta, \end{cases} \quad \text{where} \quad \beta > 0. \tag{1.28}$$

Using (1.28) we draw the phase portraits of system (1.17) as Figure 1.6.



(a) $\alpha < 0$ (b) $\alpha > 0$

Figure 1.6 The phase portraits of system (1.17) when $\alpha \neq 0$ and $\beta > 0$

Definition 1.5 In Figure 1.6, the singular point $(0, 0)$ is called the focus. When $\alpha < 0$, $(0, 0)$ is called the stable focus. When $\alpha > 0$, $(0, 0)$ is called the unstable focus.

Remark 1.2 In system (1.17), when $\alpha = 0$, it follows that

$$\begin{cases} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} = \beta\eta, \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = -\beta\xi. \end{cases} \tag{1.29}$$

From (1.29) we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = -\frac{\xi}{\eta}. \tag{1.30}$$

Thus the general solution of (1.30) is given by

$$\xi^2 + \eta^2 = c. \tag{1.31}$$

When $\alpha = 0$, the phase portrait of system (1.17) is given in Figure 1.7. Of course, when $\alpha = 0$, from (1.28) we also can get Figure 1.7.

Definition 1.6 In Figure 1.7, the singular point $(0, 0)$ is called center.

Case 4 When $b = c = 0$ and $a = d = \mu \neq 0$, from (1.19) we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}. \tag{1.32}$$

Obviously, the general solution of (1.32) is

$$y = cx.$$

Thus we obtain the phase portrait of (1.19) as Figure 1.8.

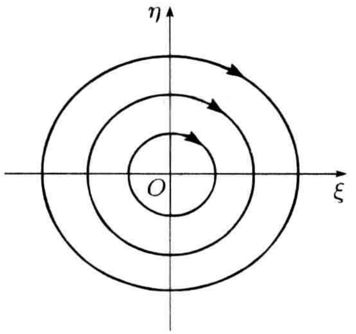
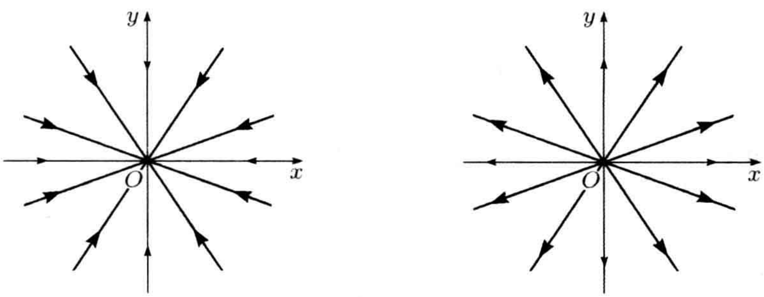


Figure 1.7 The phase portraits of system (1.17) when $\alpha = 0$ and $\beta > 0$



(a) $\mu < 0$ (b) $\mu > 0$

Figure 1.8 The phase portraits of system (1.19) when $b = c = 0$ and $a = d = \mu \neq 0$.

Definition 1.7 In Figure 1.8, the singular point (0, 0) is called the critical singular point.

1.3 Phase Portraits and Their Simulation for Some Linear Systems

Example 1.1 Draw the phase portrait for the system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -2x - 3y. \end{cases} \tag{1.33}$$

Solution Noting that $a = 0$, $b = 1$, $c = -2$, and $d = -3$ in (1.33), thus the characteristic equation is

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$$

that it,

$$\lambda^2 + 3\lambda + 2 = 0. \tag{1.34}$$

Solving (1.34), we get two eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2.$$

Under the transformations

$$\begin{cases} \xi = (d - \lambda_1)x - by, \\ \eta = (d - \lambda_2)x - by, \end{cases}$$

that is

$$\begin{cases} \xi = -2x - y, \\ \eta = -x - y, \end{cases} \tag{1.35}$$

system (1.33) becomes

$$\begin{pmatrix} \dfrac{\mathrm{d}\xi}{\mathrm{d}t} \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{1.36}$$

On ξ–η plane the phase portrait of system (1.36) is shown in Figure 1.9.

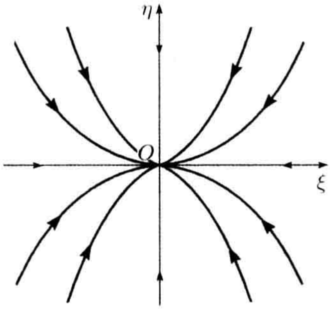


Figure 1.9 The phase portrait of system (1.36)

On the other hand, from (1.35), on x–y plane the η axis is expressed by

$$y = -2x, \tag{1.37}$$

and the ξ axis is expressed by

$$y = -x. \tag{1.38}$$

It is easy to test that $y = -2x$ and $y = -x$ are two linear solutions of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x - 3y}{y}. \tag{1.39}$$

From Figure 1.9 and (1.37), (1.38) we obtain the phase portrait of system (1.33) as Figure 1.10(a).

Remark 1.3 The two linear solutions of the equation (1.39) can be obtained using the following method.

Assume that

$$y = \alpha x \tag{1.40}$$

is a solution of the equation (1.39). Thus it follows that

$$\alpha = \frac{-2x - 3\alpha x}{\alpha x}.$$