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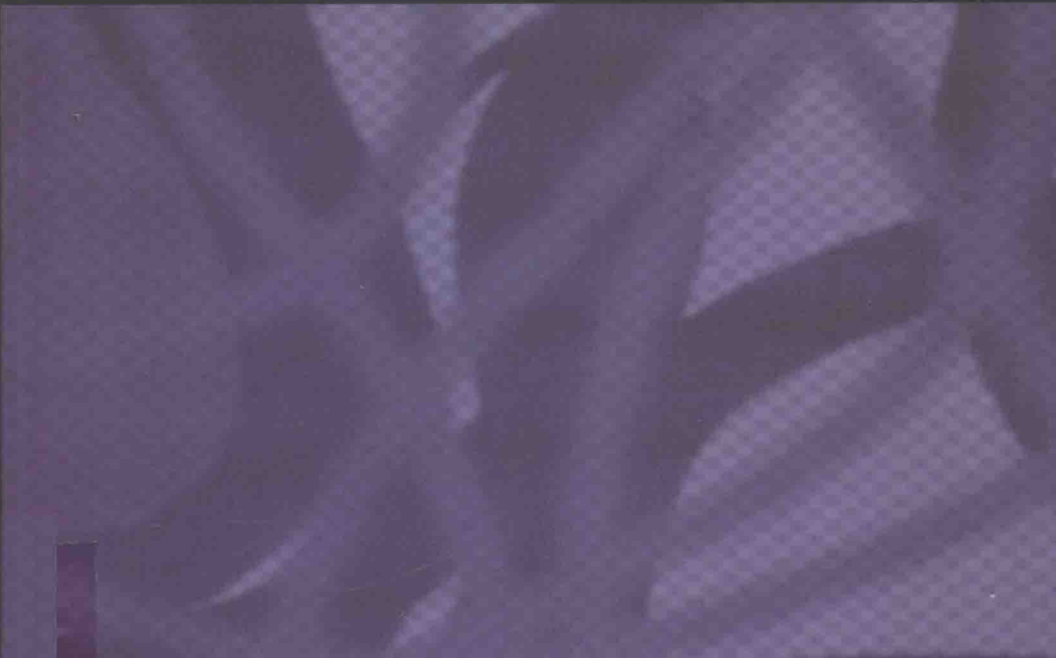
塑性数学理论

R. 希尔



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R. 希尔　著

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影印版前言

自从上世纪80年代起，世界图书出版公司北京公司一直致力于与世界各国知名出版商合作，是国内最早开展购权影印图书出版工作的机构。时至今日，已经持续近30年，不仅引进的品种数独占鳌头，而且包括了大量在国际上具有深远影响的经典图书，受到了国内学者和专家的认可和好评。

现在应国内广大读者的要求，在获得牛津大学出社授权的前提下，世界图书出版公司北京公司将陆续影印出版该社各类丛书中的经典图书。牛津大学出版社是世界著名出版机构之一，每年出版的书籍、刊物超过四千种，其学术著作和教科书的作者均为相关领域的著名学者，其中不乏科学研究前沿的顶尖科学家和领军人物，书籍内容涵盖了最新的科学进展的各个方面，因此一直受到国内外科研人员和高校师生的高度评价，其中已经出版的数学和物理学系列丛书，如*Oxford Graduate Texts in Mathematics*，*Oxford Graduate Texts*，*Oxford Lecture Series in Mathematics and Its Applications*和*Oxford Mathematical Monographs*在国内有着广泛的影响，受到普遍好评。

毫无疑问，考虑到我国的国情以及科学教育发展的迫切需要，这项工作的最大受益者将是那些经济尚不富裕，但却渴望学习知识，想及时了解最新科学技术成果的国内高校和研究机构中的莘莘学子，相对原版，影印版的价格他们更容易接受。在这里，中国的读者和我们出版公司要特别感谢牛津大学出版社以传播科技知识为重，授权世界图书出版公司北京公司影印出版该社系列丛书中的部分图书。我们相信，这些图书的引进，不仅会受到数学物理等相关专业的教师和研究生的欢迎，相关领域的科研人员也将会从中受益。

前　言

尽管金属塑性应变的第一个数学解释已经在80年前由Lévy和Saint-Venant概述过了，但是直到1920年，仍有一些重要的进展没有被论述。塑性理论仍然是一个年轻的学科，很少有认真的学生。我写本书的目的是希望工程师和应用数学工作者能够注意到这个值得学习和研究的领域，在金属工艺学方面它有着日益增长的应用。

为了给全书一种额外的统一性，为阐明理论的应用而选择的实例按照边值问题从简单到复杂的原则编排。这些例子中很多涉及到金属成形，比如滚压和拉伸，这不单是考虑到我自己的研究兴趣，而且是因为正是在这里金属塑性已被最明确地证实。我尤其注重为工业领域的研究人员提供所需的数据，只要是看来适当的地方，我都会介绍化简和凝练解析结果的近似。我没有简单地罗列该领域的方方面面，而是有目的地选取一些内容，然后根据现有的可靠的知识全面而真实地加以论述。

本书假设读者熟悉弹性理论的基本知识。附录给出了两个数学技巧，即下标及求和约定和双曲微分方程的解，也许大多数工程师并不熟悉它们。第一个可以迅速掌握，这是为了简化应力–应变关系的讨论而必须采用的。第二个在力学的很多分支都会出现，该方法的好处是简单和直接。

考虑到这是一个快速发展的学科，文献中的很多结果都是模糊不清和矛盾的，因此在本书的许多论述中，我不可避免地给出了我自己的判断。在本书第一稿阶段，几个重要的理论结果出现了，我尽最大努力去涵盖它们。想要辨认我自己的论述没有困难，所有引用他人的结果我都尽可能地给出具体的参考文献，尤其是那些1949年秋季以前出现在科学期刊中的论文。

由衷感谢牛津大学出版社职员的通力合作。感谢我的妻子对我的帮助和鼓励，没有她本书是不可能完成的。

R. H.

1950年5月

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I

INTRODUCTION

1. Definition of the subject

'THEORY of plasticity' is the name given to the mathematical study of stress and strain in plastically deformed solids, especially metals. This follows the well-established precedent set by the 'theory of elasticity', which deals with methods of calculating stress and strain in elastically deformed solids, and not, as a literal interpretation suggests, with the physical explanation of elasticity. The relation of plastic and elastic properties of metals to crystal structure and cohesive forces belongs to the subject now known as 'metal physics'.

The theory of plasticity takes as its starting-point certain experimental observations of the macroscopic behaviour of a plastic solid in uniform states of combined stress. The task of the theory is twofold: first, to construct explicit relations between stress and strain agreeing with the observations as closely and as universally as need be; and second, to develop mathematical techniques for calculating non-uniform distributions of stress and strain in bodies permanently distorted in any way. At the present time metals are the only plastic solids for which there is enough data to warrant the construction of a *general* theory. For this reason the theory is related specifically to the properties of metals, though it *may* apply to other potentially plastic materials (e.g. ice, clay, or rock).

By contrast with many other plastic solids the most striking attribute of a metal is its capacity for cold-work. At ordinary temperatures, and under favourable applied stresses, a dimensional change of twentyfold can easily be obtained with a ductile metal, for example by compressing or shearing a cylinder of copper. More severe strains are enforced locally when a metal billet is extruded or pierced. The theory of plasticity is, therefore, especially concerned with technological forming processes such as the rolling of strip, extrusion of rods and tubes, drawing of wire, and deep-drawing of sheet. The purpose of the analysis is to determine the external loads, the power consumption, and the non-uniform strain and hardening due to the cold-working. Intense plastic strains are also produced locally in many standard mechanical tests of metals, for instance indentation by a conical die, the bending of a notched bar, or the extension of a tensile specimen past the necking point. A rational account of the physical significance of these tests requires a knowledge of the state of stress and the extent of the plastic zone.

At the other extreme, where the subjects of elasticity and plasticity meet, a typical application is to predict the critical loading which just causes a structural member to yield plastically at its weakest point. Between these extremes come problems where the plastic and elastic strains are of a similar order of magnitude, as in a beam partly overstrained by bending or twisting, or in a pressure vessel strengthened by an initial permanent expansion.

2. Historical outline

The scientific study of the plasticity of metals may justly be regarded as beginning in 1864. In that year Tresca published a preliminary account of experiments on punching and extrusion, which led him to state that a metal yielded plastically when the maximum shear stress attained a critical value. Criteria for the yielding of plastic solids, mainly soils, had been proposed before, for example by Coulomb (1773), and had been applied by Poncelet (1840) and Rankine (1853) to problems such as the calculation of earth-pressure on retaining walls; there appears, however, to have been no earlier important investigation for metals. Tresca's yield criterion was applied by Saint-Venant to determine the stresses in a partly plastic cylinder subjected to torsion or bending (1870) and in a completely plastic tube expanded by internal pressure (1872) (the first step towards the solution for a partly plastic tube was taken by Turner in 1909). Saint-Venant also set up a system of five equations governing the stresses and strains in two-dimensional flow, and, recognizing that there is no one-one relation between stress and total plastic strain, postulated that the directions of the maximum shear strain-*rate* coincided at each moment with the directions of the maximum shear stress. In 1871 Lévy, adopting Saint-Venant's conception of an ideal plastic material, proposed three-dimensional relations between stress and rate of plastic strain.

There seems to have been no further significant advance until the close of the century when Guest investigated the yielding of hollow tubes under combined axial tension and internal pressure, and obtained results broadly in agreement with the maximum shear-stress criterion. During the next decade many similar experiments were performed, mainly in England, with slightly differing conclusions. Various yield criteria were suggested, but for many metals, as later and more accurate work was to show, the most satisfactory was that advanced by von Mises

(1913) on the basis of purely mathematical considerations; it was interpreted by Hencky some years afterwards as implying that yielding occurred when the elastic shear-strain energy reached a critical value. Von Mises also independently proposed equations similar to Lévy's.

Between the two wars the subject was actively developed by German writers. In 1920 and 1921 Prandtl showed that the two-dimensional plastic problem is hyperbolic, and calculated the loads needed to indent a plane surface and a truncated wedge by a flat die. Parallel experiments by Nadai were in accord with these calculations, but it has been shown recently that Prandtl's work is defective in certain respects. The general theory underlying Prandtl's special solutions was supplied in 1923 by Hencky, who also discovered simple geometrical properties of the field of slip-lines in a state of plane plastic strain. It was some time, however, before the equations governing the variation of the velocity of flow along slip-lines were obtained (Geiringer, 1930) and even longer before the correct approach to the solution of plane problems was clarified (1945–9). In 1923 Nadai investigated, both theoretically and experimentally, the plastic zones in a twisted prismatic bar of arbitrary contour. The effective application of plastic theory to technological processes began in 1925 when von Karman analysed, by an elementary method, the state of stress in rolling. In the following year Siebel, and soon afterwards Sachs, put forward similar theories for wire-drawing.

It was not until 1926, when Lode measured the deformation of tubes of various metals under combined tension and internal pressure, that the Lévy–Mises stress-strain relations were shown to be valid to a first approximation. However, Lode's results indicated certain divergences, and these were afterwards confirmed by the more controlled experiments of Taylor and Quinney (1931). The theory was now generalized in two important directions: first by Reuss (1930) who made allowance for the elastic component of strain, following an earlier suggestion by Prandtl; second by Schmidt (1932) and Odquist (1933) who showed, in slightly different ways, how work-hardening could be brought within the framework of the Lévy–Mises equations. The first generalization was broadly confirmed by experiments of Hohenemser (1931–2), and the second by investigations of Schmidt. Thus, by 1932, a theory had been constructed, reproducing the main plastic and elastic properties of an isotropic metal at ordinary temperatures, and substantially in accord with observation. However, from then until the early 1940's little progress was made in the solution of special problems. Further generalizations were formulated (for example, by von Mises in 1928 and by Melan in 1938), but mathematical expediency and lack of accurate data combined to render them, for the time being, academic.

Meanwhile a rival theory proposed by Hencky in 1924 was favoured for its analytic convenience in problems where the plastic strain was small, despite its conflict with experience in establishing a one-one relation between stress and strain. This theory was given prominence by Nadai in his book on plasticity (1931), and was afterwards extensively employed by the Russian school (1935 onwards). Hencky's equations lead to approximately correct results only for certain loading-paths, but many writers have applied them without discrimination.

The war stimulated research in England and America, through problems such as the calculation of the stresses in autofrettaged gun-barrels and of the forces resisting a shot penetrating armour plate. Since then the subject has been intensively studied in many countries, and the advances made are such that the present book is largely an account of the work of the five years 1945–9.

3. Physical background

During the construction of the theory frequent reference is made to the plastic properties of metal single crystals and polycrystalline aggregates. It is assumed that the reader has a general knowledge of these, and only a brief résumé of the relevant properties is given here. For broader and more detailed accounts, presented from other standpoints, specialist works on metal physics should be consulted.†

(i) *Single crystals.* In a freshly grown metal crystal, isolated from external disturbances and of the highest purity, the atoms are disposed in equilibrium under their mutual forces in a regular three-dimensional array, with a periodic structure characteristic of the metal. Most of the well-known metals have a lattice structure which is either face-centred cubic (copper, aluminium, lead, silver, gold), body-centred cubic (alpha iron, vanadium, tungsten, and the alkali metals), or hexagonal close-packed (zinc, magnesium, cadmium). According to current theory, the cohesive forces binding the atoms together are such that a perfect metal crystal could sustain, with only slight displacements of the atoms from their normal positions, very much greater applied stresses than an actual

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† C. S. Barrett, *Structure of Metals* (McGraw-Hill Book Co., 1943); W. Boas, *An Introduction to the Physics of Metals and Alloys* (John Wiley & Sons, Inc., New York, 1947); A. H. Cottrell, *Theoretical Structural Metallurgy* (Edward Arnold & Co., London, 1948); C. F. Elam, *Distortion of Metal Crystals* (Clarendon Press, Oxford, 1935); W. Hume-Rothery, *Atomic Theory for Students of Metallurgy* (Institute of Metals, London, 1946); E. Schmid and W. Boas, *Kristallplastizität* (Julius Springer, Berlin, 1935); F. Seitz, *The Physics of Metals* (McGraw-Hill Book Co., 1943).

crystal is observed to do. Thus, in a freshly prepared crystal, sensitive measurements show that elastic (or reversible) deformation has ceased after a *macroscopic* shear strain of order 10^{-4}, whereas a perfect crystal should be capable of an elastic shear strain of order 10^{-1}. The discrepancy is attributed to faults or disturbances in the lattice structure, formed during growth or introduced by subsequent handling. It is thought that the faults are separated by distances of the order of 1,000 atomic spacings, and that each extends over a volume containing perhaps 100–1,000 atoms. Certain of the faults are considered to be of a kind that 'weaken' the crystal, and are known as 'dislocations'. Mathematical studies of conjectural atomic arrangements in a dislocation indicate that a very small applied stress would cause it to move (as a geometrical entity) through an otherwise perfect crystal. The resultant effect of the passage of a dislocation is a relative displacement of the parts of the crystal bordering its path by an amount equal to one or two atomic spacings. In this way the movement through the crystal of many dislocations produces an overall strain without affecting the main lattice structure; the substantial preservation of the structure during plastic deformation is confirmed by X-ray examination. That the strain is plastic and irreversible is attributed to the 'trapping' of the dislocations at other faults which are not mobile under the external stresses. Additional dislocations are thought to be created or liberated during the deformation, but despite this a continually increasing stress is usually needed to enforce plastic strain (strain-hardening); the movement of free dislocations is progressively impeded by the local disordering of the lattice at points where trapped dislocations accumulate. The increase in potential energy of the deformed crystal is only a small fraction (of order one-tenth) of the work done by the applied stress; the remainder appears as vibrational energy of the atoms in or near moving dislocations (whose speed must be close to that of sound) and is ultimately dissipated as heat throughout the crystal.

If the external temperature is sufficiently great, the activation energy needed to move existing dislocations may also be provided by thermal agitation; the effect of the applied stress is mainly to give direction to the resultant flow (transient creep). If the stress is removed, and the crystal is held at a sufficiently high temperature for a certain period, thermal fluctuations assist the atoms over their potential barriers towards the original regular array, which is the configuration of greatest stability. Ultimately the imperfections created during the previous deformation are removed, and the crystal is said to have been 'annealed'.

In the theory of plasticity the strain is regarded as macroscopically uniform, but on a microscopic scale it is known that the plastic distortion is largely confined to narrow bands (slip-lines) which extend through the crystal and are presumably created by the passage, along closely grouped planes, of large numbers of dislocations, many of which become trapped or mutually locked. These bands are perhaps some 100 atoms thick, and the planes to which they are parallel are known as slip-, or glide-, planes: they are often the crystallographic planes most densely packed with atoms. The average spacing of the slip-bands depends on the amount and rate of strain, and on the metal, but is normally of the order of 10,000 atomic distances. The lattice between the slip-bands is still virtually perfect and only distorted elastically; the strain there is greater than the overall strain at the elastic limit since the applied stress has increased because of the hardening; moreover, since the slip-bands extend over only a small proportion of the total volume, even after quite large strains, the elastic moduli of the crystal as a whole are little affected by the plastic deformation.

When a crystal in the form of a wire is stretched under tension the cross-section becomes elliptical. Macroscopically, the deformation may be described by saying that the crystal has undergone a shear in a certain direction over a certain set of parallel slip-planes, together with a rotation bringing these planes more nearly parallel to the axis of the wire. Only a limited number of active slip-directions have ever been observed. In face-centred cubic metals at ordinary temperatures there are apparently only four possible slip-planes (the octahedral planes), in each of which there are three possible slip-directions (lying in the cubic planes); slip occurs in a body-centred cubic metal in many more ways, but in a hexagonal metal only over the basal planes and along the digonal axes. The tension needed to deform the wire plastically varies greatly with the orientation of the crystal to the axis, but it is found that a slip-direction is activated only when a certain critical value (the yield stress) is attained by the component of shear stress acting over the slip-plane and in the slip-direction; this is of order 100 gm./mm.^{2} in an annealed crystal at ordinary temperatures. The yield stress is approximately the same for all the different slip-directions in a crystal in a given state, and is independent of the type of test (for example, whether tension or compression). For ordinary strain-rates and for temperatures where creep is negligible, the yield stress is a function mainly of the amount of previous plastic distortion. In particular, the same relationship is obtained between shear stress and shear strain irrespective of which set of slip-planes is operating, and, if the applied stress is subsequently changed so that another set of planes is operated, the new shear-hardening curve is a continuation of the previous one. It appears, therefore, that the disordering of the lattice affects all planes equally whether or not they are active; this phenomenon is known as latent hardening. When a gradually increasing stress is applied to a crystal (either in its original state or plastically deformed), it is observed that the active slip-planes are those on which the critical value of the shear stress is first attained. Double slip begins when the rotation of the crystal brings another set of planes into the position where the corresponding shear stress is equal to the current yield value; several sets can be operated simultaneously under combined stresses.

(ii) *Polycrystalline aggregate.* A metal, in its generally used form, is a compact aggregate of crystal grains with varying shapes and orientations, each grain having grown from a separate nucleus in the melt. The metal may be considered macroscopically isotropic when the orientations are randomly distributed and when the average dimensions of the individual crystals are small compared with the dimensions of the whole specimen (for example, 10^{-3}–10^{-2} cm. compared with 1–100 cm.). Nevertheless, the properties of an aggregate are not always simply statistical averages of the properties of a single crystal, taken over all orientations. While this is approximately true of properties which depend mainly on the bulk structure, such as the coefficient of thermal expansion or the elastic moduli, it is not necessarily true of plastic phenomena.

Theory and experiment suggest that the transition from one orientation to another in neighbouring grains takes place through a layer only a few atoms thick. In this transition zone, or grain boundary, the atoms take up equilibrium positions which are a compromise between the normal positions in each of the two lattices. These atoms have a higher free energy than atoms within the grains, and are consequently main agents of viscous flow and intercrystalline fracture at high temperatures. They are also thought to be potential sources of dislocations, but at the same time hindrances to the passage of others, in greater or lesser degree. Being the centre of a stress concentration extending over many atoms, a dislocation arriving at a boundary can activate slip on a skew plane in a neighbouring crystal, and this should be easier when many slip-planes of near orientation are available, as in face-centred or body-centred cubic metals. Indeed, the shear-hardening curve of a polycrystalline cubic metal does not exceed the mean curve for single crystals of arbitrary