Probability and Its Applications

David Nualart

The Malliavin Calculus and Related Topics

Second Edition

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To my wife Maria Pilar

Preface to the second edition

There have been ten years since the publication of the first edition of this book. Since then, new applications and developments of the Malliavin calculus have appeared. In preparing this second edition we have taken into account some of these new applications, and in this spirit, the book has two additional chapters that deal with the following two topics: Fractional Brownian motion and Mathematical Finance.

The presentation of the Malliavin calculus has been slightly modified at some points, where we have taken advantage of the material from the lectures given in Saint Flour in 1995 (see reference [248]). The main changes and additional material are the following:

In Chapter 1, the derivative and divergence operators are introduced in the framework of an isonormal Gaussian process associated with a general Hilbert space H. The case where H is an L^2-space is trated in detail afterwards (white noise case). The Sobolev spaces $\mathbb{D}^{s,p}$, with s is an arbitrary real number, are introduced following Watanabe's work.

Chapter 2 includes a general estimate for the density of a one-dimensional random variable, with application to stochastic integrals. Also, the composition of tempered distributions with nondegenerate random vectors is discussed following Watanabe's ideas. This provides an alternative proof of the smoothness of densities for nondegenerate random vectors. Some properties of the support of the law are also presented.

In Chapter 3, following the work by Alòs and Nualart [10], we have included some recent developments on the Skorohod integral and the associated change-of-variables formula for processes with are differentiable in future times. Also, the section on substitution formulas has been rewritten and an Itô-Ventzell formula has been added, following [248]. This formula allows us to solve anticipating stochastic differential equations in Stratonovich sense with random initial condition.

There have been only minor changes in Chapter 4, and two additional chapters have been included. Chapter 5 deals with the stochastic calculus with respect to the fractional Brownian motion. The fractional Brownian motion is a self-similar Gaussian process with stationary increments and variance t^{2H}. The parameter $H \in (0,1)$ is called the Hurst parameter. The main purpose of this chapter is to use the the Malliavin Calculus techniques to develop a stochastic calculus with respect to the fractional Brownian motion.

Finally, Chapter 6 contains some applications of Malliavin Calculus in Mathematical Finance. The integration-by-parts formula is used to compute "greeks", sensitivity parameters of the option price with respect to the underlying parameters of the model. We also discuss the application of the Clark-Ocone formula in hedging derivatives and the additional expected logarithmic utility for insider traders.

August 20, 2005 *David Nualart*

Preface

The origin of this book lies in an invitation to give a series of lectures on Malliavin calculus at the Probability Seminar of Venezuela, in April 1985. The contents of these lectures were published in Spanish in [245]. Later these notes were completed and improved in two courses on Malliavin calculus given at the University of California at Irvine in 1986 and at École Polytechnique Fédérale de Lausanne in 1989. The contents of these courses correspond to the material presented in Chapters 1 and 2 of this book. Chapter 3 deals with the anticipating stochastic calculus and it was developed from our collaboration with Moshe Zakai and Etienne Pardoux. The series of lectures given at the Eighth Chilean Winter School in Probability and Statistics, at Santiago de Chile, in July 1989, allowed us to write a pedagogical approach to the anticipating calculus which is the basis of Chapter 3. Chapter 4 deals with the nonlinear transformations of the Wiener measure and their applications to the study of the Markov property for solutions to stochastic differential equations with boundary conditions. The presentation of this chapter was inspired by the lectures given at the Fourth Workshop on Stochastic Analysis in Oslo, in July 1992. I take the opportunity to thank these institutions for their hospitality, and in particular I would like to thank Enrique Cabaña, Mario Wschebor, Joaquín Ortega, Süleyman Üstünel, Bernt Øksendal, Renzo Cairoli, René Carmona, and Rolando Rebolledo for their invitations to lecture on these topics.

We assume that the reader has some familiarity with the Itô stochastic calculus and martingale theory. In Section 1.1.3 an introduction to the Itô calculus is provided, but we suggest the reader complete this outline of the classical Itô calculus with a review of any of the excellent presentations of this theory that are available (for instance, the books by Revuz and Yor [292] and Karatzas and Shreve [164]).

In the presentation of the stochastic calculus of variations (usually called the Malliavin calculus) we have chosen the framework of an arbitrary centered Gaussian family, and have tried to focus our attention on the notions and results that depend only on the covariance operator (or the associated Hilbert space). We have followed some of the ideas and notations developed by Watanabe in [343] for the case of an abstract Wiener space. In addition to Watanabe's book and the survey on the stochastic calculus of variations written by Ikeda and Watanabe in [144] we would like to mention the book by Denis Bell [22] (which contains a survey of the different approaches to the Malliavin calculus), and the lecture notes by Dan Ocone in [270]. Readers interested in the Malliavin calculus for jump processes can consult the book by Bichteler, Gravereaux, and Jacod [35].

The objective of this book is to introduce the reader to the Sobolev differential calculus for functionals of a Gaussian process. This is called the analysis on the Wiener space, and is developed in Chapter 1. The other chapters are devoted to different applications of this theory to problems such as the smoothness of probability laws (Chapter 2), the anticipating stochastic calculus (Chapter 3), and the shifts of the underlying Gaussian process (Chapter 4). Chapter 1, together with selected parts of the subsequent chapters, might constitute the basis for a graduate course on this subject.

I would like to express my gratitude to the people who have read the several versions of the manuscript, and who have encouraged me to complete the work, particularly I would like to thank John Walsh, Giuseppe Da Prato, Moshe Zakai, and Peter Imkeller. My special thanks go to Michael Röckner for his careful reading of the first two chapters of the manuscript.

March 17, 1995 *David Nualart*

Contents

Introduction

The Malliavin calculus (also known as the stochastic calculus of variations) is an infinite-dimensional differential calculus on the Wiener space. It is tailored to investigate regularity properties of the law of Wiener functionals such as solutions of stochastic differential equations. This theory was initiated by Malliavin and further developed by Stroock, Bismut, Watanabe, and others. The original motivation, and the most important application of this theory, has been to provide a probabilistic proof of Hörmander's "sum of squares" theorem.

One can distinguish two parts in the Malliavin calculus. First is the theory of the differential operators defined on suitable Sobolev spaces of Wiener functionals. A crucial fact in this theory is the integration-by-parts formula, which relates the derivative operator on the Wiener space and the Skorohod extended stochastic integral. A second part of this theory deals with establishing general criteria in terms of the "Malliavin covariance matrix" for a given random vector to possess a density or, even more precisely, a smooth density. In the applications of Malliavin calculus to specific examples, one usually tries to find sufficient conditions for these general criteria to be fulfilled.

In addition to the study of the regularity of probability laws, other applications of the stochastic calculus of variations have recently emerged. For instance, the fact that the adjoint of the derivative operator coincides with a noncausal extension of the Itô stochastic integral introduced by Skorohod is the starting point in developing a stochastic calculus for nonadapted processes, which is similar in some aspects to the Itô calculus. This anticipating stochastic calculus has allowed mathematicians to formulate and discuss stochastic differential equations where the solution is not adapted to the Brownian filtration.

The purposes of this monograph are to present the main features of the Malliavin calculus, including its application to the proof of Hörmander's theorem, and to discuss in detail its connection with the anticipating stochastic calculus. The material is organized in the following manner:

In Chapter 1 we develop the analysis on the Wiener space (Malliavin calculus). The first section presents the Wiener chaos decomposition. In Sections 2,3, and 4 we study the basic operators D, δ, and L, respectively. The operator D is the derivative operator, δ is the adjoint of D, and L is the generator of the Ornstein-Uhlenbeck semigroup. The last section of this chapter is devoted to proving Meyer's equivalence of norms, following a simple approach due to Pisier. We have chosen the general framework of an isonormal Gaussian process $\{W(h), h \in H\}$ associated with a Hilbert space H. The particular case where H is an L^2 space over a measure space $(T, \mathcal{B}, \mu)$ (white noise case) is discussed in detail.

Chapter 2 deals with the regularity of probability laws by means of the Malliavin calculus. In Section 3 we prove Hörmander's theorem, using the general criteria established in the first sections. Finally, in the last section we discuss the regularity of the probability law of the solutions to hyperbolic and parabolic stochastic partial differential equations driven by a space-time white noise.

In Chapter 3 we present the basic elements of the stochastic calculus for anticipating processes, and its application to the solution of anticipating stochastic differential equations. Chapter 4 examines different extensions of the Girsanov theorem for nonlinear and anticipating transformations of the Wiener measure, and their application to the study of the Markov property of solution to stochastic differential equations with boundary conditions.

Chapter 5 deals with some recent applications of the Malliavin Calculus to develop a stochastic calculus with respect to the fractional Brownian motion. Finally, Chapter 6 presents some applications of the Malliavin Calculus in Mathematical Finance.

The appendix contains some basic results such as martingale inequalities and continuity criteria for stochastic processes that are used along the book.