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Sheaves on Manifolds

流形上的层

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Sheaves on Manifolds

With a Short History
«Les débuts de la théorie des faisceaux»
By Christian Houzel

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Preface

For a long time after its introduction by Leray, sheaf theory was mainly applied to the theory of functions of several complex variables or to algebraic geometry, until it became a basic tool for almost all mathematicians, and cohomology a natural language for many people.

However, while there exists an extensive literature dealing with cohomology of sheaves (e.g. the famous book by Godement) or even with derived functors, there are in fact very few books developing sheaf theory within the beautiful framework of derived categories although its necessity is becoming more and more evident. Most of the constructions of the theory take on their full strength in this context, or even, do not make sense outside of it. This is particularly evident for the Poincaré-Verdier duality, which appeared in the sixties, as well as for the Sato microlocalization, introduced in 1969, which is only beginning to be fully understood.

Since the seventies, other fundamental ideas have emerged and sheaf theory (on manifolds) naturally includes the "microlocal" point of view. Our aim is to present here a self-contained work, starting from the beginning (derived categories and sheaves), dealing in detail with the main features of the theory, such as duality, Fourier transformation, specialization and microlocalization, micro-support and contact transformations, and also to give two main applications. The first of these deals with real analytic geometry, and includes the concepts of constructible sheaves, subanalytic cycles, Euler-Poincaré indices, Lefschetz formula, perverse sheaves, etc. The second one is the theory of linear partial differential equations, including D-modules, microfunctions, elliptic and micro-hyperbolic systems, and complex quantized contact transformations.

With this book we hope to illustrate the deep links that tie together branches of mathematics at first glance seemingly disconnected, such as for example here, algebraic topology and linear partial differential equations. At the same time, we want to emphasize the essentially geometrical nature of the problems encountered (most obvious in the involutivity theorem for sheaves), and to show how efficient the algebraic tools introduced by Grothendieck are in solving them, even for an analyst.

Of course, many important applications of the theory are just touched upon, such as for instance the theory of microdifferential systems (complete monographs on the topic are however available now), others are simply omitted, such as representation theory and equivariant sheaf theory.

Finally, we want to express our thanks to C. Houzel who agreed to write a short history of sheaf theory, to L. Illusie who helped us when preparing the "Historical Notes", to those who went through various parts of the book and made constructive comments, especially E. Andronikof, A. Arabia, J-M. Delort, E. Leichtnam and J-P. Schneiders, and also to Catherine Simon at Paris-Nord University and the secretarial staff of the RIMS at Kyoto, who had the patience to type the manuscripts.

May 1990 M. Kashiwara and P. Schapira

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Introduction

The aim of this book is to give a self-contained exposition of sheaf theory.

Sheaves were created during the last world war by Jean Leray, while he was a prisoner of war in a German camp.

The purpose of sheaf theory is quite general: it is to obtain global information from local information, or else to define "obstructions" which characterize the fact that a local property does not hold globally any more: for example a manifold is not always orientable, or a differential equation can be locally solvable, but not globally. Hence, sheaf theory is a wide generalization of a part of algebraic topology (e.g. singular homology) which corresponds to *constant sheaves* or, more generally, to *locally constant sheaves*. There are many natural examples of sheaves, such as orientation sheaves, sheaves of differentiable or holomorphic functions, sheaves of solutions of systems of differential equations, constructible sheaves obtained as direct images, etc.

It was not clear at the beginning however whether such a general theory could have any application, until it was successfully applied to the theory of functions of several complex variables and one can imagine that the original work of Leray would have remained far from accessible without the substantial work developed in the fifties by Cartan, Serre, and later Grothendieck (cf. the Short History by Houzel, below).

Sheaf theory takes on its full strength when combined with the tools of homological algebra. In fact, Leray also introduced the notion of spectral sequences which, together with that of derived functors of Cartan-Eilenberg, leads naturally to the theory of derived categories, due to Grothendieck. After having been long reserved to some specialists of algebraic geometry, the theory of derived categories began to be fully recognized as a basic tool of mathematics. In particular, it would certainly not have been possible without it to give such a beautiful generalization of Poincaré's duality, as did Verdier in the sixties (after related work of Grothendieck in the framework of étale cohomology), or to treat systematically what Grothendieck calls "the six operations" on sheaves, that is, the functors Rf_*, f^{-1}, $Rf_!$, $f^!$, $\otimes^L$, and $R\mathcal{H}om$. As we shall see, this formalism leads to deep and powerful formulas which interpret, in a general context, classical results. We have already mentioned the Poincaré duality, but there are many other topics such as the Lefschetz fixed point formula or the Euler-Poincaré index. Of course these functors are an abstract version of classical operations on functions: direct image for integration, inverse image for composition, tensor product for product. (These three operations give six operations by "duality", an operation which has no counterpart for functions, but which we shall introduce here for constructible functions.)

At this stage, we have briefly explained what one could roughly call "the classical theory of sheaves". But a new and fundamental idea, due to Mikio Sato, appeared in 1969, which was to give a new perspective to sheaf theory, namely "the microlocal point of view", and indeed it is one of the aims of this book to develop sheaf theory within this new framework. Sato's main interest was the study of analytical singularities of solutions of systems of linear differential equations. Already in 1959 he had used local cohomology to define the sheaf of hyperfunctions and to interpret them as sums of boundary values of holomorphic functions. Ten years later he introduced the sheaf of microfunctions, to recognize "from which direction the boundary values come". To perform this, Sato introduced the functor ν_M of *specialization* (along a submanifold M of a manifold X), and its *Fourier transform* the functor μ_M of *microlocalization*. These functors send the derived categories of sheaves on X to the derived category of sheaves on the normal and conormal bundles to M in X respectively, and they allow us to analyze precisely a sheaf on a neighborhood of M, taking into account all normal (or conormal) directions to M.

When trying to apply Sato's theory to the study of microhyperbolic systems, the present authors gradually realized that the only information they were using was the characteristic variety of the system (in the cotangent bundle to a complex manifold X), and that it was possible to forget the complex structure of X and even the fact that the subject was partial differential equations. What they were doing was nothing more than non-characteristic deformations (of the complex of holomorphic solutions sheaves of the system), in the "non-forbidden" directions, that is, across non-characteristic real hypersurfaces. Note that these techniques of non-characteristic deformation had already appeared before, but here the authors were dealing with micro-differential systems, and really needed *microlocal geometry*. In particular, they introduced the γ-topology, a kind of microlocal cut-off. Later, in 1982, by abstracting their previous work on microhyperbolic systems, they introduced the notion of micro-support of a sheaf: roughly speaking, a point p of the cotangent bundle T^*X to a real manifold X does not belong to $SS(F)$, the micro-support of a sheaf F, if F has no cohomology supported by half-spaces whose conormals are close to p. This new definition allows us to study sheaves "microlocally" and, in particular, to make contact transformations (the natural transformations on cotangent bundles) operate on sheaves, similarly as quantized contact transformations operate on microfunctions in Sato-Kawai-Kashiwara [1] or on Fourier distributions in Hörmander [2].

The idea of micro-support is closely related to Morse theory. As is well-known, if ϕ is a real function on a compact real manifold X, the cohomology groups of the spaces $\{x \in X; \phi(x) < t\}$ are isomorphic as long as t does not meet a critical value of ϕ. If one considers the similar problem for the cohomology groups of a sheaf F on X, one obtains the corresponding result with the help of the micro-support of F. For complex manifolds, the micro-support of constructible sheaves may also be described using the *vanishing-cycle functor* of Grothendieck-Deligne. Notice that the vanishing-cycle functor appears as a particular case of Sato's microlocalization functor.

The micro-support has a deep geometrical meaning, and we shall prove that this set is involutive. This gives in particular a purely real and geometric proof of the classical corresponding result for the characteristic variety of systems of differential equations.

As we shall see all along in this book, the microlocal point of view on sheaves deepens the theory of sheaves and leads to many applications. Let us discuss here only a few examples.

(a) Many morphisms in sheaf theory become isomorphisms if some microlocal condition is satisfied. Consider for example a morphism of manifolds $f : Y \to X$, and let F be a sheaf (or better, a complex of sheaves) on X. Then there exists a canonical morphism

$$\omega_{Y/X} \otimes f^{-1}F \to f^{!}F \ , \tag{i.1}$$

($\omega_{Y/X}$ is the relative dualizing complex). It is well-known that this morphism is an isomorphism if f is smooth, but in fact there is a stronger result: this morphism is an isomorphism as soon as "f is non-characteristic for F"; (if f is a closed embedding, this means that the intersection of $SS(F)$ and the conormal bundle to Y in X is contained in the zero-section).

(b) Let X be a complex manifold, and $\mathcal{M}$ a system of linear differential equations on X, that is, a left coherent $\mathcal{D}_X$-module, where $\mathcal{D}_X$ denotes the sheaf of rings of holomorphic differential operators on X. Let

$$F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \tag{i.2}$$

be the complex of sheaves of holomorphic solutions of the system. The geometrical situation of linear partial differential equations with analytic coefficients is neatly expressed by the formula below, whose proof relies basically on the Cauchy-Kowalevski theorem:

$$SS(F) = \mathrm{char}(\mathcal{M}) \ . \tag{i.3}$$

Here $\mathrm{char}(\mathcal{M})$ denotes the characteristic variety of the system.

If one applies the result of (a) to the case where Y is a real analytic manifold and X a complexification of Y, then one finds that if $\mathcal{M}$ is elliptic, the complex of real analytic solutions and the complex of hyperfunction solutions of $\mathcal{M}$ are isomorphic. Hence, we get a purely sheaf-theoretical proof of a classical result of analysis.

(c) A complex of sheaves F on a real analytic manifold is weakly constructible if there exists a subanalytic stratification of X such that all cohomology groups are locally constant on the strata. This condition will be shown to be equivalent to a microlocal one, namely that the micro-support of F is a subanalytic Lagrangian subset of T^*X. Moreover if some finiteness condition is satisfied, one can associate to F a *Lagrangian cycle* supported by $SS(F)$, and one obtains the global Euler-Poincaré index of F on X as the intersection number of this cycle with a cycle naturally associated to the zero-section. By this result, we have an efficient microlocal tool to calculate indices.

In this book, we hope to convince the reader that this point of view is crucially relevant. Starting from the beginning (derived categories), we shall present both the classical theory ("the six operations") and the microlocal theory (micro-localization, micro-support, contact transformations). Then we shall apply the machinery to the study of constructible sheaves on real manifolds, and finally deal briefly with its applications to linear partial differential equations.

In more detail, the contents of the book are as follows.

Chapter I contains the basic facts about homological algebra which are necessary for the rest of the book and in particular the theory of derived categories (with the exception of the notion of t-structures, postponed until Chapter X).

Chapters II and III contain the "classical" notions on sheaves in the language of derived categories, including the six operations, as well as the Fourier-Sato transformation, which interchanges sheaves (more precisely, objects of the derived category of sheaves) on a vector bundle, and sheaves on the dual vector bundle.

Chapter IV is devoted to microlocalization. After recalling the geometric construction of the normal deformation of a submanifold M in a manifold X, we define the specialization functor ν_M, which sends sheaves on X to sheaves on the normal bundle $T_M X$, and its Fourier-Sato transform, the microlocalization functor μ_M. We also define a natural generalization of μ_M, the functor μhom and we study the functorial properties of all these functors.

In **Chapter V** we introduce the micro-support of sheaves. After proving a global extension theorem for sheaves in terms of the geometry of their micro-support, we make use of the γ-topology to cut-off sheaves "microlocally". Then we study the behavior of the micro-support under the functorial operations in the non-characteristic case (for inverse images) or proper case (for direct images). As an application, *the Morse inequalities* for sheaves are obtained.

In **Chapter VI**, we use the micro-support to localize the derived category of sheaves $\mathbf{D}^b(X)$ with respect to a subset Ω of T^*X, (one obtains new triangulated categories, $\mathbf{D}^b(X;\Omega)$), and to define "microlocal" inverse or direct images. Next, we extend the results of Chapter V to the general case by studying the behavior of the micro-support with respect to the functorial operations. In particular, we prove that the micro-support of $\mu_M(F)$ is contained in the normal cone of $SS(F)$ along T_M^*X. This inequality is a sheaf-theoretical version of a theorem on micro-hyperbolic systems, and will be used all along the book. This chapter also contains a crucial result, *the involutivity theorem* for micro-supports.

In **Chapter VII** we perform contact transformations for sheaves. If χ is a contact transformation between two open subsets Ω_X and Ω_Y of T^*X and T^*Y then, under suitable conditions, one can construct an isomorphism $\mathbf{D}^b(X;\Omega_X) \simeq$

$\mathbf{D}^b(Y;\Omega_Y)$, and this isomorphism is compatible with μhom. When calculating the image of the constant sheaf A_M on a submanifold M of X, one is led naturally to the notion of *pure sheaves* along a smooth Lagrangian manifold. A pure sheaf is "generically" (and microlocally) isomorphic to some sheaf $L_M\,[d]$ for some A-module L and some shift d, but the calculation of the shift requires the whole machinery of the inertia index of a triplet of Lagrangian planes. In this chapter we calculate in particular the shift of the microlocal composition of kernels.

In **Chapter VIII**, we make a detailed study of constructible sheaves on real manifolds. We introduce the (microlocal) notion of a μ-stratification, and then prove that a sheaf is weakly constructible if and only if its micro-support is subanalytic and isotropic (hence, Lagrangian). Then we can apply the preceding results to study the functorial operations on constructible sheaves. On a complex manifold, we prove that a sheaf is constructible if it is so on the real underlying manifold, and moreover if its micro-support is invariant by the action of $\mathbb{C}^\times$. From this, we deduce a theorem for non-proper direct images. Finally, we show that the nearby-cycle and the vanishing-cycle functors are particular case of the specialization and the microlocalization functors.

The notions of subanalytic chains and cycles are introduced in **Chapter IX**, with the help of the dualizing complex. Then, using the functor μhom, we associate to a constructible sheaf F its characteristic cycle and we show that the intersection number of this Lagrangian cycle with the zero-section of T^*X gives the global Euler-Poincaré index of F on X. We also calculate local Euler-Poincaré indices, and make the link between Lagrangian cycles and constructible functions on X, thus obtaining a new calculus on these functions. In this chapter, we also give a Lefschetz fixed point formula for constructible sheaves.

Chapter X develops the theory of perverse sheaves. After recalling the notion of t-structures, we define the perverse sheaves (i.e. perverse complexes) on a real manifold, and show that they form an abelian category. Then we study perverse sheaves on complex manifolds, give a microlocal characterization of perversity, and prove that perversity is preserved by various functorial operations.

In **Chapter XI**, we show briefly how to apply the theory of sheaves to the study of systems of linear partial differential equations. After a short review of the theory of $\mathcal{O}_X$ and $\mathcal{D}_X$-modules, we prove one inclusion in (i.3), and deduce that the complex of holomorphic solutions of a holonomic $\mathcal{D}_X$-module is perverse. We also introduce the sheaves of hyperfunctions and microfunctions, and deduce from (i.3) some basic results on the microfunction solutions of elliptic or hyperbolic systems. In the course of this chapter, we also make quantized contact transformations operate on the sheaf $\mathcal{O}_X$. This result has many important applications which shall not be discussed here.

We end this book with a short **Appendix** in which we collect all results (with some proofs) that we need on symplectic geometry, especially on the inertia index.

Each chapter opens with a short introduction, and includes exercises of varying difficulty. Some of these exercises (especially in Chapter I) are auxiliary results used in the course of the book. In general, the proof of such exercises is straightforward, otherwise a hint is given.

We have as far as possible avoided giving bibliographical references within the text. Instead, we have chosen to end each chapter with a few historical comments. The reason is that most of the time a theorem has a long and complicated history, and it would be tedious to quote each time everyone who contributed to a result. On the other hand, it seems improper to quote only the person who has initiated the subject or who has put it into final form.

The origin and the beginnings of sheaf theory are rather intricate, and this book benefits from the historical work of Christian Houzel who has agreed to contribute a detailed account of this part of the history.

A Short History: Les débuts de la théorie des faisceaux

by Christian Houzel

1. Le cours de Leray (1945)

Pendant qu'il était prisonnier de guerre à l'Oflag XVII en Autriche, Jean Leray a fait un cours de topologie algébrique à l'Université de captivité qu'il avait contribué à organiser. C'est un sujet qu'il avait déjà abordé en 1934 dans son article avec J. Schauder sur l'extension en dimension infinie de la notion de degré d'application et du théorème du point fixe de Brouwer [33]. Leray avait besoin d'un tel théorème dans des espaces fonctionnels pour obtenir l'existence de solutions des équations non linéaires rencontrées en hydrodynamique (pour lesquelles les solutions ne sont pas nécessairement régulières ni uniques).

Le cours de Leray a été publié à la fin de la guerre en 1945 dans le Journal de Liouville [29]. La topologie algébrique y est développée sur des bases nouvelles, évitant les hypothèses d'orientabilité ou de linéarité locale et les méthodes de subdivision ou d'approximation simpliciale. L'accent est mis sur la cohomologie, qui n'avait été clairement distinguée de l'homologie que juste avant la guerre [47], en particulier après les travaux de de Rham [37]; la cohomologie d'un espace à coefficients dans un anneau a toujours une structure multiplicative, et la structure multiplicative en homologie, avec laquelle on travaillait dans le cas des variétés compactes orientables, se déduit de celle de la cohomologie par dualité de Poincaré. Leray rebaptise "homologie" la cohomologie et parle de "groupes de Betti" quand il s'agit de l'homologie. Pour définir la cohomologie d'un espace topologique E à coefficients dans un anneau A, il s'inspire du procédé de Čech [9], mais il remplace la notion ensembliste de recouvrement par une notion mieux adaptée à la topologie algébrique: celle de couverture. Pour définir une couverture on se donne d'abord un "complexe abstrait", suite de groupes commutatifs libres de type fini correspondant aux diverses dimensions p et munis de bases $(X^{p\alpha})_\alpha$, avec la donnée d'un opérateur cobord $X^{p\alpha} \mapsto \dot{X}^{p\alpha}$ élément du groupe de dimension $p + 1$ (on étend par linéarité aux autres éléments du groupe), soumis à l'axiome que le cobord d'un cobord est nul. On rend "concret" le complexe en associant à chaque $X^{p\alpha}$ un support $|X^{p\alpha}|$, partie non vide de E; on impose l'axiome que si $X^{q\beta}$ est adhérent à $X^{p\alpha}$ (c'est-à-dire lui est relié par une suite finie d'éléments de base dont chacun intervient dans le cobord du précédent), le support de $X^{q\beta}$ est contenu dans celui de $X^{p\alpha}$. Le complexe concret K est une couverture si les supports sont fermés, pour tout point x de E, le sous-complexe formé par les éléments dont le support contient x est un simplexe (sa cohomologie est triviale) et la somme K^0 des éléments de dimension 0 est un cocycle (dit (co-)cycle unité). Les classes de cohomologie de E à coefficients dans A sont celles des formes L^p de couvertures quelconques K de E (combinaisons linéaires à coefficients dans A des éléments de base de K en dimension p), en convenant d'identifier L^p avec "l'intersection" $L^p.K'^0$ chaque fois que K' est une autre couverture (on définit l'intersection à l'aide du complexe produit tensoriel; le support de $X^{p\alpha} \otimes X'^{q\beta}$ est l'intersection des supports de $X^{p\alpha}$ et de $X'^{q\beta}$ et on passe au quotient en annulant les éléments de support vide). Dans le cas d'un espace normal, on peut calculer la cohomologie en prenant seulement les couvertures d'une famille stable par intersection et contenant, pour tout recouvrement ouvert fini ρ de E, une couverture dont les supports sont "ρ-petits"; dans le cas d'un espace compact, on peut se contenter d'une seule couverture si ses supports sont "simples" (c'est-à-dire cohomologiquement triviaux). Leray étend les résultat de Hopf [23] sur certaines variétés orientables compactes au cas d'espaces topologiques compacts. Il développe la théorie de la dualité permettant de récupérer les groupes de Betti. La première partie de son cours se termine sur l'introduction du nombre de Lefschetz d'une application continue de E dans lui-même, dans le cas où E est compact connexe et admet un recouvrement fini "convexoïde" (c'est-à-dire par des fermés simples dont les intersections sont vides ou simples).

Dans la deuxième partie, Leray compare le nombre de Lefschetz de $\xi : E \to E$ à celui de la restriction de ξ à un fermé stable par ξ. Il introduit les "pseudo-cycles" (éléments de la limite projective des cohomologies de parties B compactes de E) pour étendre au cas non compact des résultats démontrés précédemment dans le cas compact. Il définit des couvertures ("dallages") à partir de décompositions cellulaires de variétés différentiables et il établit la dualité de Poincaré dans le cas orientable et sans bord. La troisième partie, qui est l'origine du travail de Leray en topologie algébrique, définit l'indice total $i(O)$ des solutions d'une équation $x = \xi(x)$ dans un ouvert O de E (où $\xi : F \to E$ est une application continue définie dans un fermé F qui contient $\bar{O}$); on suppose E "convexoïde" (compact connexe et admettant un recouvrement par des fermés simples dont les intersections finies sont vides ou simples et dont les intérieurs forment une base de la topologie) et on considère la cohomologie à coefficients entiers. L'indice total $i(O)$ est défini si l'équation considérée n'a aucune solution sur la frontière de O et il ne dépend que de la restriction de ξ à $\bar{O}$; il est invariant par homotopie (sur ξ) et est égal au nombre de Lefschetz dans le cas où $O = E$. Si toutes les solutions de l'équation dans O appartiennent à une réunion d'ouverts disjoints O_α contenus dans O, $i(O)$ est la somme des $i(O_\alpha)$. Dans le cas d'une solution isolée x, on définit son indice comme l'indice total $i(V)$ où V est un voisinage assez petit de x, et si O ne contient que des solutions isolées, $i(O)$ est la somme des indices de ces solutions. Un théorème d'unicité s'obtient dans la théorie des sillages avec grande résistance au courant en établissant que toute solution de l'équation est isolée et d'indice 1. Leray définit encore un indice pour des équations d'une forme un peu plus générale, et il applique sa théorie à l'équation de Fredholm. Il considère aussi le cas d'équations de la forme $x = \xi(x, x')$ où $\xi : F \to E$ est une application