Otmar Scherzer · Markus Grasmair
Harald Grossauer · Markus Haltmeier
Frank Lenzen

APPLIED MATHEMATICAL SCIENCES 167

Variational Methods in Imaging

成像中的变分法

Springer

世界图书出版公司
www.wpcbj.com.cn

Otmar Scherzer
Markus Grasmair
Harald Grossauer
Markus Haltmeier
Frank Lenzen

Variational Methods in Imaging

With 72 Figures

Springer

图书在版编目（CIP）数据

成像中的变分法 = Variational methods in imaging：英文/（奥）斯科泽（Scherzer，O.）著. —影印本. —北京：世界图书出版公司北京公司，2013.3
ISBN 978-7-5100-5842-4

Ⅰ.①成… Ⅱ.①斯… Ⅲ.①成象处理—变分法—英文 Ⅳ.①O177

中国版本图书馆 CIP 数据核字（2013）第 035493 号

书　　名：	Variational Methods in Imaging
作　　者：	Otmar Scherzer，Markus Grasmair，et al.
中 译 名：	成像中的变分法
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河市国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@wpcbj.com.cn
开　　本：	24 开
印　　张：	14
版　　次：	2013 年 3 月
版权登记：	图字：01-2013-0911
书　　号：	978-7-5100-5842-4　　定　　价：59.00 元

Otmar Scherzer
otmar.scherzer@uibk.ac.at

Markus Grasmair
markus.grasmair@uibk.ac.at

Harald Grossauer
harald.grossauer@uibk.ac.at

Markus Haltmeier
markus.haltmeier@uibk.ac.at

Frank Lenzen
frank.lenzen@uibk.ac.at

All affiliated with:
Department of Mathematics
University of Innsbruck
Techniker Str. 21a/2
6020 Insbruck
Austria

Editors

Reprint from English language edition:
Variational Methods in Imaging
by O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, F. Lenzen

This book is dedicated to *Zuhair Nashed* on the occasion of his 70th birthday. Zuhair has collaborated with Heinz Engl, University of Linz, Austria. Heinz Engl in turn has supervised Otmar Scherzer, who was also supervised afterwards by Zuhair during several long- and-short term visits in the USA. Finally, Markus Grasmair was supervised by Otmar Scherzer during his PhD studies, and the thesis was also evaluated by Zuhair. Three generations of mathematicians in Austria congratulate Zuhair and his family on his 70th birthday.

Otmar Scherzer also dedicates this book to his family: Roswitha, Anna, Simon, Heide, Kurt, Therese, Franz, Paula, and Josef.
Markus Haltmeier dedicates this book to his family.
Frank Lenzen dedicates this book to Bettina, Gisela, Dieter, and Ulli.

Preface

Imaging is an interdisciplinary research area with profound applications in many areas of science, engineering, technology, and medicine. The most primitive form of *imaging* is *visual inspection*, which has dominated the area before the technical and computer revolution era. Today, computer imaging covers various aspects of *data filtering*, *pattern recognition*, *feature extraction*, *computer aided inspection*, and *medical diagnosis*. The above mentioned areas are treated in different scientific communities such as *Imaging*, *Inverse Problems*, *Computer Vision*, *Signal* and *Image Processing*, ..., but all share the common thread of recovery of an object or one of its properties.

Nowadays, a core technology for solving imaging problems is *regularization*. The foundations of these approximation methods were laid by Tikhonov in 1943, when he generalized the classical definition of *well-posedness* (this generalization is now commonly referred to as *conditional well-posedness*). The heart of this definition is to specify a *set of correctness* on which it is known *a priori* that the considered problem has a unique solution. In 1963, Tikhonov [371, 372] suggested what is nowadays commonly referred to as Tikhonov (or sometimes also Tikhonov–Phillips) regularization. The abstract setting of regularization methods presented there already contains all of the variational methods that are popular nowadays in imaging. Morozov's book [277], which is the English translation of the Russian edition from 1974, is now considered the first standard reference on Tikhonov regularization.

In the early days of regularization methods, they were analyzed mostly theoretically (see, for instance, [191, 277, 278, 371–373]), whereas later on numerics, efficient solutions (see, for instance, the monographs [111, 204, 207, 378]), and applications of regularization methods became important (see, for instance, [49, 112–114]).

Particular applications (such as, for instance, segmentation) led to the development of specific variational methods. Probably the most prominent among them is the Mumford–Shah model [276, 284], which had an enormous impact on the analysis of regularization methods and revealed challenges for the efficient numerical solution (see, e.g., [86, 88]). However, it is notable that the Mumford–Shah method also reveals the common features of the abstract form of Tikhonov regularization. In 1992, Rudin, Osher, and Fatemi published *total variation regularization* [339]. This paper had an enormous impact on theoretical mathematics and applied sciences. From an analytical point of view, properties of the solution of regularization functionals have been analyzed (see, for instance, [22]), and efficient numerical algorithms (see [90, 133, 304]) have been developed.

Another stimulus for regularization methods has come from the development of non-linear parabolic partial differential equations for *image denoising* and *image analysis*. Here we are interested in two types of evolution equations: *parabolic subdifferential inclusion* equations and *morphological* equations (see [8, 9, 194]). Subdifferential inclusion equations can be associated in a natural way with Tikhonov regularization functionals. This for instance applies to *anisotropic diffusion filtering* (see the monograph by Weickert [385]). As we show in this book, we can associate *non-convex* regularization functionals with morphological equations.

Originally, Tikhonov type regularization methods were developed with the emphasis on the stable solution of *inverse problems*, such as tomographical problems. These inverse problems are quite challenging to analyze and to solve numerically in an efficient way. In this area, mainly simple (quadratic) Tikhonov type regularization models have been used for a long time. In contrast, the underlying physical model in image analysis is simple (for instance, in denoising, the identity operator is inverted), but sophisticated regularization techniques are used. This discrepancy between the different scientific areas led to a split.

The abstract formulation of Tikhonov regularization can be considered in *finite dimensional* space setting as well as in *infinite dimensional function space* setting, or in a combined *finite-infinite* dimensional space setting. The latter is frequently used in spline and wavelet theory. Moreover, we mention that Tikhonov regularization can be considered in a *deterministic* setting as well as in a *stochastic* setting (see, for instance, [85, 231]).

This book attempts to bridge the gap between the two research areas of image analysis and imaging problems in inverse problems and to find a common language. However, we also emphasize that our research is biased toward *deterministic* regularization and, although we use statistics to motivate regularization methods, we do not make the attempt to give a stochastic analysis.

For applications of imaging, we have chosen examples from our own research experience, which are *denoising*, *telescope imaging*, *thermoacoustic imaging*, and *schlieren tomography*. We do not claim that these applications are most representative for imaging. Certainly, there are many other active research areas and applications that are not touched in this book.

Of course, this book is not the only one in the field of *Mathematical Imaging*. We refer for instance to [26, 98]. Imaging from an inverse problems point of view is treated in [49]. There exists also a vast number of proceedings and edited volumes that are concerned with mathematical imaging; we do not provide detailed references on these volumes. Another branch of imaging is mathematical methods in tomography, where also a vast amount of literature exists. We mention exemplarily the books [232, 288, 289].

The objective of this book certainly is to bridge the gap between regularization theory in image analysis and in inverse problems, noting that both areas have developed relatively independently for some time.

Acknowledgments

The authors are grateful for the support of the Austrian Science Foundation (FWF), which supported the authors during writing of the book. The relevant supporting grants are Y-123 INF, FSP 92030, 92070, P18172-N02, S10505.

Moreover, Otmar Scherzer is grateful to the Radon Institute in Linz and the available research possibilities there.

The authors thank the Infmath group in Innsbruck and the Imaging group in Linz for their proofreading. We are grateful to many researchers that stimulated our research and spared much time for discussion.

Otmar Scherzer acknowledges the possibility to teach preliminary parts of the book in summer schools in Vancouver (thanks to Ian Frigaard), in Jyväskylä (thanks to Kirsi Majava), and at CMLA, Paris (thanks to Mila Nikolova).

The authors are grateful to GE Medical Systems Kretz Ultrasound AG for providing the ultrasound data frequently used in the book as test data. Moreover, the authors thank Vaishali Damle and Marcia Bunda of Springer New York for their constant support during the preparation of the book.

Innsbruck,
2008

Markus Grasmair, Harald Grossauer, Markus Haltmeier,
Frank Lenzen, Otmar Scherzer

Contents

Part I

Fundamentals of Imaging

1

Case Examples of Imaging

In this chapter, we study several imaging examples from our own research experience. The first example concerns the problem of *denoising*. The other examples are related to *inverse problems*, which in general are defined as problems of recovering the cause for an observed effect (see [152]).

1.1 Denoising

One of the most important problems in digital image processing is *denoising*. Noise is usually considered as undesired perturbation in an image. However, it appears during every data acquisition process, for instance during recording with CCD sensors (see [359]).

Denoising is the process of reducing spurious noise in an image. It is either used to make images look "nicer" or as a preprocessing step for *image analysis* and *feature extraction.*

In order to highlight the importance of denoising for image analysis, we apply a *segmentation* and an *edge detection* algorithm to the ultrasound data shown in Fig. 1.1. It can be seen from Figs. 1.2 and 1.3 that after filtering in a preprocessing step, the implementation of these algorithms yields clearly better results.

- The task of segmentation is to retrieve all pixels belonging to an object of interest in a given image.
  As an example, we consider segmentation of the vein in the ultrasound image Fig. 1.1, which is the circular, dark domain in the center. To that end we use the following *region-growing algorithm* based on *intensity thresholding* (see [336]): Given an intensity threshold c and a *seed pixel* p with an intensity less than or equal to c, we start with the initial region $R^0 := \{p\}$ and iteratively obtain regions R^{i+1} from R^i by adding pixels that are *neighboring* R^i and whose intensities are less than or equal to c. The

O. Scherzer et al., *Variational Methods in Imaging*,

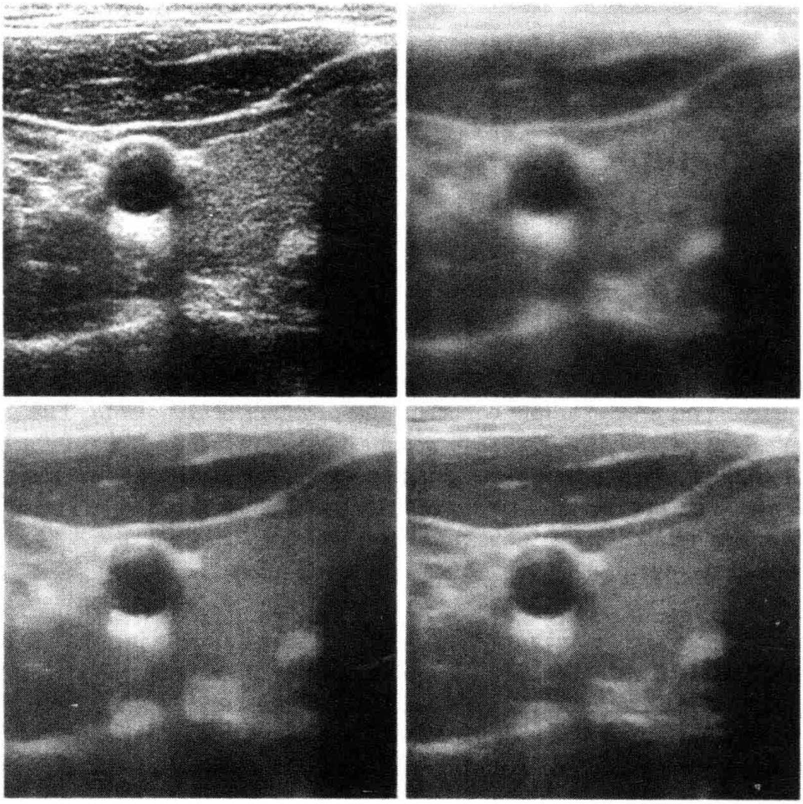

Fig. 1.1. Results of different variational regularization techniques for denoising ultrasound data (*top left*), which are described in Chapter 4.

region growing stops if no more pixels satisfying these two conditions can be found. Figure 1.2 shows the result of the region-growing algorithm applied to the original and filtered data in Fig. 1.1. The results imply that the segmentation is unsatisfactory if the algorithm is applied to unfiltered data.

- Another example that reveals the importance of denoising as preprocessing step in image analysis is edge detection. Here the goal is to extract the boundaries of objects or regions in the image.

  One widely used method for edge detection is the *Sobel operator*: Let

$$G_x = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix}, \quad G_y = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

  We denote the discrete convolution (see [184, Sect. 3.4]) of an image $\mathbf{u}$, interpreted as real-valued matrix, with the masks G_x and G_y by $G_x * \mathbf{u}$ and $G_y * \mathbf{u}$, respectively. The Sobel operator is given by

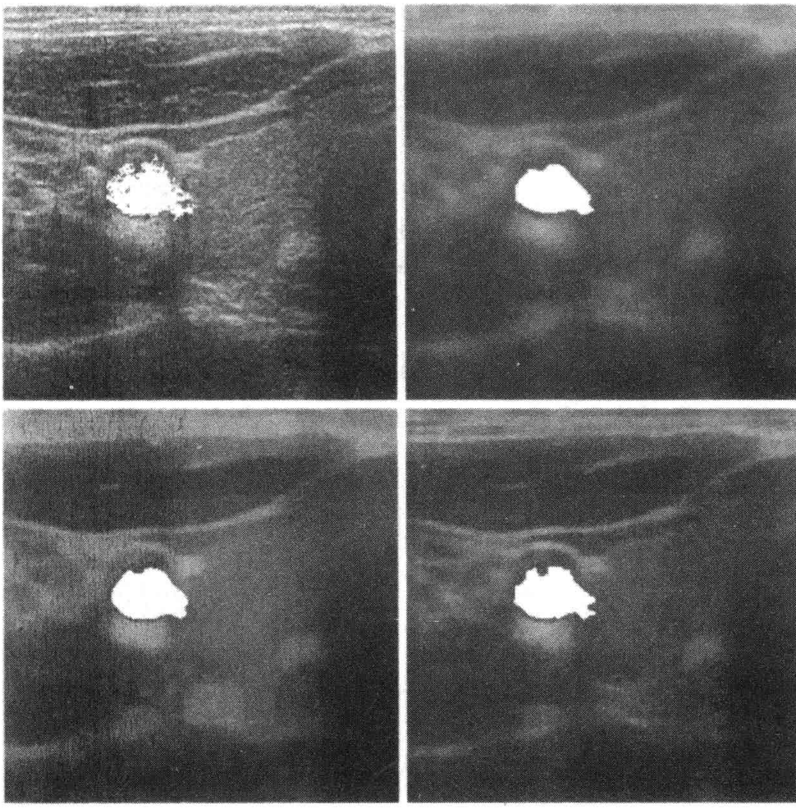

Fig. 1.2. Segmentation of the vein in the ultrasound image Fig. 1.1. The white regions indicate the results of a region-growing algorithm applied to the original data (*top left*) and the different smoothed images. Segmentation of the original data provides a region with fuzzy boundary. When the algorithm is applied to filtered data, the results show a more regular shape that better reflects the vein's true boundary.

$$G : \mathbf{u} \mapsto \sqrt{(G_x * \mathbf{u})^2 + (G_y * \mathbf{u})^2} \, .$$

The value $(G\mathbf{u})_{ij}$ is large near edges and small in homogeneous regions of the image. As can be seen from Fig. 1.3, the edge detector gives significantly better results for the filtered than for the unfiltered data, where spurious edges appear.

Among the variety of denoising techniques, two classes are of importance for this book: *variational methods*, which are discussed in Chapter 4, and *evolutionary partial differential equations*, which are discussed in Chapter 6.

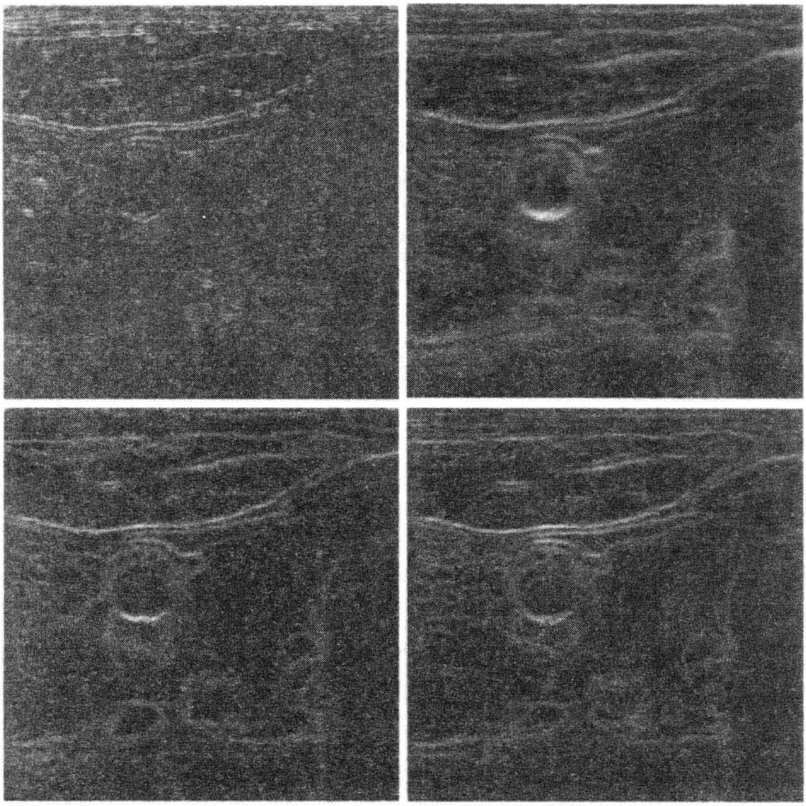

Fig. 1.3. Edge detection with the Sobel operator. The images show the value of the Sobel operator applied to the original (*top left*) and filtered data. Using filtered data improves the quality of detection, as spurious edges created by noise are suppressed.

1.2 Chopping and Nodding

Chopping and nodding (see [51, 148, 230, 252, 333]) is a common approach for the removal of background noise in infrared observations of the sky with ground-based telescopes.

The basic assumption is that the background noise can be decomposed into two components, the first of which mainly depends on the time of acquisition of the image, whereas the second, *residual noise*, varies in time at a slower rate and mainly depends on the optical path of light through the telescope.

We denote by $\mathbf{x} \in S^2$ the position in the sky the telescope, located at $0 \in \mathbb{R}^3$, is originally pointing to. Here S^2 denotes the unit sphere in $\mathbb{R}^3$. From this position $\mathbf{x}$, a signal u_1 is recorded.

Then a *chopping* procedure is performed, which consists in tilting the secondary mirror of the telescope by a certain angle (see Fig. 1.4). After tilting, the telescope points to a position $\mathbf{y} \in S^2$, and a signal u_2 is recorded. The