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ON QUASI-PRIMITIVE RINGS (III)

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Introduction

It is well known that every non-zero nil semi-simple right Artinian ring R can be expressed uniquely as the direct sum of a finite number of minimal right ideals. To obtain deeper information we naturally ask what kind of rings R can contain identity and be expressed as a direct sum of a finite number of right ideals which need not to be minimal. In order to proceed we introduce here the notion of M-Artinian rings which extends the notion of the ordinary Artinian rings.

In this paper we shall give a structure of M-Artinian rings. In general, the M-Artinan rings are not the quasi-primitive rings as stated in [1]. Therefore we'd better modify slightly the definition of the quasi-primitive rings and extend it to the so-called weak quasi-primitive rings which contain the M-Artinian rings and also hold all the results of quasi-primitive rings we have obtaind in [1, 2].

§ 1. M-Artinian rings

In this section we introduce the concept of M-Artinian ring and give its structure.

Definition 1.1. *Let R be an associative ring and M its subset. M is said to be an $\mathscr{S}$-subset if and only if the following conditions are satisfied:*

(i) $0\notin M$,

(ii) *let e be an idempotent of R. If $ae=a\in M$ or $ea=a\in M$, then $e\in M$,*

(iii) *if e_1 and e_2 are two pairwise orthogonal idempotent elements of M, then $e_1Re_2\cap M\neq\emptyset$.*

For example, let $M=R\backslash 0$, then it is clear that M satisfies all conditions of definition 1.1.

Definition 1.2. *A right (left) ideal L of R is said to be an M-right (left) idael, if $M\cap L\neq\emptyset$. An M-right (left) ideal L is called M-minimal, if there exists an M-right (left) ideal L' such that $L\cap M\supseteq L'\cap M$, then it must be $L\cap M=L'\cap M$.*

Definition 1.3. *Let L_1 and L_2 be two right (left) ideals of R, L_1 and L_2 are*

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called M-equivalent if and only if $L_1 \cap M = L_2 \cap M$.

According to definition 1.3, we can introduce an equivalence relation which can classify all right (left) ideals of R into pairwise disjoint equivalence classes.

Proposition 1.1. *Let* $\mathbf{C}$ *be an equivalence class in the meaning of definition* 1.3, *then there exists a smallest element in* $\mathbf{C}$.

Proof Denote L_i by elements of $\mathbf{C}$ and let $L = \bigcap_{L_i \in \mathbf{C}} L_i$. Since L_i, $L_j \in \mathbf{C}$, we have $L_i \cap M = L_j \cap M$. Hence $L \cap M = L_i \cap M$. It is easy to see $L \in \mathbf{C}$. This completes our proof.

For the sake of simplity we call the smallest element of $\mathbf{C}$ in definition 1.1 the smallest element of class.

Definition 1.4. *Let L be an M-right (left) ideal of R. An elment* $a \in M \cap L$ *is called an inductive element if and only if a satisfies the following conditions:*

(i) $aL = L$ $(La = L)$,

(ii) *let* $e \in R$ *and* $ae = a$ $(a = ea)$, *then* $e^2 - e$ *must be nilpotent.*

Let L be a right ideal, $e \in R$. We denote $e^{\perp L} = \{r \in L \mid er = 0\}$ the right annihilator of e. Similarly we have ${}^{L\perp}e = \{r \in L \mid re = 0\}$ for left ideal L.

Definition 1.5. *Let R be an associative ring, and M be an* $\mathscr{S}$*-subset of R. R is said to be an M-Artinian ring (or left M-Artinian ring) if and only if R satisfies the following conditions:*

(i) *R satisfies the minimal condition for M-right (left) ideals, i.e., if* $L_1 \cap M \sqsupseteq L_2 \cap M \sqsupseteq \cdots$ *is a descending chain for* $L_i \cap M$ *where* L_i *are M-right (left) ideals, then there exists a positive integer n such that* $L_n \cap M = L_{n+1} \cap M = \cdots$.

(ii) *every smallest element L of class whose elements are equivalent to an M-minimal right (left) ideal must have an inductive element of L.*

(iii) *let L be an M-right (left) ideal, if* $L \cap M$ *has an idempotent e such that* $eR \neq L$, *then* $e^{\perp L}$ $({}^{\perp L}e)$ *must be M-right (left) ideal.*

Now we give an example for M-Artinian ring.

Any nil semi-simple Artinian ring must be M-Artinian. In fact, if we take $M = R \setminus 0$, then is clear that M is an $\mathscr{S}$-subset of R. Hence every M-right ideal is a non-zero right ideal and the minimum condition for M-right ideals is the minimum condition for right ideals. On the other hand, let L be a minimal right ideal of semi-simple Artinian ring, then it is clear that every non-zero element a of L must be an inductive element of L, this means that the condition (ii) of definition 1.5 is satisfied. The condition (iii) can be easily obtained by $e^{\perp L} \neq 0$.

Now we are going to study the structure of M-Artinian ring.

Lemma 1.1. *Let R be an M-Artinian ring. Then every M-right ideal L of R contains an idempotent element which contains also in M.*

Proof It follows from the assumption of R that every M-right ideal L must contain an M-minimal right ideal L_0. Without less of generality, it can be assumed that L_0 is the smallest element of class. By the assumption of definition 1.5, it is clear that L_0 has an inductive element a such that $a\in M\cap L_0$ as well as $aL_0=L_0$ and an element $e_0\in L_0$ such that $a=ae_0$. Hence $e_0^2-e_0$ is nilpotent. Let $x_0=e_0^2-e_0$, then $ax_0=0$ and there exists an integer $n_0>0$ such that $x_0^{n_0-1}\neq 0$, $x_0^{n_0}=0$. Set $e_1=e_0+x_0-2e_0x_0$, we have $e_1\in L_0$. From $ae_1=ae_0=a$ it follows that e_1 does not nilpotent. On the other hand, let $x_1=e_1^2-e_1$, then $x_1\in L_0$, $x_1=4x_0^3-3x_0^2$. Hence there exists a positive integer $n_1<n_0$ such that $x_1^{n_1}=0$. Let $e_2=e_1+x_1-2e_1x_1$, then $e_2\in L_0$ and $ae_2=ae_1$. It is easy to show that $e_2^2-e_2=4x_1^3-3x_1^2$. Now let $x_2=e_2^2-e_2$, then there exists a positive integer $n_2<n_1$ such that $x_2^{n_2}=0$. We can go on in this way and after a finite steps we must stop. Hence there exists a positive integer m such that $e_m^2-e_m=0$ and $e_m=e_{m-1}+x_{m-1}-2e_{m-1}x_{m-1}$. According to the inductive method it is clear that $e_m\in L_0$, $ae_m=ae_{m-1}=\cdots=ae_0=a\in M\cap L$. Therefore from the property of $\mathscr{S}$-subset M it follows $e_m\in L\cap M$.

Lemma 1.2. *Let R be an M-Artinian ring, then every M-right ideal L must be $L=eR$ where $e^2=e\in M$.*

Proof From lemma 1.1 it follows that L has an idempotent element e in M. If $L\neq eR$, then $e^{\perp L}$ must be an M-right ideal by assumption. Now we will show that there exists an idempotent element e' such that $e'R=L$. In fact, if all idempotent elements e_i of $L\cap M$ have the property $L\neq e_iR$ then $e_i^{\perp L}$ are all M-right ideals and $e_i^{\perp L}\neq 0$. Now let $\Sigma=\{e_i^{\perp L}\mid e_i^2=e_i\in L\cap M\}$, then by the assumption there exists an minimal element $e^{*\perp L}$ in Σ such that $e^{*\perp L}$ contains an M-minimal right ideal L_0. Without loss of generality we can assume that L_0 is a minimal element of class. Then by the proof of the lemma 1.1 we know that there exists an element $a\in M\cap L_0$ such that $aL_0=L_0$ and an idempotent element $e_0\in M\cap L_0$ such that $ae_0=a$, $e_0R=L_0$. Thus we have $e^*e_0=0$. Putting $e'=e^*-e_0e^*+e_0$ it is clear that $e'\in L$, $ae'=ae_0=a\in M$. Therefore $e'\in M$ by the property of $\mathscr{S}$-subset. On the hand, we know $e'^2=e'\neq 0$. Since $e'e_0=e_0^2=e_0\neq 0$ we have $e_0\notin e'^{\perp L}$, it follows $M\cap e'^{\perp L}\subsetneqq M\cap e^{*\perp L}$ from $e^*e'=e^{*2}=e^*$, $e_0\in e^{*\perp L}$. Since $e'\in L\cap M$, hence $e'^{\perp L}\in\Sigma$, but this contradict to the fact that $e^{*\perp L}$ is M-minimal in Σ. Hence there exists $e^2=e\in L\cap M$ such that $L=eR$.

Corollary. *There exists only one element in any class* $\mathbf{C}$ *of M-right ideals.*

Proof If $L_1, L_2\in\mathbf{C}$, then by lemma 1.2 $L_1=e_1R$, $L_2=e_2R$ and $e_1, e_2\in L_1\cap M=L_2\cap M$. Hence $L_1=L_2$.

Lemma 1.3. *Let $L_0\subset L$ are M-right ideals and $L_0\neq L$. If $L_0=e_0R$, $e_0^2=e_0\in M$, $L=eR$, $e^2=e\in M$, then there exists an M-right ideal $L_1=eR$ such that $L_0\oplus L_1=L$, where $e_1^2=e_1$, $e_0e_1=e_1e_0=0$.*

Proof Put $e_1=e-e_0e$, then $e_1\in L$. Since $e_0\in L$, then $ee_0=e_0$. Therefore we have $e_1^2=e_1$, $e_1e_0=e_0e_1=0$. Let $e'=e_0+e_1$, then $e'\in L$. Since $e'R=(e_0+e_1)R=e_0R\oplus e_1R$, $L=eR=(e_1+e_0e_1)R\subseteq e'R\subseteq L$, hence we have $L=e'R=L_0\oplus L_1$.

Now we attempt to show $e_1\in M$. Since $e_0\in M\cap L_0$, $L_0\neq L$, $e_0^{\perp L}$ is M-right ideal by definition 1.5. It is easy to see that $e_0^{\perp L}=e_1R$. Hence e_1R is an M-right ideal. Therefore there exists an element $a\in e_1R\cap M$. It follows that $e_1a=a\in M$. By the property of $\mathscr{S}$-subset we have $e_1\in M$.

Theorem 1.1. *Let R be an M-Artinian ring. Then $R=e_1R\oplus e_2R\oplus\cdots\oplus e_nR$, where e_iR are all M-minimal right ideals, $e_i^2=e_i\in M$. If ${}^{R\perp}R=0$ then R contains the identity.*

Proof By lemma 1.1 R has an M-minimal right ideal $L_1=e_1R$, $e_1^2=e_1\in M$. By lemma 1.2 and 1.3 $R=L_1\oplus L_1'$, where $L_1'=e_1'R$, $e_1'^2=e_1'\in M$ and $e_1'e_1=e_1e_1'=0$. Now we consider L_1'. By lemma 1.3 there exist M-minimal right ideals $L_2=e_2^*R$, $e_2^{*2}=e_2^*\in M$, $L_2'=e_2'R$, $e_2'^2=e_2'\in M$ such that $e_2^*e_2'=e_2'e_2^*=0$, $L_1'=L_2\oplus L_2'$, $R=L_1\oplus L_2\oplus L_2'$.

Let $e_2=e_2^*-e_2^*e_1$, then from $e_1e_2^*=e_1e_2'=0$ it follows $e_2^2=e_2$. Since $e_2e_2^*=e_2^*\in M$, we have $e_2\in M$. Thus $R=e_1R\oplus e_2R\oplus L_2'$, where $e_1e_2=e_2e_1=0$, $e_1L_2'=e_2L_2'=0$.

Now we apply the induction as follows: assume that R can be expressed as a direct sum of M-right ideals

$$R=e_1R\oplus e_2R\oplus\cdots\oplus e_kR\oplus L_k',$$

where $e_i^2=e_i\in M$, $e_ie_j=\delta_{ij}e_i$, $e_iL_k'=0$, $i,\ j=1,\ \cdots,\ k$.

By lemma 1.3 there exist M-minimal right ideals $L_{k+1}=e_{k+1}^*R$, $e_{k+1}^{*2}=e_{k+1}^*\in M$ and $L_{k+1}'=e_{k+1}'R$, $e_{k+1}'^2=e_{k+1}'\in M$ such that $L_k'=L_{k+1}\oplus L_{k+1}'$, $e_{k+1}^*e_{k+1}'=e_{k+1}'e_{k+1}^*=0$. Let $e_{k+1}=e_{k+1}^*-\sum_{i=1}^{k}e_{k+1}^*e_i$, then from $e_ie_{k+1}^*=0$, $i=1,\ \cdots,\ k$ it follows $e_{k+1}^2=e_{k+1}$, $e_{k+1}e_i=e_ie_{k+1}=0$, $e_{k+1}L_{k+1}'=0$. Since $e_{k+1}e_{k+1}^*=e_{k+1}^{*2}=e_{k+1}^*\in M$, we have $e_{k+1}\in M$ by the property of $\mathscr{S}$-subset. Therefore we have $R=e_1R\oplus\cdots\oplus e_{k+1}R\oplus L_{k+1}'$, where $e_i^2=e_i\in M$, $e_ie_j=e_i\delta_{ij}$, $e_iL_{k+1}'=0$, $i,\ j=1,\ \cdots,\ k,\ k+1$. By the induction we have $R=e_1R\oplus\cdots\oplus e_nR\oplus L_{n+1}'$ for any positive integer n.

Set $\mathscr{L}_i=e_iR\oplus e_{i+1}R\oplus\cdots\oplus e_nR\oplus L_{n+1}'$, we obtain a descending chain $R=\mathscr{L}_1\supsetneqq\mathscr{L}_2\supsetneqq\cdots\supsetneqq\mathscr{L}_k\supsetneqq\cdots$. By assumption of the condition for M-minimal right ideals there exists a positive integer m such that $\mathscr{L}_m\cap M=\mathscr{L}_{m+1}\cap M=\cdots$. Hence $R=e_1R\oplus\cdots\oplus e_mR$, where e_iR are all M-minimal right ideals $e_i^2=e_i\in M$, $e_ie_j=e_i\delta_{ij}$, $i,\ j=1,\ \cdots,\ m$.

Now we put $1=\sum_{i=1}^{m}e_i$ and denote ${}^{R\perp}1=\{r\in R\mid r1=0\}$. If $s\in R$, $r\in{}^{R\perp}1$, then $r1s=rs=0$, hence ${}^{R\perp}1={}^{R\perp}R$, and ${}^{R\perp}1=0$ by the assumption. Therefore $(x-x1)\,1=0$ for any $x\in R$, hence $x=x\,1$ and 1 is the identity of R.

Corollary. *Let R be M-Artinian ring, then there exist no nil right ideal of R which is M-right ideal.*

Theorem 1.2. *Let R be an M-Artinian ring, then there exist matrix units $\{e_{ij}\}_{n\times n}$ such that $R1=\sum_{i,j}^{n}Be_{ij}$, where $B=\{r\in R\mid re_{ij}=e_{ij}r,\ i,\ j=1,\ \cdots,\ n\}$, $1=\sum_{i=1}^{n}e_{ii}$ and $e_{ij}\in M$ for $i\geqslant j$.*

Proof $R=\sum_{i=1}^{n}\oplus e_iR$, $e_ie_j=\delta_{ij}e_i$, $i,\ j=1,\ \cdots,\ n$ by theorem 1.1 put $e_i=e_{ii}$, $i=1,\cdots,$

n. By assumption we know that $e_iRe_j \cap M = \emptyset$ for $i>j$. Now we choose an arbitrary element $e_{ij}=e_i\sigma_{ij}e_j \in e_iRe_j \cap M$, $\sigma_{ij}\in R$. Since e_iR is M-minimal we have $e_{ij}R=e_iR$ for $i>j$. Therefore there exists an element $\sigma_{ii}\in R$ such that $e_{ij}e_j\sigma_{ji}e_i=e_i$. Denote $e_{ji}=e_j\sigma_{ji}e_i$, then $e_{ij}e_{ji}=e_{ii}=e_i$ for $i>j$. We are going to show that $e_{ji}e_{ij}=e_{jj}$ for $j<i$. Since $e_{ij}e_{ji}e_{ij}=e_ie_{ij}=e_{ij}$, we have $e_{ij}(e_j-e_{ji}e_{ij})=0$. Now we first show that $e_{ij}^{\perp R}=\sum_{i\neq j}\oplus e_iR$. In fact, if we put $L_0=(1-e_j)R$, $L=e_{ij}^{\perp R}$, then we have $e^2=e\in M$, $eR=L$ by lemma 1.2. Since $(1-e_j)e_i=e_i\in M$, $i\neq j$, it follows from the property of $\mathscr{S}$-subset that $1-e_j\in M$. If $L\neq L_0$, then by lemma 1.3 there exists $e_1^2=e_1\in M$ such that $L_0\oplus L_1=L$, $L_1=e_1R$, $(1-e_j)e_1=e_1(1-e_j)=0$. Therefore $e_1R=e_je_1R\sqsubseteq e_jR$. Since e_1R and e_jR are M-minimal right ideals, we have $e_1R=e_jR$ and $L=e_1R+(1-e_j)R=e_jR+\sum_{i\neq j}e_iR=R$. This contradicts $e_j\notin e_{ij}^{\perp R}=L$. Hence $L=L_0$, $e_{ij}^{\perp R}=\sum_{i\neq j}e_iR$. It follows therefore from $e_{ij}(e_j-e_{ji}e_{ij})=0$ that $e_j-e_{ji}e_{ij}\in\sum_{i\neq j}e_iR$. Hence $e_{ji}e_{ij}=e_{jj}=e_j$. Thus we have already constructed $n\times n$ number of matrix units $\{e_{ij}\}_{n\times n}$. Now we prove the last statement. Let $a\in R$, $a_{ij}=\sum_{k=1}^{n}a_{ki}\, a\, e_{jk}$, then it is easy to see $a_{ij}e_{rs}=e_{rs}a_{ij}$. Hence $a_{ij}\in B$ and $\sum_{i,j=1}^{n}a_{ij}e_{ij}=1a1=a1$.

Theorem 1.3. *Let R be an M-Artinian ring. Then R can be expressed as $R1=\sum_{i,j}^{n}Be_{ij}$, as stated in theorem 1.2. Denote $\mathfrak{M}_\lambda=e_{\lambda\lambda}R1$, $K=e_{\lambda\lambda}Re_{\lambda\lambda}$, then*

$$\mathfrak{M}_\lambda=\sum_{j=1}^{n}\oplus Ke_{\lambda j}=\sum_{j=1}^{n}\oplus Be_{\lambda j}$$

such that $R1$ is the ring of K (or B)-endomorphisms of $\mathfrak{M}_\lambda$ and $K=e_{\lambda\lambda}B\cong B$. Similarly if we denote $A_\lambda=Re_{\lambda\lambda}$, then we have $A_\lambda=\sum_{j=1}^{n}\oplus e_{i\lambda}K$ such that R is the ring of K (or B)-endomorphisms of A_λ.

Proof By theorem 1.2 $R\cdot 1=\sum_{i,j=1}^{n}Be_{ij}$, hence

$$\mathfrak{M}_\lambda=e_{\lambda\lambda}R\cdot 1=\sum_j Be_{\lambda j}=\sum_{i,j,k}e_{\lambda\lambda}Be_{ik}e_{\lambda j}$$

$$=e_{\lambda\lambda}\sum_{j=1}^{n}Re_{\lambda j}=\sum_{j=1}^{n}e_{\lambda\lambda}Re_{\lambda\lambda}e_{\lambda j}=\sum_{j=1}^{n}Ke_{\lambda j}.$$

It follows that

$$\sum_{j=1}^{n}Ke_{\lambda j}e_{\lambda\lambda}=\sum_{j=1}^{n}Be_{\lambda j}e_{\lambda\lambda}=K=Be_{\lambda\lambda}.$$

It is easy to see $e_{\lambda\lambda}B\cong B$. We are going to show that $R\cdot 1$ is the ring of K-endomorphisms of $\mathfrak{M}_\lambda$. For this purpose we first attend to $e_{\lambda j}R\cdot 1=e_{\lambda\lambda}R\cdot 1=\mathfrak{M}_\lambda$. Hence for any element m there exists an element $r\in R$ such that $e_{\lambda j}r\cdot 1=m$. Now we take an $e_{\lambda k}$ and set $s_k=e_{kj}\cdot r\cdot 1\in R\cdot 1$. Then we have $e_{\lambda k}s_k=e_{\lambda k}e_{kj}\cdot r\cdot 1=m$. Clearly $e_{\lambda\mu}s_k=0$, $\mu\neq k$.

Let σ be an arbitrary K-endomorphism of $\mathfrak{M}_\lambda$ and $e_{\lambda k}\sigma=m_k$. Then we have s_k such that $e_{\lambda k}s_k=m_k$, $e_{\lambda\mu}s_k=0$ for $\mu\neq k$. Put $s=\sum_{k=1}^{n}s_k\in R\cdot 1$, then $e_{\lambda j}\cdot s=e_{\lambda j}\cdot\sigma$, $j=1,\ \cdots,\ n$. Therefore $\sigma=s\in R\cdot 1$.

Now we attempt to show the last assertion of our theorem. As showing above we have

$$A_\lambda = Re_{\lambda\lambda} = \sum_{i,j}^{n} Be_{ij}e_{\lambda\lambda} = \sum_{i=1}^{n} Be_{i\lambda} = \sum_{i=1}^{n} e_{i\lambda}e_{\lambda\lambda}B = \sum_{i=1}^{n} e_{i\lambda}K.$$

Moreover we can show that R is the ring of K-endomorphisms of A_λ.

§ 2. Weak quasi-primitive rings

In this section we weaken the conditions of quasi-primitive rings in [1]. For this purpose we first introduce the concept of weak quasi-primitive rings which not only contains the M-Artinian rings but also holds the results which we have obtained in [1].

All the definitions and terms, which are given in [1] and used in this section, hold precisely their original meanings.

Let $\mathfrak{M}$ be F-vector space and $\hat{F}$ the set of invertible elements of F. Clearly $\hat{F}$ is a multiplicative group.

Proposition 2.1. *A strictly cyclic R-module $\mathfrak{M}$ is F-space if and only if thes et S of all free elements of $\mathfrak{M}$ satisfies the following conditions:*

(i) *there exists a set $\{u_i\}_\Gamma$ of S such that $\mathfrak{M} = \sum_{i\in\Gamma} \oplus Fu_i$,*

(ii) *let $u_1, \cdots, u_n$ be F-linearly indepedent elements, $x \in S$, if $x, u_1, \cdots, u_n$ are F-nonlinearly indepedent, then $x = \sum_{i<\infty} f_i u_i$, $f_i \in \hat{F}$, $i = 1, \cdots, n$.*

Proof Let $\{u_i\}_{\Gamma'}$ be a basis, $\{E_i\}_{\Gamma'}$ its pairwise orthogonal projections, $u \in R$, then $u = \sum_i g_i u_i$, By the assumption of our theorem we have $g_i \in \hat{F}$, $i = 1, \cdots, n$. Clearly, for any element $E_\lambda \in \{E_j\}$, it is true that $u\ E_\lambda = 0$ or $u\ E_\lambda = g_\lambda u_\lambda$, hence $u\ E_\lambda \in S$. Thus the condition (ii) of definition 1.3 in [1] is satisfied. Next we consider its condition (iii). Indeed, if $f \in F$, $f\mathfrak{M} = \mathfrak{M}$, then there exists $u \in S$ such that $fuR = \mathfrak{M}$. Hence $fu = \sum_{i=1}^{n} g_i u_i$, $g_i \in \hat{F}$. Since $u \in S$ we know $u = \sum_{i'} g'_{i'} u_{i'}$, $g'_{i'} \in \hat{F}$. Insomuch as $\sum_{i'} fg'_{i'} u_{i'} = \sum_i g_i u_i$, we get $fg'_i = g_i$, $i = 1, \cdots, n$. Hence $f \in \hat{F}$, therefore the sufficiency is proved.

The proof of the necessity is already given in [1].

Definition 2.1. *Denote S by the set of the free elements of $\mathfrak{M}$. S_α a subset of S. Then S_α is called a set containing basic free elements if and only if S_α satisfies the following conditions:*

(i) *S_α contains a basis $\{u_i\}$ of $\mathfrak{M}$,*

(ii) *if $x_1, \cdots, x_n$ are arbitrary finite number of F-linearly indepedent elements and $x_1, \cdots, x_n, x$ are F-nonlinearly independent elements, $x \in S_\alpha$, then*

$$x = \sum_{i=1}^{n} f_i x_i, \quad f_i \in \hat{F},\ i = 1, \cdots, n.$$

Denote $M^* = \{S_\alpha\}$ by the class of all sets containing basic free elements. Then M^*

is a partially ordered set under the set theoritical inclusion relation. Let $S_0 \subset S_1 \subset \cdots \subset S_\alpha \subset \cdots$ be an ascending chain of M^* and $S^* = \bigcup S_\alpha$. We shall show $S^* \in M^*$. Indeed, S^* contains a basis, let $x_1, \cdots, x_n$ be F-linearly independent elements of S^*, $x^* \in S^*$ and $x^*, x_1, \cdots, x_n$ be F-nonlinearly indepedent, then there exists S_α which contains x^*, $x_1, \cdots, x_n$. Therefore $x^* = \sum_i g_i x_i$, $g_i \in \hat{F}$. Hence $S^* \in M^*$. Applying Zorn's lemma we know that there exists a maximal element $\hat{S}$ in M^*.

Definition 2.2. *The above maximal element $\hat{S}$ of M^* is said to be a set of local free elements. A strictly cyclic module $\mathfrak{M}$ will be calledan F-quasispace, if and only if $\mathfrak{M}$ has a set of local free elements.*

From now on we will use the symbol $\hat{S}$, a set of local free elements, instead of S in [1], the set of all free elements of $\mathfrak{M}$. Specially, the sentence "$\mathfrak{M}$ has a basis $\{u_i\}$" is understood by $\mathfrak{M} = \sum \oplus F u_i$ and $\{u_i\} \subset \hat{S}$.

Proposition 2.2. *Let $\mathfrak{M}$ be F-quasispace. Then $\mathfrak{M}$ has the following property:*

(i) *$\mathfrak{M}$ has a F-basis,*

(ii) *let $\{u_i\}$ be a basis, $u \in \hat{S}$, then $u E_i = 0$ or $v E_i \in \hat{S}$, where $\{E_i\}$ is pairwise orthogonal projections of $\{u_i\}$,*

(iii) *if $f \in F$ and an element $u \in \hat{S}$ with $fu \in \hat{S}$, then $f \in \hat{F}$.*

Proof (i) is clear. Now we attempt to show (ii). Since $\{u_i\} \subset \hat{S}$, $u \in \hat{S}$, we have $u = \sum_i g_i u_i$, $g_i \in \hat{F}$. Hence any element E_j of $\{E_i\}$ has $uE_j = 0$ or $u E_j = g_j u_j$. Now we can show $uE_j \in \hat{S}$. Indeed $g_j \in \hat{F}$, hence $g_j u_j \in \hat{S}$. Denote $\hat{S}' = \hat{S} \cup g_j u_j$, it certainly can assume $g_j u_j \notin \hat{S}$. Let $v_1, \cdots, v_n$ be F-linearly independent elements of $\hat{S}$, then $v_1, \cdots, v_n, g_j u_j$ is F-linearly independent if and only if $v_1, \cdots, v_n, u_j$ is F-linearly independent. In fact, if $f_1 v_1 + \cdots + f_n v_n + f g_j u_j = 0$, $f_i, f \in F$ and $f \neq 0$, then $f g_j \neq 0$. Hence $v_1, \cdots, v_n, u_j$ are F-non linearly independent. Conversely, if $v_1, v_2, \cdots, v_n, u_j$ are F-nonlinearly independent, then it must be $u_j = \sum \hat{g}_i v_i$, $\hat{g}_i \in \hat{F}$. Hence $g_j u_j = \sum g \hat{g}_i v_i$, therefore $v_1, \cdots$, v_n, $g_j u_j$ are F-nonlinearly independent. Now let $x_1, x_2, \cdots, x_n$ be elements of $\hat{S}'$ and F-linearly independent and let $x' \in \hat{S}'$ such that $x', x_1, \cdots, x_n$ are F-nonlinearly independent. If $x_n = g_j u_j$, then just as mentioned above, it is clear that $x_1, \cdots, x_{n-1}, u_j$ belong to $\hat{S}$ and must be F-linearly independent as well as $x', x_1, \cdots, x_{n-1}, u_j$ F-linearly independent. Of course, we may assume $x' \in \hat{S}$, hence by the structure of $\hat{S}$ we know

$$x' = \sum_{i=1}^{n-1} g_i x_i + \hat{g}_n u_j, \quad \hat{g}_i \in \hat{F},\ i = 1, \cdots, n.$$

Therefore
$$x' = \sum_{i=1}^{n-1} \hat{g}_i x_i + \{\hat{g}_n g_j\} g_j u_j$$

as reqired. Thus we assume that $x_1, x_2, \cdots, x_n$ all belong to $\hat{S}$ and $x' = g_j u_j$. Then $x_1, \cdots, x_n, u_j$ belong to $\hat{S}$. From the F-linearly dependence of $x_1, \cdots, x_n, x'$, it follows that $x_1, \cdots, x_n, u_j$ are also F-linearly dependent, hence we obtain again

$$u_j=\sum_{i=1}^{n}\hat{g}_i x_i,\quad \hat{g}_i\in\hat{F},\qquad x'=g_j u_j=\sum_{i=1}^{n} g_j g_i x_i,\quad g_j\hat{g}_i\in\hat{F}.$$

This proves $\hat{S}'\in M$ and $\hat{S}\subsetneqq\hat{S}'$. But it contradicts the maximal property of $\hat{S}$, hence it must be $g_j u_j=uE_j\in\hat{S}$.

Now we shall show (iii). By the Structure of $\hat{S}$ we know $fu=\sum_{i<\infty}\hat{f}_i u_i$, $\hat{f}_i\in\hat{F}$. Since $u\in\hat{S}$, hence $u=\sum\hat{g}_j u_j$, $\hat{g}_j\in\hat{F}$. Hence $fu=\sum f\hat{g}_j u_j=\sum\hat{f}_i u_i$. Insomuch as the property of F-spasce we see $fg_j=f_j$, Hence $f=f_j g_j^{-1}\in\hat{F}$.

Corollary. *If $u\in\hat{S}$, then $\hat{g}u\in\hat{S}$ for any non-zero element $\hat{g}\in\hat{F}$.*

Remark. *The conditions of F-space defined in* [1] *are too strong, because they demand every maximal element $\hat{S}$ to be identified with S.*

Definition 2.3. *A ring R is called a weak quasi-primitive ring if and only if R has a strictly cyclic and faithful module $\mathfrak{M}$ with the following conditions:* (i) *$\mathfrak{M}$ is an F-quasispace,* (ii) *if $u_{i_1},\cdots,u_{i_n}$ are a finite number of an F-basis of $\hat{S}$ and $r\in R$ is a given element such that $u_{i_1}r\neq 0$, $u_{i_j}r=0$, $j\neq 1$, then there exists an element $t\in R$ such that $u_{i_1}t\in\hat{S}$, $u_{i_j}t=0$.*

Example 1. Every M-Artinian ring must be a weak quasi-primitive ring.

Proof Let R be an M-Artinian ring. Then by theorem 1.3 $A_\lambda=Re_{\lambda\lambda}$ is a strictly cyclic and faithful left R-module and can be expressed as a direct sum $A_\lambda=\sum_{i=1}^{n}\oplus e_{i\lambda}K$, where $K=e_{\lambda\lambda}Re_{\lambda\lambda}$, $\{e_{ij}\}_{n\times n}$ are matrix units. We also know that R is the ring of K-endomorphisms of A_λ. Let S denote the set of all free elements of strictly cyclic R-module $A_\lambda=Re_{\lambda\lambda}$ and $M^*=\{S_\alpha\}$ the class of all sets containing basic free elements. Then it is clear that $M^*\neq\emptyset$. Since $S_0=\{e_{1\lambda},e_{2\lambda},\cdots,e_{n\lambda}\}$ is a set containing basic free elements, hence M^* must have an element $\hat{S}$ of local free elements. Hence A_λ is a K-quasispace, Since R is the ring of K-endomrphisms, hence all conditions of definition 2.3 are satisfied. Therefore R is a weak quasi-primitive ring.

Remark. *In general, M-Artinian ring is not quasi-primitive ring, but if K is a division ring, then every set of local free elements is the set of all free elements S. Hence the weak quasi-primitive rings are precisely the quasi-primitive rings, in this case the usual semi-simple Artinian rings are a good example.*

Example 2. Let $\mathfrak{M}=\sum\oplus Fu_i$ be a (left) vector space over a division ring F, Ω the complete ring of linear transformations of $\mathfrak{M}$. We divide the basis $\{u_i\}_\Gamma$ into disjoint subsets with the same cardinal number, namely $\{u_i\}_\Gamma=\bigcup_{\alpha\in\Gamma'}\{u_j\}_{I_\alpha}$, and $\{u_i\}_{I_\alpha}\cap\{u_j\}_{I_\beta}=\emptyset$, $\alpha\neq\beta$, Card. $I_\alpha=$Card. I_β, $\alpha,\beta\in\Gamma'$. Therefore $\mathfrak{M}=\sum_{\alpha\in\Gamma'}\oplus\sum_{i\in I_\alpha}\oplus Fu_i$. $\{l_\alpha\}_{\alpha\in\Gamma'}$ denote the set with $u_i l_i=u_i$, for $i\in I_\alpha$; $u_j l_\alpha=0$, for $j\notin I_\alpha$. Then it is clear that $l_\alpha l_\beta=l_\alpha\delta_{\alpha\beta}$ where $\delta_{\alpha\beta}=0$, $\alpha\neq\beta$; $\delta_{\alpha\beta}=1$, $\alpha=\beta$. Let $R=\sum_{\alpha\in\Gamma'}\oplus\Omega l_\alpha$, then R is a ring and we can choose an arbitrary element l of $\{l_\alpha\}_{\Gamma'}$. If we set $A=lR$, then it can be shown

$$A=\sum_{\alpha\in\Gamma'}\oplus K\beta_\alpha \tag{2.1}$$

where $K = lRl = l\Omega l$, $\beta_a \in A$. In fact, by the structure of $\{l_a\}_{\Gamma'}$ we see

$$\mathfrak{M} = \sum_I \oplus Fu_i \oplus N(l) = \sum_{I_a} \oplus Fu_i \oplus N(l_a), \tag{2.2}$$

where Card. I = Card. I_a, $N(l) = \{m \in \mathfrak{M} \mid ml = 0\}$. Put $u_{i'} = v_i$ for $i' \in I_a$, $i \in I$, we have $\mathfrak{M} = \sum_I \oplus Fv_i \oplus N(l_a) = \sum_I \oplus Fu_i \oplus N(l)$. Hence there exist elements σ_a, $\sigma_0 \in \Omega$ such that $u_i\sigma_a = v_i$, $v_i\sigma_0 = u_i$, $i \in I$. It is easily seen that $u_il\sigma_al_a\sigma_0l = u_i$, $v_il_a\sigma_0l\sigma_al_a = v_i$, $i \in I$. Hence

$$l\sigma_al_al_a\sigma_0l = l, \quad l_a\sigma_0ll\sigma_al_a = l_a. \tag{2.3}$$

From the above it follows $\Omega l_a = \Omega l\sigma_al_a$, hence $Rl_a = Rl\sigma_al_a$. Put $\beta_a = l\sigma_al_a$, we have $A = \sum K\Omega l_a = \sum lKl\Omega l_a = \sum KRl_a = \sum K\beta_a = \sum \oplus K\beta_a$. This proves (2.1). On the other hand, if we set $l = l_a$, then $\hat{k}_a = l\sigma_al$ is an inverse element of $\hat{k}_0 = l\sigma_0l$ in K. If $\hat{K}$ denotes the set of all invertible elements of K, then $\hat{K}$ is a multiplitical group. As example 1 we can show that there exists a set $\hat{S}$ of local free elements such that $A = \sum \oplus K\beta_a$ is a weak K-space. Let $\widetilde{\Omega}$ be the ring of K-endomorphisms of A, then any subring $\widetilde{R}$ of $\widetilde{\Omega}$ is a weak quasi-primitive ring as required, only if $\widetilde{R}$ is finite fold transitive. We can now prove that R itself is also a weak quasi-primitive ring. In fact, if we set $\mathscr{A} = l\Omega = l\Omega l\Omega = \sum K\tau_j$, then Ω must be the complete ring of endomorphisms over $K = l\Omega l$ of $\mathscr{A}$ as proved by [2]. But $A = lR \subset l\Omega = \mathscr{A}$, hence if $x_1, \cdots, x_n$ are K-linearly independent elements of A, then these are also K-linearly independent elements of $\mathscr{A}$. Therefore there exists an element $\omega \in \Omega$ such that $x_1\omega \in \hat{S}$, $x_i\omega = 0$ for $i = 2, \cdots, n$, hence $x_1\omega = \sum_{i=1}^{m} \hat{k}_i\beta_i$. Put $\varepsilon = l_1 + \cdots + l_m$, we have $x_1\omega\varepsilon = x_1\omega \in \hat{S}$ and $x_i\omega\varepsilon = 0$. But $\omega\varepsilon \in R = \sum \Omega l_a$. Hence R is finite fold transitive in space A. This has proved that R is a weak quasi-primitive ring.

From the above discussion it is easy to see that the weak quasi-primitive ring can imply the same results as the quasi-primitive ring does, provided we use $\hat{S}$ intead of S.

References

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拟本原环(III)

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摘要

本文引进了M-阿丁环的概念,它扩充了通常阿丁环的概念。同时我们还给出了M-阿丁环的结构. 此外,本文还扩充了拟本原环概念,使它能包含更广泛的环类,这就是我们在本文中所定义的弱拟本原环. 例如M-阿丁环必是拟本原环,但它一定是弱拟本原环.

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LIE GROUPS AND SOME EVOLUTION EQUATIONS

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1. In 1978, in a lecture delivered in Peking, Prof. Chern, S. S. pointed out that some important evolution equations, such as KDV equation, Sine-Gordon equation etc., might be considered as the structure equations of the 2-unimodular group $SL(2, R)$. Chern and Peng illustrated it further in the paper [1]. The purpose of our paper is to establish the relation between some evolution equations of higher order and the multidimensional scattering problems and the general linear groups $GL(n, C)$, and give them a geometric interpretation.

2. We assume that $GL(n)$ is the real or complex general linear group, a is an element of $GL(n)$, $GL(n)_a$ is the tangent space at a, da is the tangent vector at a, then

$$\omega = da \cdot a^{-1}$$

is a map from the tangent bundle of $GL(n)$ to the tangent space at the unit element e, that is the Lie algebra of $GL(n)$, and is called the right invariant differential form. The structure equation is

$$d\omega = \frac{1}{2}[\omega, \omega].$$

The Lie algebra $gl(n)$ consists of all n-matrices with the commutator.

If $g_1(x, t)$ and $g_2(x, t)$ are any two elements of the $gl(n)$, such that $[g_1, g_2] \neq 0$, then

$$\bar{\omega}(x, t) = g_1(x, t)dx + g_2(x, t)dt$$

is the family of 2-dimensional subspaces in the $gl(n)$, (dx, dt) is the coordinates of the subspace. The question is that does there exist a 2-dimensional surface $a(x, t)$ in the $GL(n)$, its tangent plane at $a(x, t)$ is $\bar{\omega}(x, t) \cdot a(x, t)$ Evidently, this is to solve the equation $da = \bar{\omega} \cdot a$.

From the relative theory, a necessary and sufficient condition that the equation be solvable is that the $\bar{\omega}$ satisfies the structure equation

$$d\bar{\omega} = \frac{1}{2}[\bar{\omega}, \bar{\omega}],$$

that is

$$(g_{1t} - g_{2x})dx \wedge dt = -[g_1, g_2]dx \wedge dt,$$

i.e.

$$g_{2x} - g_{1t} = [g_1, g_2].$$

Indeed, this is the integrability condition of the equation $da = \bar{\omega} \cdot a$.

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3. Now we shall illustrate that the equations

$$g_{2x}-g_{1t}=[g_1,\ g_2]$$

are some evolution equations, if g_1 and g_2 are selected suitably.

i) If p is a constant

$$C=2u(x,\ t)+4p^2,$$

$$g_1=\begin{pmatrix} p & u(x,\ t) \\ -1 & -p \end{pmatrix},$$

$$g_2=\begin{pmatrix} -pC-\frac{1}{2}C_x & -pC_x-\frac{1}{2}C_{xx}-uC \\ C & pC+\frac{1}{2}C_x \end{pmatrix},$$

then $$g_{2x}-g_{1t}=\begin{pmatrix} -pC_x-\frac{1}{2}C_{xx} & -u_t-pC_{xx}-\frac{1}{2}C_{xxx}-u_xC-uC_x \\ C_x & pC_x+\frac{1}{2}C_{xx} \end{pmatrix},$$

$$[g_1,\ g_2]=\begin{pmatrix} -pC_x-\frac{1}{2}C_{xx} & -2p^2C_x-pC_x+uC_x \\ C & pC_x+\frac{1}{2}C_{xx} \end{pmatrix},$$

the structure equations are reduced to the KDV equation

$$u_t+6uu_x+u_{xxx}=0.$$

ii) If p is a constant

$$g_1=\begin{pmatrix} p & u \\ -u & -p \end{pmatrix},$$

$$g_2=\begin{pmatrix} -4p^3-2pu^2 & -u_{xx}-2pu_x-4p^2u-2u^3 \\ u_{xx}-2pu_x-4p^2u-2u^3 & 4p^3+2pu^2 \end{pmatrix},$$

then the structure equations are reduced to the MKDV equation

$$u_t+6u^2u_x+u_{xxx}=0.$$

iii) If p is a constant

$$g_1=\begin{pmatrix} p & \frac{1}{2}u_x \\ -\frac{1}{2}u_x & -p \end{pmatrix},$$

$$g_2=\begin{pmatrix} \frac{1}{4p}\cos u & \frac{1}{4p}\sin u \\ \frac{1}{4p}\sin u & -\frac{1}{4p}\cos u \end{pmatrix},$$

then the structure equations are reduced to the Sine-Gordon equation

$$u_{xt}=\sin u.$$

These were illustrated in Chern's lecture.

iv) If λ is a constant

$$g_1=\begin{pmatrix} -i\lambda & \phi \\ \psi & i\lambda \end{pmatrix}, \quad g_2=\begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$

then the structure equations are reduced to the general evolution equations which are considered by AKNS in paper [2]

$$\phi C-\psi B=A_x, \quad \phi_t-2A\phi=B_x+2i\lambda B, \quad \psi_t+2A\psi=C_x-2i\lambda C.$$

As the traces of g_1 and g_2 are both equal to zero, we can consider in the 2-unimodular group. These evolution equations may be considered as the integrability conditions of the equation $da=\bar{\omega}\cdot a$. Thus, there is a one to one correspondence between the solution of the equation and the integral manifolds of $da=\bar{\omega}\cdot a$ through the unit element e.

4. In paper [3], there is a discussion of higher order evolution equations and the multidimensional scattering problems. For these equations, the number of their independent variable and unknown function are increased. Now we establish the relation between these equations and the structure equations of the general linear group.

i) If p is a constant

$$g_1(x, t)=ip\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}+(N_{ij}),$$

$$g_2(x, t)=p\begin{pmatrix} q_1 & & & \\ & q_2 & & \\ & & \ddots & \\ & & & q_n \end{pmatrix}+(\bar{N}_{ij}),$$

where $d_i(i=1, 2, \cdots, n)$ are constant, and $d_i\neq d_j(i\neq j)$, $q_i(i=1, 2, \cdots, n)$ are independent of x, $\bar{N}_{ij}=a_{ij}N_{ij}$

$$a_{ij}=\begin{cases} \dfrac{q_i-q_j}{i(d_i-d_j)}, & i\neq j, \\ 0, & i=j, \end{cases}$$

then

$$g_{1x}=(a_{ij}N_{ij,x}), \quad g_{2t}=(N_{ij,t}),$$

$$[g_1, g_2]=[N, \bar{N}]=(\sum_k(a_{kj}-a_{ik})N_{ik}N_{kj}),$$

thus, these structure equations are reduced to evolution equations

$$N_{ij,t}=a_{ij}N_{ij,x}+\sum_k(a_{ik}-a_{kj})N_{ik}N_{kj}, \quad i, j=1, 2, \cdots, n.$$

Particularly, when $n=3$, $N_{ii}=0$, $N_{ij}=\sigma_{ij}\cdot N^*_{ij}$, that is

$$N=\begin{pmatrix} 0 & N_{12} & N_{13} \\ \sigma_{21}N^*_{12} & 0 & N_{23} \\ \sigma_{31}N^*_{13} & \sigma_{32}N^*_{23} & 0 \end{pmatrix}=\begin{pmatrix} 0 & A & A \\ \sigma_{21}A^* & 0 & A \\ \sigma_{31}A^* & \sigma_{32}A^* & 0 \end{pmatrix},$$

where $\sigma_{31}=-\sigma_{21}\cdot\sigma_{32}$, also let $a_{12}=v_1$, $a_{13}=v_2$, $a_{23}=v_3$, then we obtain three wave equations

$$A_{1t}=v_1A_{1x}+\sigma_{32}(v_2-v_3)A_2A_3^*,$$
$$A_{2t}=v_2A_{2x}+(v_1-v_3)A_1A_3,$$
$$A_{3t}=v_3A_{3x}+\sigma_{21}(v_1-v_2)A_1^*A_2.$$

ii) In order to discuss those equations which have independent variables x, y and t, we must consider the family of 3-dimensional planes in the $gl(n)$. We assume

$$g_1(x,\ y,\ t)=ipD+N+S,\quad g_2(x,\ y,\ t)=S,\quad g_3(x,\ y,\ t)=Q+CS,$$

where p is a constant, B, C, D, N, S and Q all are n-matrices, B_{ij}, C_{ij} and D_{ij} all are constants, and g_1, g_2, g_3 are not coplanar, then

$$\bar{\omega}(x,\ y,\ t)=g_1dx+g_2dy+g_3\,dt$$

is the family of 3-dimensional planes in $gl(n)$. In this case, the structure equation

$$d\bar{\omega}=\frac{1}{2}[\bar{\omega},\ \bar{\omega}]$$

is reduced to

$$g_{1t}-g_{3x}=[g_3,\ g_1],\quad g_{2x}-g_{1y}=[g_1,\ g_2],\quad g_{3y}-g_{2t}=[g_2,\ g_3],$$

i.e.

$$N_t+BS_t-Q_x-CS_x=[Q,\ N]+ip[Q,\ D]+[Q,\ BS]+ip[CS,\ D]+[CS,\ N]+[CS,\ BS], \qquad ①$$
$$S_x-N_y-BS_y=ip[D,\ S]+[N,\ S]+[BS,\ S], \qquad ②$$
$$Q_y-CS_y-S_t=[S,\ Q]+[S,\ CS]. \qquad ③$$

we make $C\times$② $+B\times$③, then

$$CS_x-BS_t+[B,\ C]S_y+BQ_y-CN_y=ipC[D,\ S]+C[N,\ S]+C[BS,\ S]+B[S,\ Q]+B[S,\ CS]$$

adding ① again, we obtain

$$-[B,\ C]S^2-[B,\ C]S_y+(ip[C,\ D]+[C,\ N]+[Q,\ B])S+[Q,\ N]+ip[Q,\ D]=N_t-Q_x+BQ_y-CN_y. \qquad ④$$

If

$$[B,\ C]=0, \qquad ⑤$$
$$ip[C,\ D]+[C,\ N]+[Q,\ B]=0, \qquad ⑥$$
$$ip[Q,\ D]+[Q,\ N]+Q_x-BQ_y+CN_y=N_t, \qquad ⑦$$

then equation ④ holds for any S. Therefore, if B, C, D, Q and N satisfy ⑤, ⑥, ⑦, and S satisfies ①, ② and ③, then it holds for the integrability condition of the equation $da=\bar{\omega}\cdot a$, that is the structure equation. So there is unique integral manifold through the unit e. In particular, if

$$B=(B_{ij})=(b_i\delta_{ij}),\quad C=(C_{ij})=(c_i\delta_{ij}),$$
$$D=(D_{ij})=(d_i\delta_{ij}),\quad N_{ii}=0,$$

where b_i, c_i and d_i all are constants, then we have $[B,\ C]=0$, i.e., ⑤ is identity, but ⑥ is reduced to