欧美初等数学经典系列（第一辑）

尘封的经典

——初等数学经典文献选读（第一卷）

刘培杰数学工作室 编

- 拉格雷老成果的新运用
- 平面对称群的识别与标记
- 奇特的幂次和
- n次幂差分的欧拉公式
- 单连通平面域的剖分
- 卡特兰数的初步估值

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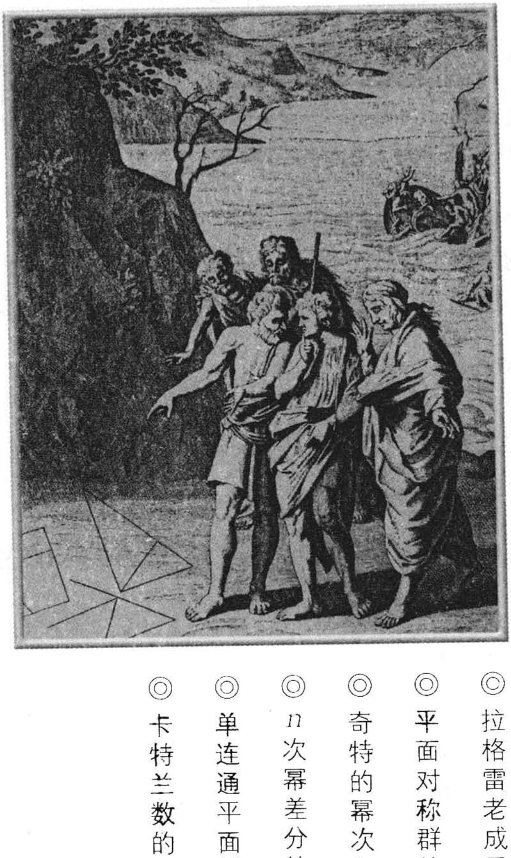

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内容提要

本书搜集初等数学的经典文献,包括"拉格雷旧成果的新运用""平面对成群的识别与标记""匈牙利的数学发展""Bonnesen 等周不等式""准割圆多项式""n 次幂差分的欧拉公式""算数级数""三角不等式""调和级数的一些收敛子级数"等在内,编辑成书,便于读者进行学习和查阅,本书适用于学生学习同时也可作为数学爱好者的兴趣读物。

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Contents

目　录

RECENT APPLICATIONS OF SOME OLD WORK OF LAGUERRE

EMIL GROSSWALD①

1. Introduction. Edmond Laguerre (1834 – 1886) is rightly considered as one of the foremost mathematicians of his time. He was a forerunner of Hadamard in the study of entire functions; the "Laguerre polynomials" are an important tool in several branches of pure and of applied mathematics, and Laguerre is also often quoted for his contributions to geometry ("theory of cycles"), algebraic equations, and continued fractions.

Nevertheless, he rates only four half-lines in the 1972 edition of the *Petit Larousse* [15], and his name is not even mentioned in such excellent surveys as

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① Emil Grosswald, a native of Romania, has a master's degree in mathematics and electrical engineering from Bucharest, a degree in electrical engineering from Paris, and a Ph. D. in mathematics (directed by H. Rademacher) from the University of Pennsylvania (1950). From 1939 to 1948 he lived successively in France, Cuba, and Puerto Rico. Except for a few years at the Institute for Advanced Study, the University of Paris, and the Technion, he has spent most of his professional life at the University of Pennsylvania and (since 1968) at Temple University. His main interests are in number theory and algebra. —*Editors*

[8], [18], [19], and [14]. To my surprise, not only does Laguerre not rate an entry, but he is not even mentioned under some other heading, in the *Encyclopaedia Britannica* (at least not in its 1954 edition) [5]. There is one brief mention of Laguerre in the four-volume *World of Mathematics* [12] in connection with the solution of a classical problem of tangent circles by the use of his "theory of cycles" and another brief mention in [2] in connection with three-dimensional analytical geometry.

Among Laguerre's numerous contributions to mathematics, is a theory of importance both to the theory of equations and to the study of the zeros of polynomials. It would be wrong to say that the theory is forgotten. In fact, G. Szegö [20] gives a sketchy proof of the main theorem, in view of some applications similar to those in the present paper; and a few years ago, Dočev used it to obtain an excellent bound for the absolute value of the zeros of Bessel Polynomials (see [4]). This theory, however, is not readily found in Laguerre's own papers. In order to obtain its more powerful results, the reader must combine several of Laguerre's papers (often very condensed notes published in the *Comptes Rendus* of the French Academy of Sciences), fortunately now collected in his two-volume *Oeuvres* [10]. Several very readable proofs of Laguerre's theorems can be found in the excellent monograph [11] by M. Marden. These are based mainly on considerations from mechanics (spherical and plane fields of forces, points of equilibrium, centers of mass, etc.). There exists also, however, a masterly presentation (quoted in [11]) of this material, together with some of its applications, in the unsurpassed work, *Aufgaben und Lehrsätze aus der Analysis* by Pólya and Szegö [16, vol. 2, Part 5, Chapter 2, Problems 101 - 120]. The reader who is willing to spend the time and effort needed to solve those 20 problems, and in this way to rediscover Laguerre's theory for himself under Pólya and Szegö's guidance, will profit greatly. His task has been made easier by the recent English translation [16a]. Such a reader is to be encouraged in his endeavor and need not continue to read the present article. Indeed, my purpose here is to make a leisurely, coherent presentation of Laguerre's theory, by following Pólya and Szegö's treatment to large extent, for the benefit of those who choose a less arduous way to become acquainted with this beautiful work. Several applications, some of them recent, follow the theoretical part.

2. The center of mass. Let us consider n complex numbers z_1, z_2, ..., z_n, represent them in the complex plane, and assume that at each of these points there is a unit mass. Then the "center of mass" of the n unit masses is given by the formula

$$\zeta = \frac{1}{n}(z_1 + z_2 + \dots + z_n)$$

It is an elementary exercise to verify that this definition has intrinsic meaning, i. e. , that the position of ζ with respect to the given points does not depend on the location of the origin and the orientation of the axes. Indeed, if each $z_j(j=1, 2, \ldots, n)$ is subject to the same translation, say $z_j \to z'_j = z_j + a$, then also $\zeta \to \zeta' = \zeta + a$. Similarly for a rotation, if $z_j \to z'_j = z_j e^{i\phi}$, then also $\zeta \to \zeta' = \zeta e^{i\phi}$. It also is clear that if $a \leqslant \operatorname{Re} z_j \leqslant b$, then also

$$a \leqslant \operatorname{Re} \zeta \leqslant b \qquad (1)$$

In fact, unless $\operatorname{Re} z_j = a$ or $\operatorname{Re} z_j = b$ for all z_j(which then would be collinear), one has strict inequalities in (1).

Let us now consider C, the smallest convex polygon that contains all the points z_j. By a rotation we may bring any side of C, say (see Fig. 1) $z_2 z_3$, into a vertical position, so that $\operatorname{Re} z_j \leqslant \operatorname{Re} z_2 = \operatorname{Re} z_3 = b$, say. Then, by (1), also $\operatorname{Re} \zeta \leqslant b$, with strict inequality, unless $\operatorname{Re} z_j = b$ for all $j = 1, 2, \ldots, n$. We may say that the line z_2, z_3 determines two half-planes, of which one does not contain any points z_j; then ζ belongs to the other half-plane. The same reasoning holds of course, for all sides of C, so that we conclude that ζ itself belongs to the intersection of those half-planes, i. e. , to C. In fact, the strict inequalities in (1) show that ζ belongs to the interior of C, unless all z_j's are collinear (in which case no interior exists).

3. The generalized center of mass. We now

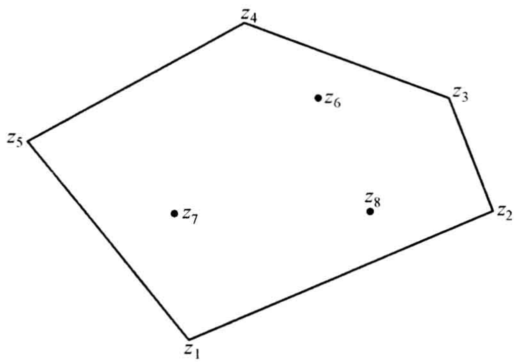

Fig. 1

proceed to generalize the concept of a "center of mass." The point (or complex number) ζ determined in section 2 will be said to be the "center of mass with respect to the point at infinity." When we want to stress this fact, we shall write ζ_∞. In other words, we define

$$\zeta = \zeta_\infty = \frac{1}{n}\sum_{j=1}^{n} z_j \qquad (2)$$

We now want to define a center of mass ζ of $z_1, z_2, \ldots, z_n$, with respect to an arbitrary point z_0 that we shall call (for want of a better name) the "pole". We do this by reducing the new problem to the old one. We first map the pole z_0 into the point at infinity by some appropriate linear fractional transformation, say

$$z \to z' = \frac{a}{z - z_0} + b \qquad (3)$$

Under (3), each z_j is mapped into

$$z'_j = \frac{a}{z_j - z_0} + b$$

while $z_0 \to z'_0 = \infty$. Next, we find the center of mass $\zeta' = \zeta'_\infty$ of the (z'_j)'s with the pole $z'_0 = \infty$ by (2), i. e.

$$\zeta' = \frac{1}{n}\sum_{j=1}^{n} z'_j$$

Finally, we map the whole configuration back, by the transformation inverse to (3). Under this inverse transformation, $z'_j = z_j$, $z'_0 = \infty \to z_0$ and ζ' is mapped into some point ζ_{z_0}, which we define as the center of mass of the points $z_j (j = 1, \ldots, n)$ with respect to the pole z_0.

While all the operations described are well defined, the question arises: does such a construction have any geometric meaning? Does ζ_{z_0} depend, as its name implies, only on the set $\{z_1, z_2, \ldots, z_n; z_0\}$? Indeed, we have used (3) in our construction, and (3) depends on the two arbitrary parameters a and b. Will not ζ_{z_0} also depend on our arbitrary choice of these parameters? Fortunately, ζ_{z_0} turns out to have an intrinsic meaning and is independent of the particular transformation (3) selected. Indeed, by (2) and (3)

$$\zeta' = \frac{1}{n}\sum_{j=1}^{n} z'_j = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{a}{z_j - z_0} + b\right) =$$

$$b + \frac{a}{n}\sum_{j=1}^{n}\frac{1}{z_j - z_0}$$

On the other hand ζ' is the image of ζ_{z_0} under (3), so that

$$\zeta' = \frac{a}{\zeta_{z_0} - z_0} + b$$

By equating the two expressions of ζ' we obtain

$$\frac{1}{\zeta_{z_0} - z_0} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z_j - z_0}$$

or, solving for ζ_{z_0}

$$\zeta_{z_0} = z_0 + n\left\{ \sum_{j=1}^{n} \frac{1}{z_j - z_0} \right\}^{-1} \tag{4}$$

This explicit formula shows that ζ_{z_0} is indeed independent of a and b, as claimed.

4. A separation theorem. We consider now some properties of ζ_{z_0}. We recall that in section 2 $\zeta = \zeta_\infty$ was a point of C. We may think of C as follows: Consider all pairs of points z_j, $z_k (1 \leqslant j, k \leqslant n)$ and the straight line determined by them. This divides the plane into two half-planes. Sometimes both half-planes contain some of the given points (e. g., for z_5, z_7 in Fig. 1), but sometimes only one contains points (as in our first example, $z_2 z_3$, or, say, z_1, z_5 in Fig. 1). In the latter case, delete the half-plane without points (for z_1, z_5 one deletes the "southwesterly" half-plane below the infinite straight line through z_1 and z_5). The intersection of all remaining half-planes is precisely C. Finally, we observe that the straight line through z_j, z_k may also be considered as a generalized circle through z_j, z_k and the point at infinity (the "pole" in that construction).

Exactly the same considerations apply to z'_1,

$z'_2, \ldots, z'_n$ with $\zeta' = \zeta'_\infty$, which lies inside the (ordinary) polygon C', the smallest convex polygon that contains all the (z'_j)'s, while $z'_0 = \infty$, the "pole" lies outside. When we map back, under the function inverse to (3), the straight lines that form the sides of C', say z'_j, z'_k, are transformed into circles through the pole z_0 (the image of $z'_0 = \infty$) and the points z_j, z_k. The image of C' is, therefore, a curvilinear polygon, that we may also construct directly as follows: we take the points $z_1, z_2, \ldots, z_n$ two by two and construct a circle through the couple z_j, z_k $(1 \leqslant j, k \leqslant n)$ and the pole z_0. This circle divides the complex plane into two circular domains (its "inside" and its "outside"). It may happen that one of these two (open) domains contains no other points $z_j (j = 1, \ldots, n)$. In this case we "delete" it mentally. The intersection of the remaining circular domains is a closed curvilinear polygon, say $C = C_{z_0}$, the inverse image of C' under (3). Then C_{z_0} divides the complex plane into two regions (Jordan curve theorem). In one of them we find z_0, the inverse image of $z'_0 = \infty$; in the other, ζ_{z_0}, the inverse image of $\zeta' = \zeta'_\infty$. Indeed, C' separates $z'_0 = \infty$ and $\zeta' = \zeta'_\infty$ and consequently its image C separates the corresponding images z_0 and ζ_{z_0}.

By dropping an unnecessary subscript, we may state the results obtained so far as a theorem.

THEOREM 1. *Let C_z be the curvilinear polygon defined above, corresponding to the points $z_1, z_2, \ldots,$*

z_n and to the pole z. Then C_z separates z from the center of mass ζ_z of $z_1, z_2, \ldots, z_n$ relative to z.

5. The Fundamental Theorem. Let $F(z) = \prod_{j=1}^{n}(z - z_j)$ be a polynomial and z an arbitrary point. Then the following theorem holds:

THEOREM 2. *The center of mass ζ_z of the zeros of $F(z)$, with respect to z is given by*

$$\zeta_z = z - nF(z)/F'(z) \tag{5}$$

Proof. Equation (5) follows immediately from (4) and

$$\frac{F'(z)}{F(z)} = \prod_{j=1}^{n} \frac{1}{z - z_j}$$

THEOREM 3. *If the zeros $z_j (j = 1, 2, \ldots, n)$ of $F(z)$ belong to any circular domain D (i. e., either all are outside or all are inside some circle Γ) and if z is outside D, then C_z is in D.*

Proof. The smallest convex polygon C' containing all the (z'_j)'s is inside any circle Γ' that contains all the (z'_j)'s. Indeed, Γ' is convex and contains all (z'_j)'s, while C' is the *smallest* convex set that contains all (z'_j)'s. We also observe that $z' = \infty$ is outside Γ'. In particular, given the circle Γ that contains all or none of the z_j's, let Γ' be the image of Γ under (3). Then Γ' separates $z' = \infty$ from C'. Hence, when we map back, Γ separates C_z (the image of C') from z (the image of $z' = \infty$). Consequently, if z is not in the circular domain D, then C_z is in D and the proof is complete.

THEOREM 4. (Laguerre [10, vol. 1, pp. 161 - 166]). *Let x be a simple zero of the polynomial $F(z)$ of degree n. The center of mass of the remaining zeros with respect to x is*

$$X = X(x) = x - 2(n-1)F'(x)/F''(x) \qquad (6)$$

Proof. Let $F(z) = (z - x)f(z)$; then $F''(z) = 2f'(z) + (z-x)f''(z)$ and $F'(z) = f(z) + (z-x)f'(z)$ so that $F'(x) = f(x)$ and $F''(x) = 2f'(x)$. We now apply Theorem 2 to the polynomial $f(z)$ of degree $n-1$ and obtain (6), thus proving Theorem 4.

Now let Γ_1 be a circle through x that contains all the other zeros of $F(z)$. By assumption x is a simple zero; hence, one can deform Γ_1 slightly into a circle Γ that leaves x outside, but whose interior D still contains all other zeros. Let C_x be the curvilinear polygon of these other zeros of $F(z)$, with respect to x as a pole. By Theorem 3 we know that $C_x \subset D$, because x is outside D. Also, by Theorem 1, C_x separates x from $X(x)$; hence, $X(x)$ is inside C_x and, a fortiori, in D. This finishes the proof of the following fundamental theorem of Laguerre:

THEOREM 5. (Laguerre [10, vol. 1, pp. 161 - 166]). *Let $f(z)$ be a polynomial of degree n and define the function*

$$X(z) = z - 2(n-1)f'(z)/f''(z) \qquad (7)$$

Let z_1 be a simple zero of $f(z)$ and consider a circle C (possibly a straight line) through the point z_1 of the