

Springer Finance Textbook

**Monique Jeanblanc**  
**Marc Yor**  
**Marc Chesney**

# **Mathematical Methods for Financial Markets**

**金融市场用的数学方法**



**Springer**

**世界图书出版公司**  
[www.wpcbj.com.cn](http://www.wpcbj.com.cn)

Monique Jeanblanc • Marc Yor • Marc Chesney

---

# Mathematical Methods for Financial Markets

## 图书在版编目 (CIP) 数据

金融市场用的数学方法 = Mathematical methods for financial markets: 英文/(法) 詹布兰科 (Jeanblanc, J.) 著. —影印本. —北京: 世界图书出版公司北京公司, 2013. 3

ISBN 978 - 7 - 5100 - 5843 - 1

I. ①金… II. ①詹… III. ①金融—经济数学—英文 IV. ①0177

中国版本图书馆 CIP 数据核字 (2013) 第 035275 号

---

书 名: Mathematical Methods for Financial Markets  
作 者: Monique Jeanblanc, Marc Yor, Marc Chesney  
中 译 名: 金融市场用的数学方法  
责任编辑: 高蓉 刘慧

---

出 版 者: 世界图书出版公司北京公司  
印 刷 者: 三河市国英印务有限公司  
发 行 者: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)  
联系电话: 010 - 64021602, 010 - 64015659  
电子信箱: kjb@wpcbj.com.cn

---

开 本: 24 开  
印 张: 32  
版 次: 2013 年 3 月  
版权登记: 图字: 01 - 2012 - 8527

---

书 号: 978 - 7 - 5100 - 5843 - 1 定 价: 99.00 元

---

## Editorial Board

*M. Avellaneda*

*G. Barone-Adesi*

*M. Broadie*

*M.H.A. Davis*

*E. Derman*

*C. Klüppelberg*

*W. Schachermayer*

---

## Preface

We translate to the domain of mathematical finance what F. Knight wrote, in substance, in the preface of his *Essentials of Brownian Motion and Diffusion* (1981): “it takes some temerity for the prospective author to embark on yet another discussion of the concepts and main applications of mathematical finance”. Yet, this is what we have tried to do in our own way, after considerable hesitation.

Indeed, we have attempted to fill the gap that exists in this domain between, on the one hand, mathematically oriented presentations which demand quite a bit of sophistication in, say, functional analysis, and are thus difficult for practitioners, and on the other hand, mainstream mathematical finance books which may be hard for mathematicians just entering into mathematical finance.

This has led us, quite naturally, to look for some compromise, which in the main consists of the gradual introduction, at the same time, of a financial concept, together with the relevant mathematical tools.

**Interlacing:** This program interlaces, on the one hand, the financial concepts, such as arbitrage opportunities, admissible strategies, contingent claims, option pricing, default risk and ruin problems, and on the other hand, Brownian motion, diffusion processes, Lévy processes, together with the basic properties of these processes. We have chosen to discuss essentially continuous-time processes, which in some sense correspond to the real-time efficiency of the markets, although it would also be interesting to study discrete-time models. We have not done so, and we refer the reader to some relevant bibliography in the Appendix at the end of this book. Another feature of our book is that in the first half we concentrate on continuous-path processes, whereas the second half deals with discontinuous processes.

**Special features of the book:** Intending that this book should be readable for both mathematicians and practitioners, we were led to a somewhat unusual organisation, in particular:

1. in a number of cases, when the discussion becomes too technical, in the Mathematics or the Finance direction, we give only the essence of the argument, and send the reader to the relevant references,
2. we sometimes wanted a given section, or paragraph, to contain most of the information available on the topic treated there. This led us to:
  - a) some forward references to topics discussed further in the book, which we indicate throughout the book with an arrow ( $\rightarrow$ )
  - b) some repetition or at least duplication of the same kind of topic in various degrees of generality. Let us give an important example: Itô's formula is presented successively for continuous path semi-martingales, Poisson processes, general semi-martingales, mixed processes and Lévy processes.

We understand that this way of writing breaks away with the academic tradition of book writing, but it may be more convenient to access an important result or method in a given context or model.

**About the contents:** At this point of the Preface, the reader may expect to find a detailed description of each chapter. In fact, such a description is found at the beginning of each chapter, and for the moment we simply refer the reader to the Contents and the user's guide, which follows the Contents.

**Numbering:** In the following, C,S,B,R are integers. The book consists of two parts, eleven chapters and two appendices. Each chapter C is divided into sections C.S., which in turn are divided into subsections C.S.B. A statement in Subsection C.S.B. is numbered as C.S.B.R. Although this system of numbering is a little heavy, it is the only way we could find of avoiding confusion between the numbering of statements and unrelated sections.

**What is missing in this book?** Besides discussing the content of this book, let us also indicate important topics that are not considered here: The term structure of interest rate (in particular Heath-Jarrow-Morton and Brace-Gatarek-Musiela models for zero-coupon bonds), optimization of wealth, transaction costs, control theory and optimal stopping, simulation and calibration, discrete time models (ARCH, GARCH), fractional Brownian motion, Malliavin Calculus, and so on.

**History of mathematical finance:** More than 100 years after the thesis of Bachelier [39, 41], mathematical finance has acquired a history that is only slightly evoked in our book, but by now many historical accounts and surveys are available. We recommend, among others, the book devoted to Bachelier by Courtault and Kabanov [199], the book of Bouleau [114] and

the collective book [870], together with introductory papers of Broadie and Detemple [129], Davis [221], Embrechts [321], Girlich [392], Gobet [395, 396], Jarrow and Protter [480], Samuelson [758], Taqqu [819] and Rogers [738], as well as the seminal papers of Black and Scholes [105], Harrison and Kreps [421] and Harrison and Pliska [422, 423]. It is also interesting to read the talks given by the Nobel prize winners Merton [644] and Scholes [764] at the Royal Academy of Sciences in Stockholm.

**A philosophical point:** Mathematical finance raises a number of problems in probability theory. Some of the questions are deeply rooted in the developments of stochastic processes (let us mention Bachelier once again), while some other questions are new and necessitate the use of sophisticated probabilistic analysis, e.g., martingales, stochastic calculus, etc. These questions may also appear in apparently completely different fields, e.g., Bessel processes are at the core of the very recent Stochastic Loewner Evolutions (SLE) processes. We feel that, ultimately, mathematical finance contributes to the foundations of the stochastic world.

**Any relation with the present financial crisis (2007-?)?** The writing of this book began in February 2001, at a time when probabilists who had engaged in Mathematical Finance kept developing central topics, such as the no-arbitrage theory, resting implicitly on the “good health of the market”, i.e.: its “natural” tendency towards efficiency. Nowadays, “the market” is in quite “bad health” as it suffers badly from illiquidity, lack of confidence, misappreciation of risks, to name a few points. Revisiting previous axioms in such a changed situation is a huge task, which undoubtedly shall be addressed in the future. However, for obvious reasons, our book does not deal with these new and essential questions.

**Acknowledgements:** We warmly thank Yann Le Cam, Olivier Le Courtois, Pierre Patie, Marek Rutkowski, Paavo Salminen and Michael Suchanecki, who carefully read different versions of this work and sent us many references and comments, and Vincent Torri for his advice on Tex language. We thank Ch. Bayer, B. Bergeron, B. Dengler, B. Forster, D. Florens, A. Hula, M. Keller-Ressel, Y. Miyahara, A. Nikeghbali, A. Royal, B. Rudloff, M. Siopacha, Th. Steiner and R. Warnung for their helpful suggestions. We also acknowledge help from Robert Elliott for his accurate remarks and his checking of the English throughout our text. All simulations were done by Yann Le Cam. Special thanks to John Preater and Hermann Makler from the Springer staff, who did a careful check of the language and spelling in the last version, and to Donatas Akmanavičius for editing work.

Drinking “sok z czarnych porzeczek” (thanks Marek!) was important while Monique was working on a first version. Marc Chesney greatly acknowledges support by both the University Research Priority Program “Finance and Financial Markets” and the National Center of Competence in Research

FINRISK. They are research instruments, respectively of the University of Zurich and of the Swiss National Science Foundation. He would also like to acknowledge the kind support received during the initial stages of this book project from group HEC (Paris), where he was a faculty member at the time.

All remaining errors are our sole responsibility. We would appreciate comments, suggestions and corrections from readers who may send e-mails to the corresponding author Monique Jeanblanc at [monique.jeanblanc@univ-evry.fr](mailto:monique.jeanblanc@univ-evry.fr).



# Springer Finance

*Springer Finance* is a programme of books addressing students, academics and practitioners working on increasingly technical approaches to the analysis of financial markets. It aims to cover a variety of topics, not only mathematical finance but foreign exchanges, term structure, risk management, portfolio theory, equity derivatives, and financial economics.

- Ammann M.*, Credit Risk Valuation: Methods, Models, and Application (2001)  
*Back K.*, A Course in Derivative Securities: Introduction to Theory and Computation (2005)  
*Barucci E.*, Financial Markets Theory. Equilibrium, Efficiency and Information (2003)  
*Bielecki T.R. and Rutkowski M.*, Credit Risk: Modeling, Valuation and Hedging (2002)  
*Bingham N.H. and Kiesel R.*, Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives (1998, 2nd ed. 2004)  
*Brigo D. and Mercurio F.*, Interest Rate Models: Theory and Practice (2001, 2nd ed. 2006)  
*Buff R.*, Uncertain Volatility Models – Theory and Application (2002)  
*Carmona R.A. and Tehranchi M.R.*, Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective (2006)  
*Dana R.-A. and Jeanblanc M.*, Financial Markets in Continuous Time (2003)  
*Deboeck G. and Kohonen T. (Editors)*, Visual Explorations in Finance with Self-Organizing Maps (1998)  
*Delbaen F. and Schachermayer W.*, The Mathematics of Arbitrage (2005)  
*Elliott R.J. and Kopp P.E.*, Mathematics of Financial Markets (1999, 2nd ed. 2005)  
*Fengler M.R.*, Semiparametric Modeling of Implied Volatility (2005)  
*Filipović D.*, Term-Structure Models (2009)  
*Fusai G. and Roncoroni A.*, Implementing Models in Quantitative Finance: Methods and Cases (2008)  
*Jeanblanc M., Yor M. and Chesney M.*, Mathematical Methods for Financial Markets (2009)  
*Geman H., Madan D., Pliska S.R. and Vorst T. (Editors)*, Mathematical Finance – Bachelier Congress 2000 (2001)  
*Gundlach M. and Lehrbass F. (Editors)*, CreditRisk<sup>+</sup> in the Banking Industry (2004)  
*Jondeau E.*, Financial Modeling Under Non-Gaussian Distributions (2007)  
*Kabanov Y.A. and Safarian M.*, Markets with Transaction Costs (2008 forthcoming)  
*Kellerhals B.P.*, Asset Pricing (2004)  
*Külpmann M.*, Irrational Exuberance Reconsidered (2004)  
*Kwok Y.-K.*, Mathematical Models of Financial Derivatives (1998, 2nd ed. 2008)  
*Malliavin P. and Thalmaier A.*, Stochastic Calculus of Variations in Mathematical Finance (2005)  
*Meucci A.*, Risk and Asset Allocation (2005, corr. 2nd printing 2007)  
*Pelsser A.*, Efficient Methods for Valuing Interest Rate Derivatives (2000)  
*Prigent J.-L.*, Weak Convergence of Financial Markets (2003)  
*Schmid B.*, Credit Risk Pricing Models (2004)  
*Shreve S.E.*, Stochastic Calculus for Finance I (2004)  
*Shreve S.E.*, Stochastic Calculus for Finance II (2004)  
*Yor M.*, Exponential Functionals of Brownian Motion and Related Processes (2001)  
*Zagst R.*, Interest-Rate Management (2002)  
*Zhu Y.-L., Wu X., Chern I.-L.*, Derivative Securities and Difference Methods (2004)  
*Ziegler A.*, Incomplete Information and Heterogeneous Beliefs in Continuous-time Finance (2003)  
*Ziegler A.*, A Game Theory Analysis of Options (2004)

---

## User's Guide

This book consists of two parts: the first part concerns continuous-path processes, and the second part concerns jump processes.

### Part I:

Chapter 1 introduces the main results for continuous-path processes and presents many examples, including in particular Brownian motion.

Chapter 2 presents the main tools in finance: self-financing portfolios, valuation of contingent claims, hedging strategies.

Chapter 3 contains some useful information about hitting times and their laws. Closed form expressions are given in the case of (geometric) Brownian motion.

Chapter 4 discusses finer properties of Brownian motion, e.g., local times, bridges, excursions and meanders.

Chapter 5 is devoted mainly to the presentation of one-dimensional diffusions, thus extending the scope of Chapter 4. Filtration problems are also studied.

Chapter 6 focuses on Bessel processes and applications to finance.

### Part II:

Chapter 7 is concerned with models of default risk, which involve stochastic processes with a single jump.

Chapter 8 introduces Poisson and compound Poisson processes, which are standard examples of jump processes.

Chapter 9 contains general theory of semi-martingales and aims at unifying results obtained in Chapters 1 and 8.

Chapter 10 presents some jump-diffusion processes and their applications to Finance.

Chapter 11 gives basic results about Lévy processes.

Chapter 12 consists of a list of useful formulae found throughout this book.

## Notation and Abbreviations

We shall use the standard notation and abbreviations.

We shall use **increasing** instead of nondecreasing and **positive** instead of non-negative.

u.i.	: uniformly integrable (for a family of r.v.'s)
BM	: Brownian motion
r.v.	: random variable
e.m.m.	: equivalent martingale measure
a.s.	: almost surely
w.r.t.	: with respect to
w.l.g.	: without loss of generality
SDE	: Stochastic Differential Equation
BSDE	: Backward Stochastic Differential Equation
PRP	: Predictable Representation Property
MCT	: Monotone Class Theorem
$x \vee y$	$= \sup(x, y)$
$x \wedge y$	$= \inf(x, y)$
$x \cdot y$	: scalar product of the vectors $x, y \in \mathbb{R}^d$
$H \star X$	: stochastic integral of the process $H$ with respect to the semi-martingale $X$
$x^+$	$= x \vee 0$
$x^-$	$= (-x) \vee 0$
$\partial_x f$	$= \frac{\partial}{\partial x} f = f_x$
$C_b^n$	: set of functions with continuous bounded derivatives up to $n$ -th order
$\mu(t), \mu_t$	: A function (or a process) evaluated at time $t$ . If $\mu$ is a deterministic function, $\mu(t)$ is preferably used; if $\mu$ is a process, when the subscript is not too large, $\mu_t$ is preferred
$\int_a^b ds f(s)$	$= \int_a^b f(s) ds$ when it seems convenient
$X \stackrel{\text{law}}{=} Y$	: the random variables (or the processes) $X$ and $Y$ have the same law
$X \stackrel{\text{mart}}{=} Y$	: the process $X - Y$ is a local martingale
$X \in \mathcal{F}$	: $X$ is a $\mathcal{F}$ -measurable r.v., i.e., $X \in L^0(\mathcal{F})$
$X \in b\mathcal{F}$	: $X$ is a bounded $\mathcal{F}$ -measurable random variable
$\mathcal{N}(x)$	$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$ , the cumulative function for a standard Gaussian law

Other notation can be found in the glossary at the end of the volume.

At the end of the book, the reader will find an extended bibliography, and a list of references, sorted by thema, followed by an index of authors, in which the page number where each author is quoted is specified.

In the text, some important words are in boldface. These words are also found in the subject index. Some notation can be found in the notation index.

To complete this guide, we emphasize some particular features of this book, already mentioned in the Preface:

- in some cases, proofs are sketched and/or omitted, but precise references are given;
- forward references to topics discussed further in the book are indicated with the arrow  $\mapsto$  ;
- we proceed by generalization: an important case/process is discussed, followed (a little later) by a general study.

Throughout this book, the symbol  $\square$  indicates the end of a proof, the symbol  $\triangleleft$  indicates the end of an exercise and the symbol  $\blacktriangleright$  is used to separate a long proof into different parts.

Section 2.1 refers to Chapter 2, Section 1, and Subsection 4.3.7 refers to Chapter 4, Section 3, Subsection 7. Theorem (Proposition, Lemma) 3.2.1.4 is the 4th in Chapter 3, Section 2, Subsection 1.

*Begin at the beginning, and go on till you come to the end. Then, stop.*

*Lewis Carroll, Alice's Adventures in Wonderland.*

Monique Jeanblanc  
Université d'Evry  
Dépt. Mathématiques  
rue du Père Jarlan  
91025 Evry CX  
France  
monique.jeanblanc@univ-evry.fr

Marc Chesney  
Universität Zürich  
Inst. Schweizerisches  
Bankwesen (ISB)  
Plattenstr. 14  
8032 Zürich  
Switzerland

Marc Yor  
Université Paris VI  
Labo. Probabilités et Modèles  
Aléatoires  
175 rue du Chevaleret  
75013 Paris  
France

ISBN 978-1-85233-376-8  
DOI 10.1007/978-1-84628-737-4

e-ISBN 978-1-84628-737-4

Springer Dordrecht Heidelberg London New York

British Library Cataloguing in Publication Data  
A catalogue record for this book is available from the British Library

Library of Congress Control Number: 2009936004

Mathematics Subject Classification (2000): 60-00; 60G51; 60H30; 91B28

© Springer-Verlag London Limited 2009

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms of licenses issued by the Copyright Licensing Agency. Enquiries concerning reproduction outside those terms should be sent to the publishers.

The use of registered names, trademarks, etc., in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant laws and regulations and therefore free for general use.

The publisher makes no representation, express or implied, with regard to the accuracy of the information contained in this book and cannot accept any legal responsibility or liability for any errors or omissions that may be made.

Reprint from English language edition:  
**Mathematical Methods for Financial Markets**  
by Monique Jeanblanc, Marc Yor, Marc Chesney  
Copyright © 2009, Springer London  
Springer London is a part of Springer Science+Business Media  
All Rights Reserved

This reprint has been authorized by Springer Science & Business Media for distribution in China Mainland only and not for export therefrom.

---

# Contents

---

## Part I Continuous Path Processes

---

<b>1</b>	<b>Continuous-Path Random Processes: Mathematical Prerequisites</b>	<b>3</b>
1.1	Some Definitions	3
1.1.1	Measurability	3
1.1.2	Monotone Class Theorem	4
1.1.3	Probability Measures	5
1.1.4	Filtration	5
1.1.5	Law of a Random Variable, Expectation	6
1.1.6	Independence	6
1.1.7	Equivalent Probabilities and Radon-Nikodým Densities	7
1.1.8	Construction of Simple Probability Spaces	8
1.1.9	Conditional Expectation	9
1.1.10	Stochastic Processes	10
1.1.11	Convergence	12
1.1.12	Laplace Transform	13
1.1.13	Gaussian Processes	15
1.1.14	Markov Processes	15
1.1.15	Uniform Integrability	18
1.2	Martingales	19
1.2.1	Definition and Main Properties	19
1.2.2	Spaces of Martingales	21
1.2.3	Stopping Times	21
1.2.4	Local Martingales	25
1.3	Continuous Semi-martingales	27
1.3.1	Brackets of Continuous Local Martingales	27
1.3.2	Brackets of Continuous Semi-martingales	29
1.4	Brownian Motion	30
1.4.1	One-dimensional Brownian Motion	30
1.4.2	$d$ -dimensional Brownian Motion	34

1.4.3	Correlated Brownian Motions .....	34
1.5	Stochastic Calculus .....	35
1.5.1	Stochastic Integration .....	36
1.5.2	Integration by Parts .....	38
1.5.3	Itô's Formula: The Fundamental Formula of Stochastic Calculus .....	38
1.5.4	Stochastic Differential Equations .....	43
1.5.5	Stochastic Differential Equations: The One-dimensional Case .....	47
1.5.6	Partial Differential Equations .....	51
1.5.7	Doléans-Dade Exponential .....	52
1.6	Predictable Representation Property .....	55
1.6.1	Brownian Motion Case .....	55
1.6.2	Towards a General Definition of the Predictable Representation Property .....	57
1.6.3	Dudley's Theorem .....	60
1.6.4	Backward Stochastic Differential Equations .....	61
1.7	Change of Probability and Girsanov's Theorem .....	66
1.7.1	Change of Probability .....	66
1.7.2	Decomposition of $\mathbb{P}$ -Martingales as $\mathbb{Q}$ -semi-martingales ..	68
1.7.3	Girsanov's Theorem: The One-dimensional Brownian Motion Case .....	69
1.7.4	Multidimensional Case .....	72
1.7.5	Absolute Continuity .....	73
1.7.6	Condition for Martingale Property of Exponential Local Martingales .....	74
1.7.7	Predictable Representation Property under a Change of Probability .....	77
1.7.8	An Example of Invariance of BM under Change of Measure .....	78
2	<b>Basic Concepts and Examples in Finance</b> .....	79
2.1	A Semi-martingale Framework .....	79
2.1.1	The Financial Market .....	80
2.1.2	Arbitrage Opportunities .....	83
2.1.3	Equivalent Martingale Measure .....	85
2.1.4	Admissible Strategies .....	85
2.1.5	Complete Market .....	87
2.2	A Diffusion Model .....	89
2.2.1	Absence of Arbitrage .....	90
2.2.2	Completeness of the Market .....	90
2.2.3	PDE Evaluation of Contingent Claims in a Complete Market .....	92
2.3	The Black and Scholes Model .....	93
2.3.1	The Model .....	94

2.3.2	European Call and Put Options	97
2.3.3	The Greeks	101
2.3.4	General Case	102
2.3.5	Dividend Paying Assets	102
2.3.6	Rôle of Information	104
2.4	Change of Numéraire	105
2.4.1	Change of Numéraire and Black-Scholes Formula	106
2.4.2	Self-financing Strategy and Change of Numéraire	107
2.4.3	Change of Numéraire and Change of Probability	108
2.4.4	Forward Measure	108
2.4.5	Self-financing Strategies: Constrained Strategies	109
2.5	Feynman-Kac	112
2.5.1	Feynman-Kac Formula	112
2.5.2	Occupation Time for a Brownian Motion	113
2.5.3	Occupation Time for a Drifted Brownian Motion	114
2.5.4	Cumulative Options	116
2.5.5	Quantiles	118
2.6	Ornstein-Uhlenbeck Processes and Related Processes	119
2.6.1	Definition and Properties	119
2.6.2	Zero-coupon Bond	123
2.6.3	Absolute Continuity Relationship for Generalized Vasicek Processes	124
2.6.4	Square of a Generalized Vasicek Process	127
2.6.5	Powers of $\delta$ -Dimensional Radial OU Processes, Alias CIR Processes	128
2.7	Valuation of European Options	129
2.7.1	The Garman and Kohlhagen Model for Currency Options	129
2.7.2	Evaluation of an Exchange Option	130
2.7.3	Quanto Options	132
3	<b>Hitting Times: A Mix of Mathematics and Finance</b>	135
3.1	Hitting Times and the Law of the Maximum for Brownian Motion	136
3.1.1	The Law of the Pair of Random Variables $(W_t, M_t)$	136
3.1.2	Hitting Times Process	138
3.1.3	Law of the Maximum of a Brownian Motion over $[0, t]$	139
3.1.4	Laws of Hitting Times	140
3.1.5	Law of the Infimum	142
3.1.6	Laplace Transforms of Hitting Times	143
3.2	Hitting Times for a Drifted Brownian Motion	145
3.2.1	Joint Laws of $(M^X, X)$ and $(m^X, X)$ at Time $t$	145
3.2.2	Laws of Maximum, Minimum, and Hitting Times	147
3.2.3	Laplace Transforms	148
3.2.4	Computation of $\mathbf{W}^{(\nu)}(\mathbb{1}_{\{T_\nu(X) < t\}} e^{-\lambda T_\nu(X)})$	149



3.2.5	Normal Inverse Gaussian Law .....	150
3.3	Hitting Times for Geometric Brownian Motion .....	151
3.3.1	Laws of the Pairs $(M_t^S, S_t)$ and $(m_t^S, S_t)$ .....	151
3.3.2	Laplace Transforms .....	152
3.3.3	Computation of $\mathbb{E}(e^{-\lambda T_a(S)} \mathbb{1}_{\{T_a(S) < t\}})$ .....	153
3.4	Hitting Times in Other Cases .....	153
3.4.1	Ornstein-Uhlenbeck Processes .....	153
3.4.2	Deterministic Volatility and Nonconstant Barrier .....	154
3.5	Hitting Time of a Two-sided Barrier for BM and GBM .....	156
3.5.1	Brownian Case .....	156
3.5.2	Drifted Brownian Motion .....	159
3.6	Barrier Options .....	160
3.6.1	Put-Call Symmetry .....	160
3.6.2	Binary Options and $\Delta$ 's .....	163
3.6.3	Barrier Options: General Characteristics .....	164
3.6.4	Valuation and Hedging of a Regular Down-and-In Call Option When the Underlying is a Martingale .....	166
3.6.5	Mathematical Results Deduced from the Previous Approach .....	169
3.6.6	Valuation and Hedging of Regular Down-and-In Call Options: The General Case .....	172
3.6.7	Valuation and Hedging of Reverse Barrier Options .....	175
3.6.8	The Emerging Calls Method .....	177
3.6.9	Closed Form Expressions .....	178
3.7	Lookback Options .....	179
3.7.1	Using Binary Options .....	179
3.7.2	Traditional Approach .....	180
3.8	Double-barrier Options .....	182
3.9	Other Options .....	183
3.9.1	Options Involving a Hitting Time .....	183
3.9.2	Boost Options .....	184
3.9.3	Exponential Down Barrier Option .....	186
3.10	A Structural Approach to Default Risk .....	188
3.10.1	Merton's Model .....	188
3.10.2	First Passage Time Models .....	190
3.11	American Options .....	191
3.11.1	American Stock Options .....	192
3.11.2	American Currency Options .....	193
3.11.3	Perpetual American Currency Options .....	195
3.12	Real Options .....	198
3.12.1	Optimal Entry with Stochastic Investment Costs .....	198
3.12.2	Optimal Entry in the Presence of Competition .....	201
3.12.3	Optimal Entry and Optimal Exit .....	204
3.12.4	Optimal Exit and Optimal Entry in the Presence of Competition .....	205