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Surveys of Modern Mathematics

Isolated Singular Points on Complete Intersections (Second Edition)

完全交上的孤立奇点（第二版）

E.J.N. Looijenga

高等教育出版社
HIGHER EDUCATION PRESS

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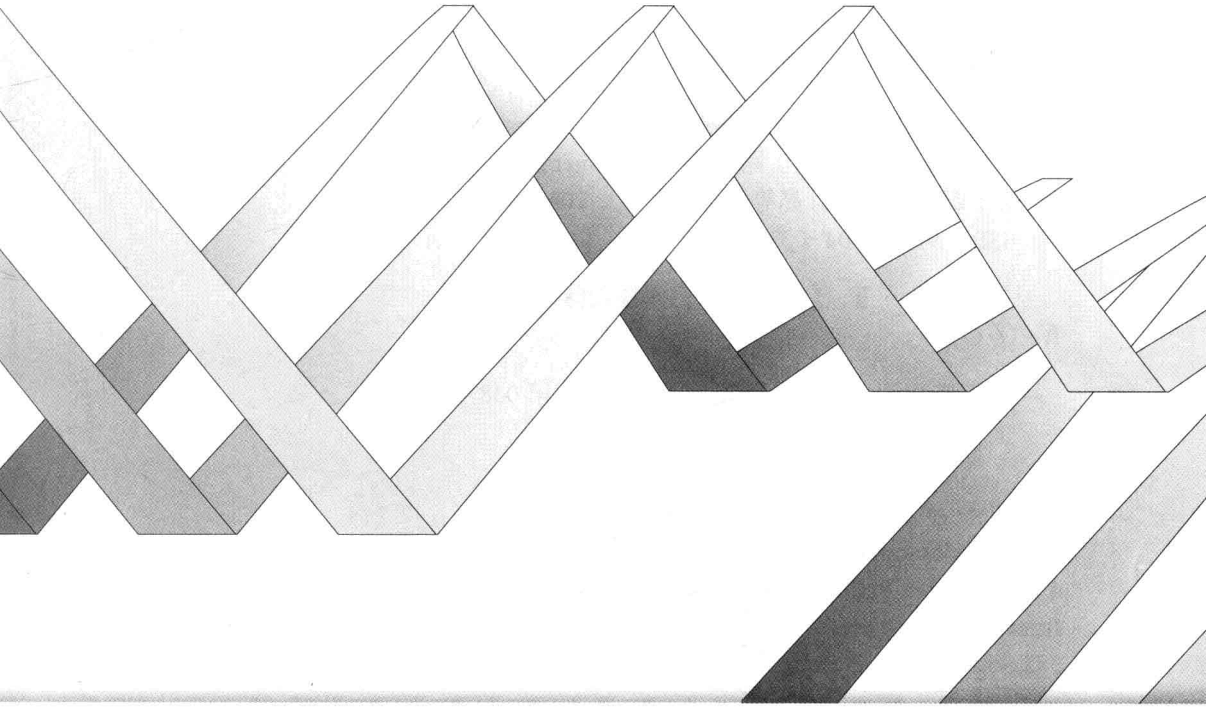

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WANQUAN JIAO SHANG DE GULI QIDIAN

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Surveys of Modern Mathematics

Surveys of Modern Mathematics

Mathematics has developed to a very high level and is still developing rapidly. An important feature of the modern mathematics is strong interaction between different areas of mathematics. It is both fruitful and beautiful. For further development in mathematics, it is crucial to educate students and younger generations of mathematicians about important theories and recent developments in mathematics. For this purpose, accessible books that instruct and inform the reader are crucial. This new book series "Surveys of Modern Mathematics" (SMM) is especially created with this purpose in mind. Books in SMM will consist of either lecture notes of introductory courses, collections of survey papers, expository monographs on well-known or developing topics.

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Series Editors

To Elisabeth

Preface to the Second Edition

Almost three decades have passed since this monograph saw the light of day and in the intervening years. Its subject matter has not only matured, but also spread out in new directions. Yet this book is still often used as a reference and so when Higher Education Press and International Press offered me to produce a TeX source file from the original type script that I could revise, I happily accepted. That immediately faced me with the question to what length I should go with bringing it up to date. A proper job would have meant writing another book, an idea I quickly discarded is not just for reasons of time, but also for lack of energy, if not competence. Going halfway did not appeal to me either, and so in the end I decided to stick to the contents of the first edition and confine myself essentially to matters of presentation. This occasionally led to simplifications of proofs, slightly stronger statements, rearrangement of the material, minor additions and, I hope, clearer exposition. There are also some—fortunately modest—changes of notation. I had to make some minor corrections as well and pray that the conversion of a typescript into an edited TeX file did not introduce a new set of errors.

Given the rather restricted purpose of the original list of references and the now common availability of forward search tools as provided by the online databases MathSciNet and ZMATH, I refrained from adding new items, except for some conference proceedings. Of course full reference data were supplied for papers that were announced as to appear in the original edition.

E. Looijenga
Utrecht, August 2012

Preface to the First Edition

In the spring term of 1980 I gave a course on singularities at Yale University (while supported by NSF grant MCS 7905018), which provided the basis of a set of notes prepared for the first two years of the Singularity Intercity Seminar (1980—1982, at Leiden, Nijmegen and Utrecht, jointly run with Dirk Siersma and Joseph Steenbrink). These notes developed into the present book. As a consequence, the aim and prerequisites of the seminar and this book are almost identical.

The purpose of the seminar was to introduce its participants to isolated singularities of complex spaces with particular emphasis on complete intersection singularities. When started we felt that no suitable account was available on which our seminar could be based, so it was decided that I should supply notes, to be used by both the lecturers (in preparing their talks) and the audience. This was quite a purifying process: many errors and inaccuracies of the first draft were thus detected (and often corrected).

The prerequisites consisted of some algebraic and analytic geometry (roughly covering the contents of the books of Mumford (1976) and Narasimhan (1966)), some algebraic topology (as in Spanier (1966) and Godement (1958)) and some facts concerning Stein spaces. Given this background, my goal was to prove every assertion in the text. This has been achieved except for the coherence theorem (8.7) and some assertions in the descriptive Chapter 1. An exception should also be made for the paragraphs marked with an asterisk (*): they generally give useful information which however is not indispensable for what follows and so may be skipped. Perhaps the whole first chapter could have been marked this way. It gives interesting examples of isolated singularities (or of constructions thereof) with the purpose to indicate the position of complete intersection singularities among them and to describe material to which the theory is going to apply. It is mainly for the latter reason that this chapter should not be skipped entirely.

As each chapter has its own introduction, I shall not review the chapters separately, nor the whole book. I believe that the first seven chapters (with the exception of Section 5.C) can be used as a basis for a course on the subject, assuming the audience has approximately the background mentioned above. The contents of Section 5.C and the last two chapters are somewhat more advanced and moreover Chapter 9 is of more specialized nature. Some of the results may be new (at least do not appear in this form in the literature), examples are the discussion (1.25), parts of Chapter 4 (Theorem (4.15) and Corollary (4.11)), the monodromy theorem Section 5.C, (6.17), the variation extension (7.17)–(7.19), and the Sections 9.A and 9.C. The references at the end should be regarded as a list of sources I consulted and not as a bibliography which pretends to be complete in any respect.

Although all the sources I used are cited, I want to single out some papers which were particularly useful for me: Lamotke (1975) for Chapter 3, Teissier (1976) for Chapter 4, Lê (1973, 1978) for Chapter 5 and Greuel (1975, 1980) for Chapters 8 and 9. As I mentioned, the book also benefitted from criticism of the lecturers in the seminar. I mention in particular, C. Cox, W. Janssen, P. Lemmens, P. Lorist, F. Menting, G. Pellikaan, J. Stevens, D. van Straten and E. van Wijngaarden. Also, comments from my co-organizers, Dirk Siersma and Jozef Steenbrink, were very helpful. I take the occasion to thank the Dutch Organization for the Advancement of Pure research (ZWO) for sponsoring the seminar and for supporting three of its participants. I am greatly indebted to W. Janssen for careful proofreading—his accurate job eliminated many errors and obscurities—and help with the exposition. I thank Ms. Ellen van Eldik for producing a beautiful camera-ready typescript. Finally, I express my thanks to my wife, Elisabeth, for her continuous support during the writing of this book.

E. Looijenga
Nijmegen, August 1983

A Few Notational Conventions

If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism between ringed spaces, then by definition we have a homomorphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X, an observation we only make in order to show our notation for a sheaf pull-back. Hence, if $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then $f^{-1}\mathcal{F}$ stands for its ordinary sheaf pull-back (so that it will be a sheaf of $f^{-1}\mathcal{O}_Y$-modules), but we shall reserve the notation $f^*\mathcal{F}$ for $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$. Notational logic dictates that we then denote $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on stalks likewise: $f_x^{-1} : \mathcal{O}_{Y,f(y)} \to \mathcal{O}_{X,x}$, but we dare not deviate too much from convention and write $f_x^*\phi \in \mathcal{O}_{X,x}$ for the image of $\phi \in \mathcal{O}_{Y,f(y)}$ under this ring homomorphism.

As a rule we use for natural pairings (such as between a vector space and its dual) what physicists call the bra-ket notation, $\langle \mid \rangle$, whereas a bilinear form is often denoted by a centered dot separating the variables: $(\cdot)$.

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Chapter 1

Examples of Isolated Singular Points

Loosely speaking, an analytic germ (X, x) in $\mathbb{C}^N$ is called a complete intersection if the minimal number of equations by which it can be defined equals its codimension in $\mathbb{C}^N$. Although such germs will be our principal object of study, we must realize that quite often analytic germs are not given to us as the common zero set of a specific set of equations. In such cases it is unreasonable to expect these germs to be complete intersections. We illustrate this by giving several constructions of singular germs, most of which fail to yield complete intersections in general. Some of the germs which happen to be complete intersections will reappear when we make a beginning of the classification in Chap. 7.

Another goal of this chapter is to make the reader acquainted with several interesting examples to which the theory we are going to develop may be applied. We do not always provide full proofs of the properties attributed to these singularities. The reader shouldn't feel uneasy about this, for in such cases we will not make any use of them.

1.A Hypersurface singularities

(1.1). Let X be an analytic set in an open $U \subset \mathbb{C}^{n+1}$ and let $x \in X$. The ideal $\mathcal{I}_{X,x}$ of holomorphic functions at x vanishing on X is principal and non-zero if and only if each irreducible component of the germ (X, x) is of dimension n (see for instance Whitney (1972), Chap. 2, Thm.'s 10 C,D). We then say that (X, x) is a *hypersurface germ.* If $f \in \mathfrak{m}_{U,x}$ generates $\mathcal{I}_{X,x}$, then the fact that $\mathcal{I}_X$ is a coherent $\mathcal{O}_U$-module implies that there is an open neighborhood U' of x in U such that f converges on U' and $\mathcal{I}_X|U' = f\mathcal{O}_{U'}$.

(1.2) Proposition. *In this situation, the following are equivalent:*

(i) *There exists a neighborhood of x in U which is non-singular except possibly in $\{x\}$.*

(ii) $\left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x} \supset \mathfrak{m}_{U,x}^k$ *for some k.*

(iii) $\left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x} \supset \mathfrak{m}_{U,x}^k$ *for some k.*

(iv) $\dim_{\mathbb{C}} \mathcal{O}_{U,x} / \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x} < \infty$.

(v) $\dim_{\mathbb{C}} \mathcal{O}_{U,x} / \left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x} < \infty$.

Proof. (i)⇒(ii). If y is a non-singular point of $X \cap U'$, then $\frac{\partial f}{\partial z_\nu}(y) \neq 0$ for some ν, for f generates $\mathcal{I}_{X,y}$. So the common zero set of $f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}$ is contained in the singular locus $C_{X \cap U'}$ of $X \cap U'$. By assumption, either $x \notin C_{X \cap U'}$ or x is an isolated point of $C_{X \cap U'}$. Hence by the local analytic Nullstellensatz, the radical of $\left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x}$ is either $\mathcal{O}_{U,x}$ or $\mathfrak{m}_{U,x}$ This clearly implies (ii).

(ii)⇒(iii). Let $Y \subset U'$ denote the common zero set of $\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}$. It will be enough to show that we have $(Y, x) \subset (X, x)$ (an inclusion of germs) because the Nullstellensatz will then imply that $\left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x}$ and $\left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x}$ have the same radical. If $(Y, x) \not\subset (X, x)$, then we can find a germ of an analytic curve $s : (\mathbb{C}, 0) \to (Y, x)$ with $s(t) \notin X$ for $t \neq 0$. But

$$\frac{d}{dt}(f \circ s) = \sum_{\nu=0}^{n} \left(\frac{\partial f}{\partial z_\nu} \circ s\right) \frac{ds_\nu}{dt} = 0$$

and so $f \circ s$ is constant equal to $f \circ s(0) = 0$. This contradicts our assumption that $s(t) \notin X$ for $t \neq 0$.

(iii)⇒(iv), (iv)⇒(v) and (ii)⇒(i) are easy, while (v)⇒(ii) will follow from Lemma (1.3) below applied to $M = \mathcal{O}_{U,x} / \left(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_0}\right) \mathcal{O}_{U,x}$. □

(1.3) Lemma. *If M is an $\mathcal{O}_{U,x}$-module of finite $\mathbb{C}$-dimension d, then $\mathfrak{m}_{U,x}^d$ annihilates M.*

Proof. Put $d_k := \dim_{\mathbb{C}} M / \mathfrak{m}_{U,x}^k M$, $k = 0, 1, 2, \ldots$ so that $0 = d_0 \leq d_1 \leq \cdots \leq d_k \leq \cdots \leq d$. Then for some $k \leq d$, we have $d_k = d_{k+1}$. This means that $\mathfrak{m}_{U,x}^k M = \mathfrak{m}_{U,x}^{k+1} M$. Since $\mathfrak{m}_{U,x}^k M \subset M$ is of finite $\mathbb{C}$-dimension, $\mathfrak{m}_{U,x}^k M$ is a Noetherian $\mathcal{O}_{u,x}$-module so that Nakayama's lemma applies: it follows that $\mathfrak{m}_{U,x}^k M = 0$. □

(1.4). If one of the (equivalent) conditions of Lemma (1.2) is satisfied we say that (X, x) is an *isolated hypersurface singularity.* The dimension occurring in Prop. ((1.2)-iv) is called the *Milnor number* of X at x and is denoted $\mu(X, x)$. Milnor (1968) originally defined this number in a topological manner but we will see in Chap. 5 that the two definitions agree. The dimension in Prop. ((1.2)-v) will be interpreted in Chap. 6 when we investigate the deformation theory of (X, x). We follow Greuel and call it the *Tjurina number* of (X, x), denoted $\tau(X, x)$. Clearly $\tau(X, x) \leq \mu(X, x)$ and we have equality if and only if $f \in \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x}$. This is for instance the case if f is *weighted homogeneous.* This means that there exist positive rational numbers $d_0, \ldots, d_n$ such that if we give z_ν weight d_ν, then f is a linear combination of monomials $z_0^{i_0} \cdots z_n^{i_n}$ of weight 1 (i.e., with $i_0 d_0 + \cdots + i_n d_n = 1$). Then it is easily checked that

$$f(z) = \sum_{\nu=0}^{n} d_\nu z_\nu \frac{\partial f}{\partial z_\nu}.$$

The condition that $f \in \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x}$ is coordinate invariant and hence also satisfied for an f which is weighted homogeneous with respect to some other system of local coordinates for $(\mathbb{C}^{n+1}, x)$. According to Saito (1971) there is a converse to this: if f defines an isolated hypersurface singularity and $f \in \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right) \mathcal{O}_{U,x}$ then f is weighted homogeneous with respect to some coordinate system.

1.B Complete intersections

(1.5) (Local complete intersection). Let (U, x) be a complex manifold germ of dimension N and $(X, x) \subset (U, x)$ an analytic subgerm of dimension n. Then (X, x) cannot be defined as the common zero set of fewer than $N - n$ holomorphic functions at x. If we can do it with $N - n$ such functions, then we say that X is a *geometric complete intersection* at x. This is a non-trivial condition. For example, the union of the (z_1, z_2)-plane and the (z_3, z_4)-plane in $\mathbb{C}^4$ is not a geometric complete intersection at the origin (see Gunning (1974), p. 159). Similarly, if $\mathcal{I} \subset \mathfrak{m}_{U,x}$ is an ideal which defines a germ of dimension n, then we say that $\mathcal{I}$ defines a *complete intersection* at x if $\mathcal{I}$ admits $N - n$ generators. We can characterize such systems of generators algebraically:

(1.6) (Cohen-Macaulay property). Let (U, x) be a smooth germ of dimension N as above. Then an ideal $\mathcal{I} \subset \mathcal{O}_{U,x}$ defines a complete intersection of dimension n if and only if it is generated by $\mathcal{O}_{U,x}$*-sequence* $f_1, \ldots, f_{N-n}$ of length $N - n$. This means that f_1 is not a zero divisor of $\mathcal{O}_{U,x}$ (which amounts to $f_1 \neq 0$) and f_j is not a zero divisor of $\mathcal{O}_{U,x}/(f_1, \ldots, f_{j-1})\mathcal{O}_{U,x}$ for $j = 2, \ldots, N - n$. If either condition is fulfilled, $\mathcal{A} := \mathcal{O}_{U,x}/\mathcal{I}$ is a Cohen-Macaulay ring of dimension n. In particular, $\dim(\mathcal{A}/\mathfrak{p}) = n$ for any associated prime ideal $\mathfrak{p}$ of $\mathcal{A}$. For a proof, see for instance Matsumura (1980), Thm. 30. Since a $\mathcal{O}_{U,x}$-sequence $f_1, \ldots, f_{N-n}$ can always be completed to one of length N, $\mathcal{A}$ will not have a zero divisor if $n > 0$.

The property that an ideal $\mathcal{I} \subset \mathfrak{m}_{U,x}$ defines a complete intersection at x only depends on the $\mathbb{C}$-algebra $\mathcal{A}$, as will follow from Lemmas (1.7) and (1.9) below. Let us start with the basic:

(1.7) Lemma. *Let $\mathcal{I} \subset \mathcal{O}_{U,x}$ be an ideal and put $\mathcal{A} := \mathcal{O}_{U,x}/\mathcal{I}$. If $\mathfrak{m}_\mathcal{A}$ denotes the maximal ideal to $\mathcal{A}$ (the image of $\mathfrak{m}_{U,x}$ in $\mathcal{A}$), then there exists a smooth subgerm $j : (U_0, x) \subset (U, x)$ of dimension $\dim_\mathbb{C} \mathfrak{m}_\mathcal{A}/\mathfrak{m}_\mathcal{A}^2$ such that $j^*\mathcal{I} \subset \mathfrak{m}_{U',x}^2$ and $\mathcal{A}$ maps isomorphically onto $\mathcal{O}_{U_0,x}/j^*\mathcal{I}$.*

Proof. We abbreviate $\mathfrak{m}_{U,x}$ by $\mathfrak{m}$. Put $r := \dim_\mathbb{C}(\mathcal{I}+\mathfrak{m}^2)/\mathfrak{m}^2$ and choose $z_1, \ldots, z_r \in \mathcal{I}$ such that their images in $(\mathcal{I} + \mathfrak{m}^2)/\mathfrak{m}^2$ give a basis. Then the ideal in $\mathcal{O}_{U,x}$ generated by $(z_1, \ldots, z_r)$ defines a smooth subgerm of codimension r (because their differentials are linearly independent at x). If we denote this subgerm by $j : (U_0, x) \subset (U, x)$, then $j^*\mathcal{I} \subset \mathfrak{m}_{U_0,x}^2$ and $\mathcal{A}$ maps isomorphically onto $\mathcal{O}_{U_0,x}/j^*\mathcal{I}$. In particular, the cotangent space of (U_0, x), $\mathfrak{m}_{U_0,x}/\mathfrak{m}_{U_0,x}^2$, is identified with the 'Zariski cotangent space' $\mathfrak{m}_\mathcal{A}/\mathfrak{m}_\mathcal{A}^2$ and so $\dim(U_0, x) = \dim_\mathbb{C} \mathfrak{m}_\mathcal{A}/\mathfrak{m}_\mathcal{A}^2$. □

(1.8). For this reason $\dim_{\mathbb{C}} \mathfrak{m}_{\mathcal{A}}/\mathfrak{m}_{\mathcal{A}}^2$ is called the *embedding dimension of* $\mathcal{A}$. It is the smallest dimension N for which $\mathcal{A}$ can be realized as a quotient algebra of $\mathbb{C}\{z_1, \dots, z_N\}$. The difference between $\dim_{\mathbb{C}} \mathfrak{m}_{\mathcal{A}}/\mathfrak{m}_{\mathcal{A}}^2$ and the (Krull) dimension of $\mathcal{A}$ is called the *embedding codimension of* $\mathcal{A}$.

(1.9) Lemma. *Suppose* (U, x) *and* (U', x') *are smooth germs of the same dimension and let* $\mathcal{I} \subset \mathcal{O}_{U,x}$ *and* $\mathcal{I}' \subset \mathcal{O}_{U',x'}$ *be ideals. Then an isomorphism* $\phi : \mathcal{O}_{U',x'}/\mathcal{I}' \to \mathcal{O}_{U,x}/\mathcal{I}$ *is always induced by an isomorphism* $\Phi : (U, x) \xrightarrow{\cong} (U', x')$.

Proof. Upon replacing (U, x) and (U', x') by the smooth subgerms found in Lemma (1.7), we easily reduce to the case that we have $\mathcal{I} \subset \mathfrak{m}_{U,x}^2$ and $\mathcal{I}' \subset \mathfrak{m}_{U',x'}^2$. Now pick a coordinate system $(z_1, \dots, z_e)$ for (U', x') and choose $\phi_\nu \in \mathfrak{m}_{U,x}$, such that its reduction mod $\mathcal{I}$ is just the image of $z'_\nu + \mathcal{I}'$ under ϕ. Let $\Phi : (U, x) \to (U', x')$ be defined by $\Phi^* z_\nu = \phi_\nu$. Since $(\phi_1, \dots, \phi_e)$ maps onto a basis of $\mathfrak{m}_{U,x}/\mathfrak{m}_{U,x}^2$ it follows from the inverse function theorem that $\Phi : (U, x) \to (U', x')$ is an isomorphism. Clearly, $\Phi^*(\mathcal{I}') \subset \mathcal{I}$. Since Φ^* induces ϕ^*, we have in fact $\Phi^*(\mathcal{I}') = \mathcal{I}$. □

(1.10) Corollary. *Let* (U, x) *be a smooth germ and* $\mathcal{I} \subset \mathcal{O}_{U,x}$ *an ideal. Then the difference of* $\dim(U, x)$ *and the minimal number of (ideal) generators of* $\mathcal{I}$ *only depends on the isomorphism type of the* $\mathbb{C}$*-algebra* $\mathcal{O}_{U,x}/\mathcal{I}$.

Proof. If we pass to from (U, x) to (U_0, x) as in Lemma (1.7), then this difference evidently does not change. If we combine this with Lemma (1.9) the corollary follows. □

(1.11) Definition. A local analytic algebra $\mathcal{A}$ (i.e., a $\mathbb{C}$-algebra isomorphic to one of the form $\mathcal{O}_{\mathbb{C}^N,0}/\mathcal{I}$) is called a *(local) complete intersection algebra* if for some surjective $\mathbb{C}$-algebra homomorphism $\pi : \mathbb{C}\{z_1, \dots, z_N\} \twoheadrightarrow \mathcal{A}$, the kernel of π defines a complete intersection in $(\mathbb{C}^N, 0)$ in the previous sense (by the preceding, this is then so for *any* such π).

(1.12) (Notion of an icis). The case that concerns us most is when we are given a smooth germ (U, x) and (X, x) is given by an ideal $\mathcal{I} \subset \mathcal{O}_{U,x}$ for which (X, x) is smooth away from $\{x\}$. This means that if $f_1, \dots, f_{N-n}$ is a set of generators of $\mathcal{I}$ (with $n = \dim \mathcal{A}$) then the common zero set of $f_1, \dots, f_{N-n}$ and the $(N-n)$-form $df_1 \wedge \cdots \wedge df_{N-n}$ is contained in $\{x\}$. This is also equivalent to the condition that given a coordinate system $(z_1, \dots, z_N)$ for (U, x), then the ideal in $\mathcal{O}_{U,x}$ generated by $f_1, \dots, f_{N-n}$ and the determinants of the $(N-n) \times (N-n)$ submatrices of the Jacobian matrix $\left(\frac{\partial f_j}{\partial z_\nu}\right)$ contains a power of $\mathfrak{m}_{U,x}$. We then say that (X, x) endowed with its local $\mathbb{C}$-algebra $\mathcal{A}$ is an *isolated complete intersection singularity* (so this includes the case that $\mathcal{A}$ is regular). Henceforth we shall abbreviate this as *icis*. We will often use this in expressions like: $(f_1, \dots, f_{N-n}) : (U, x) \to (\mathbb{C}^{N-n}, 0)$ or $f : (U, x) \to (S, o)$ (with (S, o) a smooth germ of dimension $N - n$) or $\mathcal{I} \subset \mathcal{O}_{U,x}$, or $\mathcal{A}$ defines an icis.

(1.13) Proposition. *An ideal* $\mathcal{I} \subset \mathcal{O}_{U,x}$ *defines a complete intersection* (X, x) *of dimension* $n > 0$ *is its own radical.*