Mathematics Monograph Series 25

Bifurcation Theory of Limit Cycles

Han Maoan

（极限环分支理论）

SCIENCE PRESS
Beijing

Supported by the National Fund for Academic Publication in Science and Technology

Mathematics Monograph Series 25

Han Maoan

Bifurcation Theory of Limit Cycles

(极限环分支理论)

Science Press
Beijing

Responsible Editor: Li Xin

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

Printed in Beijing

ISBN 978-7-03-036140-0

Dedicated to Professor Ye Yanqian (1923–2008)

Preface

In this book we present the bifurcation theory of limit cycles of planar systems with multiple parameters. The theory studies the changes of orbital behavior in the phase space, especially the number of limit cycles as we vary the parameters in the system. This theory has been considered and developed by many mathematicians starting with Poincaré who first introduced the notion of limit cycles. A fundamental step towards modern bifurcation theory in differential equations occurred with the definition of structural stability and the classification of structurally stable systems in the plane in 1937 developed by Andronov, Leontovich and Pontryain. A further development of the theory had taken different directions, such as selecting bifurcation sets of codimension one for primary bifurcations and of arbitrary codimension in the general case for degenerate bifurcations, and finding the number of limit cycles in Hopf bifurcation or by perturbing Hamiltonian systems. In the two-dimensional case, as was proved in Andronov et al.[2], rough systems compose an open and dense set in the space of all systems on a plane, and the non-rough systems fill the boundaries between different regions of structural stability in this space. The bifurcation theory studies orbital behavior of the non-rough systems under perturbations.

As asked by D. Hilbert in his 16th problem[107], the main task in the study of a given planar system is the number and location of limit cycles. Many studies have concentrated on perturbations of Hamiltonian systems. For this kind of systems, an important tool used to find the number of limit cycle is the so-called Melnikov function or Abelian integral in the case of polynomial equations. The function can be used to study the number of limit cycles which are produced from a center point, a homoclinic loop, a heteroclinic loop or an annulus consisting of a family of periodic orbits under perturbations.

The present book focuses on an in-depth study of limit cycles with general methods of both local and global bifurcations for small perturbations of Hamiltonian systems with the help of Melnikov functions.

The book consists of five chapters. In the first chapter, some basic notations related to limit cycles are first introduced, such as Poincaré map, stability and multiplicity of a limit cycle. Then fundamental properties of limit cycles are established, say, invariance of stability and multiplicity under changes of variables. With the help of Poincaré map, some simple bifurcation phenomena near a non-hyperbolic limit cycle are analyzed under perturbations. The topic of the second chapter is Hopf bifurcation. Poincaré map near a focus, and the stability, order and focus values related to a focus are first defined. Then three different ways to discuss the stability and the order of a focus, and to study the bifurcation problem of limit cycles near a focus are introduced. The relationships among these methods are also given. Analytical methods to study Hopf bifurcation of Liénard systems are presented, followed by interesting applications to some Liénard systems of special form and to general quadratic systems. The degenerate Hopf bifurcation near an elementary center is particularly studied by using the coefficients of the expansion of the first order Melnikov function at the center. The general form of Z_q equivariant systems on the plane is introduced and classified.

Chapter three concerns with general perturbations of Hamiltonian systems, or near-Hamiltonian systems for short. The notion of cyclicity of a near-Hamiltonian system with multiple parameters at a center, a periodic orbit or a homoclinic loop are defined, and a general method to find lower or upper bound of these cyclicities is established. A Hamiltonian system will have a nilpotent critical point when at least two singular points meet together. For example, an elementary center and a hyperbolic saddle become a cusp when they meet together. A nilpotent critical point of a Hamiltonian system could be a cusp, nilpotent center or nilpotent saddle. A cusp or nilpotent saddle can be located on a homoclinic or heteroclinic loop. A limit cycle, under perturbation, may appear in a neighborhood of a nilpotent center or a homoclinic or heteroclinic loop with a cusp or nilpotent saddle. The problem of limit cycle bifurcation is studied in detail by perturbing a nilpotent center or a homoclinic or heteroclinic loop with a cusp or nilpotent saddle. The main idea is also to make understand the analytical property of the first order Melnikov function at the corresponding Hamiltonian value.

As we knew, in Hopf bifurcation a limit cycle is created from a weak focus when the focus changes its stability. This idea can be developed to homoclinic bifurcation. That is to say, limit cycles can be found by perturbing and changing the stability of a homoclinic loop. For the purpose, the problem of determining the stability of a homoclinic loop needs to be solved. The same method can also be used to find limit cycles in a neighborhood of a heteroclinic loop with two saddles. Chapter four provides a general theory of homoclinic bifurcation, giving a way to solve these problems. Some sufficient conditions are provided for the existence of multiple limit cycles near a homoclinic, double homoclinic or heteroclinic loop, or even some types of compound loop consisting of homoclinic and heteroclinic orbits.

In the last chapter, chapter 5, an interesting application of bifurcation methods is presented to general polynomial systems on the plane. Based on the results of some polynomial systems with degrees 3, 4, 5 and 6, a lower bound of the maximal number of limit cycles is obtained for all polynomial systems of degree greater than 6.

This book has been used for three years in my class of graduate students as a text book of the course Bifurcation Theory of Limit Cycles so far. They found and corrected mistakes during their study. I am grateful to all of them. I especially thank Dr. Yang Junmin who helped me make computations of many examples and as well as all of the figures in the book.

Han Maoan
August, 2012

Contents

Chapter 1

Limit Cycle and Its Perturbations

1.1 Basic notations and facts

Consider a planar system defined on a region $G \subset \mathbb{R}^2$ of the form

$$\dot{x} = f(x), \tag{1.1.1}$$

where $f : G \longrightarrow \mathbb{R}^2$ is a C^r function, $r \geqslant 1$. Then for any point $x_0 \in G$ (1.1.1) has a unique solution $\varphi(t, x_0)$ satisfying $\varphi(0, x_0) = x_0$. Let $\varphi^t(x_0) = \varphi(t, x_0)$. The family of the transformations $\varphi^t : G \longrightarrow \mathbb{R}^2$ satisfies the following properties

(i) $\varphi^0 = \mathrm{Id}$;

(ii) $\varphi^{t+s} = \varphi^t \circ \varphi^s$.

The function φ is called the flow generated by (1.1.1) or by the vector field f. Let $I(x_0)$ denote the maximal interval of definition of $\varphi(t, x_0)$ in t. If $x_0 \in G$ is such that $\varphi(t, x_0)$ is constant for all $t \in I(x_0)$, then $f(x_0) = 0$. In this case, x_0 is called *a singular point* of (1.1.1). A point that is not singular is called *a regular point.*

For any regular point $x_0 \in G$, the solution $\varphi(t, x_0)$ determines two planar curves as follows

$$\gamma^+(x_0) = \{\varphi(t, x_0) : t \in I(x_0), t \geqslant 0\}, \quad \gamma^-(x_0) = \{\varphi(t, x_0) : t \in I(x_0), t \leqslant 0\},$$

which are called *the positive, negative orbit* of (1.1.1) through x_0 respectively. The union $\gamma(x_0) = \gamma^+(x_0) \cup \gamma^-(x_0)$ is called *the orbit* of (1.1.1) through x_0. The theorem about the existence and uniqueness of solutions ensures that there is one and only one orbit through any point in G. Thus, it is easy to prove that any different orbits do not intersect each other. *A periodic orbit* of (1.1.1) is an orbit that is a closed curve. The minimal positive number satisfying $\varphi(T, x_0) = x_0$ is said to be *the period* of the periodic orbit $\gamma(x_0)$. Obviously, $\gamma(x_0)$ is a periodic orbit of period T if and only if the corresponding representation $\varphi(t, x_0)$ is a periodic solution of the same period.

Definition 1.1.1 A periodic orbit of (1.1.1) is called a *limit cycle* if it is the only periodic orbit in a neighborhood of it. In other words, a limit cycle is an isolated periodic orbit in the set of all periodic orbits.

Now let us assume that (1.1.1) has a limit cycle $L : x = u(t), 0 \leqslant t \leqslant T$. Since (1.1.1) is autonomous, for any given point $p \in L$ we may suppose $p = u(0)$, and hence, $u(t) = \varphi(t,p)$. Further, for definiteness, let L be oriented clockwise. Introduce a unit vector below

$$Z_0 = \frac{1}{|f(p)|}(-f_2(p), f_1(p))^{\mathrm{T}}.$$

Then there exists a cross section l of (1.1.1) which passes through p and is parallel to Z_0. Clearly, a point $x_0 \in l$ near p can be written as $x_0 = p + aZ_0$ with $a = (x_0 - p)^{\mathrm{T}} Z_0 \in \mathbb{R}$ small.

Lemma 1.1.1 *There exist a constant $\varepsilon > 0$ and C^r functions P and $\tau : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ with $P(0) = 0$ and $\tau(0) = T$ such that*

$$\varphi(\tau(a), p + aZ_0) = p + P(a)Z_0 \in l, \quad |a| < \varepsilon. \tag{1.1.2}$$

Proof Define $Q(t,a) = [f(p)]^{\mathrm{T}}(\varphi(t, p + aZ_0) - p)$. We have

$$Q(T,0) = 0, \quad Q_t(T,0) = |f(p)|^2 > 0.$$

Note that Q is C^r for (t,a) near $(T,0)$. The implicit function theorem implies that a C^r function $\tau(a) = T + O(a)$ exists satisfying

$$Q(\tau(a), a) = 0 \quad \text{or} \quad [f(p)]^{\mathrm{T}}(\varphi(\tau(a), p + aZ_0) - p) = 0.$$

It follows that the vector $\varphi(\tau(a), p + aZ_0) - p$ is parallel to Z_0. Hence, it can be rewritten as $\varphi(\tau(a), p + aZ_0) - p = P(a)Z_0$, where

$$P(a) = Z_0^{\mathrm{T}}(\varphi(\tau(a), p + aZ_0) - p). \tag{1.1.3}$$

It is obvious that $P \in C^r$ for $|a|$ small with $P(0) = 0$. This ends the proof.

The above proof tells us that the function τ is the time of the first return to l. By Definition 1.1.1, the periodic orbit L is a limit cycle if and only if $P(a) \neq a$ for $|a| > 0$ sufficiently small.

Definition 1.1.2 The function $P : (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by (1.1.2) is called a Poincaré *map* or *return map* of (1.1.1) at $p \in l$.

For convenience, we sometimes use the notation $P : l \longrightarrow l$.

Definition 1.1.3 The limit cycle L is said to be *outer stable* (*outer unstable*) if for $a > 0$ sufficiently small,

$$a(P(a) - a) < 0 (> 0).$$

The limit cycle L is said to be *inner stable* (*inner unstable*) if the inequality above holds for $-a > 0$ sufficiently small. A limit cycle is called *stable* if it is both inner and outer stable. A limit cycle is called *unstable* if it is not stable.

For example, if L is stable, then the orbits near it behave like the phase portrait as shown in Figure 1.1.1.

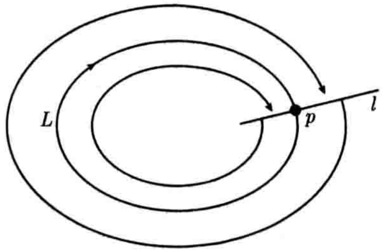

Figure 1.1.1 Behavior of a stable limit cycle

Let $P^k(a)$ denote the kth iterate of a under P. It is evident that $\{P^k(a)\}$ is monotonic in k and $P^k(a) > 0(< 0)$ for $a > 0(< 0)$. Thus, it is easy to see that L is outer stable if and only if $P^k(a) \to 0$ as $k \to \infty$ for all $a > 0$ sufficiently small. Similar conclusions hold for outer unstable, inner stable and inner unstable cases.

Remark 1.1.1 If the limit cycle L is oriented anti-clockwise we can define its stability in a similar manner by using the Poincaré map P defined by (1.1.2). For instance, it is said to be inner stable (inner unstable) if $a(P(a) - a) < 0(> 0)$ for $a > 0$ sufficiently small.

Definition 1.1.4 The limit cycle L is said to be *hyperbolic or of multiplicity one* if $P'(0) \neq 1$. It is said to have *multiplicity* k, $2 \leqslant k \leqslant r$, if $P'(0) = 1, P^{(j)}(0) = 0,\ j = 2, \cdots, k-1,\ P^{(k)}(0) \neq 0$.

By Definition 1.1.3, one can see that L is stable (unstable) if $|P'(0)| < 1\ (> 1)$.

Example 1.1.1 Consider a system given by

$$\begin{aligned} \dot{x_1} &= x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x_2} &= x_1 + x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{1.1.4}$$

The system has the form

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1$$

in polar coordinates (r, θ) with $x = (r\cos\theta, r\sin\theta)$. Thus, one can find (1.1.4) has a flow of the form

$$\varphi(t, x_0) = r(t)(\cos\theta(t), \sin\theta(t))^{\mathrm{T}}, \tag{1.1.5}$$

where

$$r(t) = r_0(r_0^2 + (1 - r_0^2)\mathrm{e}^{-2t})^{-\frac{1}{2}}, \quad \theta(t) = t + \theta_0,$$

$$x_0 = r_0(\cos\theta_0, \sin\theta_0)^{\mathrm{T}}, \quad r_0 > 0, \quad 0 \leqslant \theta_0 < 2\pi.$$

For $p = (1, 0)^{\mathrm{T}}$, we have a periodic orbit $L = \{(x_1, x_2)^{\mathrm{T}} | x_1^2 + x_2^2 = 1\}$ which has a representation

$$L: \quad x = \varphi(t, p) = (\cos t, \sin t)^{\mathrm{T}}, \quad 0 \leqslant t \leqslant 2\pi,$$

with $Z_0 = (-1, 0)^{\mathrm{T}}$. Then $p + aZ_0 = (1-a, 0)^{\mathrm{T}}$. Hence, $x_0 = p + aZ_0$ if and only if $r_0 = 1 - a$, $\theta_0 = 0$.

Taking $l = \{(x_1, 0)^{\mathrm{T}} | x_1 > 0\}$. Then noting that $\tau(0) = 2\pi$, by (1.1.5) we have $\varphi(\tau(a), p + aZ_0) \in l$ if and only if $\tau(a) = 2\pi$ for $a < 1$. Therefore,

$$\varphi(\tau(a), p + aZ_0) = (1-a)[(1-a)^2 + (2a - a^2)\mathrm{e}^{-4\pi}]^{-\frac{1}{2}}(1, 0)^{\mathrm{T}}.$$

It follows from (1.1.3) that

$$\begin{aligned} P(a) &= 1 - (1-a)[(1-a)^2 + (2a - a^2)\mathrm{e}^{-4\pi}]^{-\frac{1}{2}} \\ &= a\mathrm{e}^{-4\pi} + O(a^2) \end{aligned}$$

for $|a|$ small. By Definition 1.1.3, the limit cycle L is stable.

Next, we give formulas for $P'(0)$ and $P''(0)$. For the purpose, let

$$v(\theta) = \frac{u'(\theta)}{|u'(\theta)|} = (v_1(\theta), v_2(\theta))^{\mathrm{T}}, \quad Z(\theta) = (-v_2(\theta), v_1(\theta))^{\mathrm{T}},$$

and introduce a transformation of coordinates of the form

$$x = u(\theta) + Z(\theta)b, \quad 0 \leqslant \theta \leqslant T, \quad |b| < \varepsilon. \tag{1.1.6}$$

Lemma 1.1.2 *The transformation* (1.1.6) *carries* (1.1.1) *into the system*

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + g_1(\theta, b), \quad \frac{\mathrm{d}b}{\mathrm{d}t} = A(\theta)b + g_2(\theta, b), \tag{1.1.7}$$

where

$$\begin{aligned} A(\theta) &= Z^{\mathrm{T}}(\theta) f_x(u(\theta)) Z(\theta) = \mathrm{tr} f_x(u(\theta)) - \frac{\mathrm{d}}{\mathrm{d}\theta} \ln |f(u(\theta))|, \\ g_1(\theta, b) &= h(\theta, b)[f(u(\theta) + Z(\theta)b) - f(u(\theta))] - h(\theta, b) Z'(\theta) b, \\ g_2(\theta, b) &= Z^{\mathrm{T}}(\theta)[f(u(\theta) + Z(\theta)b) - f(u(\theta)) - f_x(u(\theta))Z(\theta)b], \\ h(\theta, b) &= (|f(u(\theta))| + v^{\mathrm{T}}(\theta) Z'(\theta) b)^{-1} v^{\mathrm{T}}(\theta), \end{aligned}$$

and $\mathrm{tr} f_x(u(\theta))$ *denotes the trace of the matrix* $f_x(u(\theta))$, *which is called the divergence of the vector field* f *evaluated at* $u(\theta)$.

Proof By (1.1.6) and (1.1.1) we have

$$(u' + Z'b)\frac{\mathrm{d}\theta}{\mathrm{d}t} + Z\frac{\mathrm{d}b}{\mathrm{d}t} = f(u + Zb). \tag{1.1.8}$$

In order to obtain (1.1.7) we need to solve $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}b}{\mathrm{d}t}$ from (1.1.8). First, multiplying (1.1.8) by v^{T} from the left-hand side and using

$$v^{\mathrm{T}} Z = 0, \quad v^{\mathrm{T}} f(u) = v^{\mathrm{T}} u' = |u'| = |f(u)|,$$

we can obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = [|f(u)| + v^{\mathrm{T}} Z' b]^{-1} v^{\mathrm{T}} f(u + Zb) = h(\theta, b) f(u + Zb).$$

Note that

$$h(\theta, b) f(u) = h(\theta, b)[f(u) + Z'b] - h(\theta, b) Z'b = 1 - h(\theta, b) Z'b.$$

It follows that

$$h(\theta, b) f(u + Zb) = h(\theta, b)[f(u + Zb) - f(u)] - h(\theta, b) Z'b + 1.$$

Then the first equation in (1.1.7) follows.

Now multiplying (1.1.8) by Z^{T} from the left and using

$$Z^{\mathrm{T}} Z = 1, \quad Z^{\mathrm{T}} f(u) = 0, \quad Z^{\mathrm{T}} Z' = v_1 v_1' + v_2 v_2' = \frac{1}{2}(|v|^2)' = 0,$$

we obtain

$$\frac{\mathrm{d}b}{\mathrm{d}t} = Z^{\mathrm{T}}[f(u + Zb) - f(u) - f_x(u) Zb] + Z^{\mathrm{T}} f_x(u) Zb.$$

By writing f and Z in their components it is direct to prove that

$$Z^{\mathrm{T}} f_x(u) Z = \mathrm{tr} f_x(u) - \frac{\mathrm{d}}{\mathrm{d}\theta} \ln |f(u)|.$$

Then the second equation of (1.1.7) follows. This finishes the proof.

Set

$$B(\theta) = [f_x(u + Zb)]_b'|_{b=0}, \quad C(\theta) = v^{\mathrm{T}}[f_x(u) Z - Z'(\theta)], \tag{1.1.9}$$

and

$$R(\theta, b) = \frac{A(\theta) b + g_2(\theta, b)}{1 + g_1(\theta, b)}.$$

Then by Lemma 1.1.2, we can write

$$R(\theta, b) = A(\theta) b + \frac{1}{2}\left[Z^{\mathrm{T}} B Z - \frac{2AC}{|f(u)|}\right] b^2 + O(b^3) \equiv A(\theta) b + \frac{1}{2} A_1(\theta) b^2 + O(b^3). \tag{1.1.10}$$

For $|b|$ small we have from (1.1.7)

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = R(\theta, b) \tag{1.1.11}$$

which is a T-periodic equation. From Lemma 1.1.2 we know that the function R is C^{r-1} in (θ, b) and C^r in b. Let $b(\theta, a)$ denote the solution of (1.1.11) with $b(0, a) = a$. Then $b(T, a)$ defines a function of a which is called a Poincaré map of (1.1.11). For the relationship of Poincaré maps of (1.1.1) and (1.1.11) we have

Lemma 1.1.3 $P(a) = b(T, a)$.

Proof Consider the equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + g_1(\theta, b(\theta, a)).$$

It has a unique solution $\theta = \theta(t, a)$ satisfying $\theta(0, a) = 0$ and $\theta(t, 0) = t$. From (1.1.10) it implies $b(\theta, 0) = 0$. This yields $\theta(T, 0) = T$, $\frac{\partial \theta}{\partial t}(T, 0) = 1$. Hence, by the implicit function theorem a unique function $\tilde{\tau}(a) = T + O(a)$ exists such that $\theta(\tilde{\tau}, a) = T$.

For $x_0 = u(0) + Z(0)a$, we have by (1.1.6)

$$\varphi(t, x_0) = u(\theta(t, a)) + Z(\theta(t, a))b(\theta(t, a), a).$$

In particular,

$$\varphi(\tilde{\tau}, x_0) = u(T) + Z(T)b(T, a) = u(0) + Z(0)b(T, a) = p + Z_0 b(T, a) \in l.$$

Thus, it follows from Lemma 1.1.1 that $\tau = \tilde{\tau}$ and $P(a) = b(T, a)$.

The proof is completed.

For $|a|$ small we can write

$$b(\theta, a) = b_1(\theta)a + b_2(\theta)a^2 + O(a^3),$$

where $b_1(0) = 1, b_2(0) = 0$. By (1.1.10) and (1.1.11) one can obtain

$$b_1' = Ab_1, \quad b_2' = Ab_2 + \frac{1}{2}A_1 b_1^2$$

which give

$$b_1(\theta) = \exp \int_0^\theta A(s)\mathrm{d}s, \quad b_2(\theta) = b_1(\theta) \int_0^\theta \frac{1}{2}A_1(s)b_1(s)\mathrm{d}s.$$

Then by Lemma 1.1.3 we have

$$P'(0) = b_1(T) = \exp \int_0^T A(s)\mathrm{d}s = \exp \int_0^T \mathrm{tr} f_x(u(t))\mathrm{d}t,$$

$$P''(0) = 2b_2(T) = b_1(T) \int_0^T A_1(s)b_1(s)\mathrm{d}s.$$

Thus, noting (1.1.10) we obtain the following theorem.

Theorem 1.1.1 *Suppose* P *is a* Poincaré *map of* (1.1.1) *at* $p \in L$. *Then*

(i) $P'(0) = \exp \oint_L \mathrm{div} f \mathrm{d}t, \quad \mathrm{div} f = \mathrm{tr} f_x$,

(ii) $P''(0) = P'(0) \int_0^T \mathrm{e}^{\int_0^t A(s)\mathrm{d}s} \left[Z^{\mathrm{T}}(t)B(t)Z(t) - \frac{2A(t)C(t)}{|f(u(t))|} \right] \mathrm{d}t.$

In particular, L is stable (unstable) if $I(L) = \oint_L \mathrm{div} f \mathrm{d}t < 0 (> 0)$.

We remark that Theorem 1.1.1 remains true in the case of counter clockwise orientation of L.

Example 1.1.2 Consider the quadratic system

$$\dot{x} = -y(1+cx) - (x^2 + y^2 - 1),$$
$$\dot{y} = x(1+cx), \quad 0 < c < 1.$$

This system has the circle $L : x^2 + y^2 = 1$ as its limit cycle. We claim that the cycle is unstable.

In fact, we have

$$I(L) = \oint_L (-2x - cy)\mathrm{d}t = \oint_L \left(\frac{c\mathrm{d}x}{1+cx} - \frac{2\mathrm{d}y}{1+cx} \right) = \iint_{x^2+y^2 \leqslant 1} \frac{2c\mathrm{d}x\mathrm{d}y}{(1+cx)^2} > 0.$$

Example 1.1.3 The system

$$\dot{x} = -y - x(x^2 + y^2 - 1)^2,$$
$$\dot{y} = x - y(x^2 + y^2 - 1)^2$$

has a unique limit cycle given by $L : (x, y) = (\cos t, \sin t), 0 \leqslant t \leqslant 2\pi$. For the system, it is easy to see that $v(\theta) = (-\sin\theta, \cos\theta)^{\mathrm{T}}, Z(\theta) = (-\cos\theta, -\sin\theta)^{\mathrm{T}}$. By Lemma 1.1.2 and (1.1.9) we then have

$$A(\theta) = 0, \quad B(\theta) = \begin{pmatrix} 8\cos^2\theta & 8\sin\theta\cos\theta \\ 8\sin\theta\cos\theta & 8\sin^2\theta \end{pmatrix}.$$

Thus from Theorem 1.1.1 it follows $P'(0) = 1, P''(0) = 16\pi$. This shows that L is a limit cycle of multiplicity 2.

From (1.1.9) and formulas for $P'(0)$ and $P''(0)$ in Theorem 1.1.1 the derivatives $P'(0)$ and $P''(0)$ are independent of the choice of the cross section l. This fact suggests that the stability and the multiplicity of a limit cycle should have the same property. Below we will prove this in detail even if the cross section l is taken as a C^r smooth curve.

1.2 Further discussion on property of limit cycles

Let L be a limit cycle of (1.1.1) as before and let l_1 be a C^r curve which has an intersection point $p_1 \in L$ with L and is not tangent to L at p_1. Then it can be represented as

$$l_1 : x = p_1 + q(a), \quad q(0) = 0, \quad \det(f(p_1), q'(0)) > 0,$$

where $q : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^2$ is C^r for a constant $\varepsilon > 0$. Note that $\det(f(p_1), q'(0)) = q'(0) \cdot (-f_2(p_1), f_1(p_1))^{\mathrm{T}}$. The condition $\det(f(p_1), q'(0)) > 0$ implies that the point $p_1 + q(a)$ is outside L if and only if $a > 0$.

Let $G(t, b, a) = \varphi(t, p_1 + q(a)) - [p_1 + q(b)]$. Then we have

$$\det \left.\frac{\partial G}{\partial(t, b)}\right|_{t=T, b=a=0} = \det(f(p_1), -q'(0)) \neq 0.$$

Hence, in the same way as Lemma 1.1.1 we can prove that there exist two C^r functions

$$P_1, \tau_1 : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}, \quad P_1(0) = 0, \quad \tau_1(0) = T$$

such that $G(\tau_1(a), P_1(a), a) = 0$, or

$$\varphi(\tau_1(a), p_1 + q(a)) = p_1 + q(P_1(a)) \in l_1. \tag{1.2.1}$$

This yields another Poincaré map $P_1 : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$.

Lemma 1.2.1 *Let P and P_1 be two* Poincaré *maps defined by* (1.1.2) *and* (1.2.1) *respectively. Then there exists a C^r function $h_1 : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ with $h_1(0) = 0, h_1'(0) > 0$ such that $h_1 \circ P = P_1 \circ h_1$.*

Proof Since $p = u(0)$ we can suppose $p_1 = u(t_1)$ for some $t_1 \in [0, T)$. Similar to Lemma 1.1.1 again, there exist two C^r functions h_1 and τ^*, both from $(-\varepsilon, \varepsilon)$ to $\mathbb{R}$, with $h_1(0) = 0$ and $\tau^*(0) = t_1$ such that

$$\varphi(\tau^*(a), p + aZ_0) = p_1 + q(h_1(a)) \in l_1. \tag{1.2.2}$$

See Figure 1.2.1.

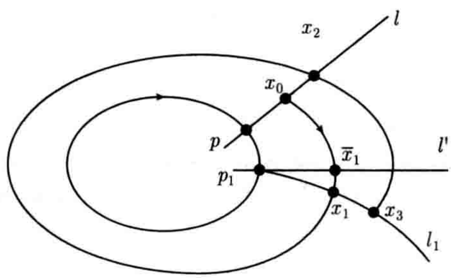

Figure 1.2.1 Two Poincaré maps

Let $x_0 = p + aZ_0, x_1 = \varphi(\tau^*(a), x_0), x_2 = \varphi(\tau(a), x_0)$. By (1.2.2) and (1.1.2) we have $x_1 = p_1 + q(a_1), a_1 = h_1(a)$ and $x_2 = p + P(a)Z_0$. Hence, by (1.2.1) and (1.2.2) we have

$$\varphi(\tau_1(a_1), x_1) = p_1 + q(P_1(a_1)), \quad \varphi(\tau^*(P(a)), x_2) = p_1 + q(a_2), \quad a_2 = h_1(P(a)).$$

On the other hand, by the flow property of φ we have

$$x_3=\varphi(\tau_1(a_1),x_1)=\varphi(\tau_1(a_1)+\tau^*(a),x_0)=\varphi(\tau^*(P(a))+\tau(a),x_0)=\varphi(\tau^*(P(a)),x_2),$$

which, together with the above, follows that $q(P_1(a_1)) = q(a_2)$ or $a_2 = P_1(a_1)$. Hence $h_1 \circ P = P_1 \circ h_1$.

It only needs to prove $h_1'(0) > 0$. Let $a \geqslant 0$. Introduce one more cross section below

$$l' : x = u(t_1) + Z(t_1)a, \quad 0 \leqslant a \leqslant \varepsilon.$$

Let $\widetilde{\tau}_1(a) = t_1 + O(a)$ be such that $\theta(\widetilde{\tau}_1, a) = t_1$. By (1.1.6) we have

$$\overline{x}_1 = \varphi(\tilde{\tau}_1, x_0) = u(t_1) + Z(t_1)b(t_1, a) \in l'.$$

Then $b(t_1, a) = |p_1\overline{x}_1|$. By the proof of Lemma 1.1.3,

$$\frac{\partial b}{\partial a}(t_1, 0) = \exp \int_0^{t_1} A(s)\mathrm{d}s > 0.$$

Consider the triangle formed by points p_1, x_1 and $\overline{x}_1$. There exists a point x^* on the orbital arc $\widehat{x_1\overline{x}_1}$ such that $f(x^*)$ is parallel to the side $x_1\overline{x}_1$. Since the arc $\widehat{x_1\overline{x}_1}$ approaches p_1 as $a \to 0$ we have $x^* \to p_1, f(x^*) \to f(p_1)$ as $a \to 0$. Hence, if we let α_1 denote the angle between sides $p_1\overline{x}_1$ and $\overline{x}_1x_1$, and α_2 the angle between sides p_1x_1 and $\overline{x}_1x_1$, then we have $\alpha_1 \to \dfrac{\pi}{2}, \alpha_2 \to \alpha_0$ as $a \to 0$, where $\alpha_0 \in \left(0, \dfrac{\pi}{2}\right]$ is the angle between the vectors $f(p_1)$ and $q'(0)$. That is, α_0 is the angle between L and l_1 at p_1. By the Sine theorem, it follows

$$\frac{|p_1\overline{x}_1|}{\sin\alpha_2} = \frac{|p_1x_1|}{\sin\alpha_1}, \quad \text{or} \quad |q(h_1(a))| = \frac{\sin\alpha_1}{\sin\alpha_2}b(t_1, a) = \frac{a}{\sin\alpha_0}\exp\int_0^{t_1} A(s)\mathrm{d}s(1+O(a)).$$

On the other hand, $q(h_1(a)) = q'(0)h_1'(0)a + O(a^2)$ which gives

$$|q(h_1(a))| = |q'(0)| \cdot |h_1'(0)|a + O(a^2), \quad a > 0.$$

Hence, we obtain

$$|h_1'(0)| = \frac{1}{|q'(0)|\sin\alpha_0}\exp\int_0^{t_1} A(s)\mathrm{d}s \neq 0.$$

Noting that $h_1(a) > 0$ for $a > 0$ we have $h_1'(0) > 0$. The proof is completed.

Corollary 1.2.1 *The stability and the multiplicity of the limit cycle L are independent of the choice of cross sections.*

Proof By Lemma 1.2.1 we have

$$h_1'(\overline{a})[P(a) - a] = P_1(h_1(a)) - h_1(a), \tag{1.2.3}$$