李彦博　肖占魁　著

Morita 系统环上的可加映射

Additive Mappings on Morita Context Rings

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· 沈 阳 ·

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Preface

Morita context rings were first introduced by Morita in [83], in order to characterize when two rings have equivalent module categories. A fundamental result is that the categories of modules over two rings with identity $\mathcal{R}$ and $\mathcal{S}$ are equivalent if and only if there exists a strict Morita context connecting $\mathcal{R}$ and $\mathcal{S}$, where "strict" implies that both Morita maps being surjective. Morita contexts have been used to the study of group actions on rings and Galois theory for commutative rings. We refer the reader to [77] for details. Moreover, some aspects of Morita context rings have been studied. For examples, in [92], Sands investigated various radicals of rings occurring in Morita contexts. R. Buchweitz investigated how to compare Hochschild cohomology of algebras related by a Morita context in [20].

Note that every ring with nontrivial idempotent is isomorphic to a Morita context ring. Thus many classes of algebras from various branches of mathematics can be viewed as Morita context rings, such as classical matrix rings, quasi-hereditary algebras, nest algebras, von Neumann algebras, incidence algebras and so on.

Morita context rings are natural generalization of the so-called triangular algebras. It is an active research area to study various mappings on triangular algebras, such as commuting mappings, Lie derivations, Jordan derivations, generalized derivations, higher derivations and non-linear mappings etcetera. These mappings have been used to study Lie isomorphisms, commutativity preserving maps, Jordan homomorphisms and Hochschild cohomology and so on. However, people pay much less attention to mappings of Morita context rings, to the best of our knowledge there were fewer articles dealing with mappings of Morita context rings before 2010. Recently, Li, Wei and Xiao jointly studied mappings of Morita context rings in [64–67, 102], which developed the theory of mappings of triangular algebras to the case of Morita context rings. The purpose of this book is to integrate the results of mappings on Morita context rings obtained by Li, Xiao and other mathematicians.

The book is divided into three chapters. We begin with the definition of Morita context rings in Chapter 1, then list examples from classical matrix algebras, path algebras, smash product, groups algebras and operator algebras. In Chapter 2, we study linear mappings on Morita context rings, including commuting mappings, Lie derivations, Jordan derivations, Jordan generalized derivations and Lie triple derivations. Note that the first important result about centralizing mappings was obtained by Posner which states that the existence of a nonzero commuting derivation on a prime algebra $\mathcal{A}$ implies that $\mathcal{A}$ is commutative. Therefore, we depict Posner's theorem in Section 1. Some related results are also given. In Section 2, we first describe the general form of commuting mappings of Morita context rings and consider the question of when all commuting mappings are proper. These work extend the main results of [24] to the case of Morita context rings. The second topic of this section is skew commuting mappings. We prove that every skew commuting map on Morita context rings under certain conditions is zero. These results not only give new perspectives to the work of [13] but also extend the main results of [24]. Moreover, a brief description about semi-centralizing mappings, k-commuting mappings and k-skew centralizing mappings on Morita context rings is also given. In Section 3, we investigate Lie derivations on Morita context rings. Our aim is to give a necessary and sufficient condition for each Lie derivation on a Morita context ring being of the standard form. In Section 4, we mainly study the question of whether there exist proper Jordan derivations on the Morita context ring $\mathcal{G}$. It is shown that if one of the bilinear pairings Φ_{MN} and Ψ_{NM} is nondegenerate, then every anti-derivation of $\mathcal{G}$ is zero. Furthermore, if the bilinear pairings Φ_{MN} and Ψ_{NM} are both zero, then every Jordan derivation of $\mathcal{G}$ is the sum of a derivation and an anti-derivation. At the end of this section, we describe a result obtained by Benkovič and Širovnik in [8]. A generalization of Jordan derivation, which was called Jordan generalized derivations is studied in Section 4. Note that we only consider the special case of Morita context rings, that is, triangular algebras. We prove that every Jordan generalized derivation on a triangular algebra is a generalized derivation. In the last section, we study Lie triple derivations of the triangular algebra $\mathcal{T}$. It is shown that under mild assumptions, each Lie triple derivation L on $\mathcal{T}$ is of standard form. That is, L can be expressed through an additive derivation and a linear functional vanishing on all second commutators of $\mathcal{T}$. Examples of Lie triple derivations on some classical triangular algebras are supplied. In Section 5, we consider local actions of linear mappings on Morita context rings. It is proved that mappings derivable (resp. Jordan derivable) at two idempotents P and Q are derivations (resp. Jordan derivations). Chapter 3 is devoted to the treatment of higher derivations and non-linear mappings. Section 1 is devoting to give a new characterization of Jordan higher derivation on associative algebras, which enables one to transfer the problems of Jordan higher derivations into the same problems concerning Jordan derivations. We establish a one to one correspondence relation between the set of all Jordan higher derivations and the set of all Jordan derivations. Applying the corresponding relation, in Section 2 we prove that every Jordan higher derivation on some operator algebras is a higher derivation. The involved operator algebras include CSL algebras, reflexive algebras, nest algebras. In Section 3, we prove that any Jordan higher derivation on a triangular algebra is a higher derivation. This extends the main result in [107] to the case of higher derivations. The kernel question in the Section 4 is whether every higher derivation on a triangular algebra is inner. We also consider some natural generalizations of higher derivations of triangular algebras, such as Jordan (triple-)higher derivations, generalized Jordan (triple-)higher derivations. In Section 5 we study nonlinear Lie higher derivations on the triangular algebra $\mathcal{T}$. Let $D = \{L_n\}_{n\in\mathbf{N}}$ be a Lie higher derivation on $\mathcal{T}$ without additive condition. Under mild assumptions, it is shown that $D = \{L_n\}_{n\in\mathbf{N}}$ is of standard form; i.e. each component $L_n (n \geqslant 1)$ can be expressed through an additive higher derivation and a non-linear functional vanishing on all commutators of $\mathcal{T}$. As applications, non-linear Lie higher derivations on some classical triangular algebras are characterized. Another class of non-linear mappings considered in this chapter is Jordan homomorphisms. We prove that every multiplicative bijective map, Jordan bijective map, Jordan triple bijective map on certain Morita context rings is additive. In the last section, we describe Jordan higher derivable points.

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Li Yanbo and Xiao Zhankui,

December 2012.

Contents

Chapter 1

Definitions and Examples of Morita Context Rings

1.1 Definitions of Morita context rings

We begin with the definition of Morita context rings. Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings

$$\Phi_{MN} : M \underset{B}{\otimes} N \longrightarrow A$$

and

$$\Psi_{NM} : N \underset{A}{\otimes} M \longrightarrow B$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc} M \underset{B}{\otimes} N \underset{A}{\otimes} M & \xrightarrow{\Phi_{MN} \otimes I_M} & A \underset{A}{\otimes} M \\ \Big\downarrow {\scriptstyle I_M \otimes \Psi_{NM}} & & \Big\downarrow \cong \\ M \underset{B}{\otimes} B & \xrightarrow{\cong} & M \end{array}$$

and

$$\begin{array}{ccc} N \underset{A}{\otimes} M \underset{B}{\otimes} N & \xrightarrow{\Psi_{NM}\otimes I_N} & B \underset{B}{\otimes} N \\ \Big\downarrow{\scriptstyle I_N\otimes\Phi_{MN}} & & \Big\downarrow{\scriptstyle \cong} \\ N \underset{A}{\otimes} A & \xrightarrow{\cong} & N\,. \end{array}$$

Let us write this Morita context as $(A, B, {}_AM_B, {}_BN_A, \Phi_{MN}, \Psi_{NM})$. If $(A, B, {}_AM_B, {}_BN_A, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| \; a \in A, m \in M, n \in N, b \in B \right\}$$

form an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication. Such an $\mathcal{R}$-algebra is usually called a *Morita context ring* of order 2 and is denoted by $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$.

In a similar way, we can define a Morita context ring of any order $n > 2$. Let $\mathcal{R}$ be a commutative ring with identity and $A_i\,(i = 1, 2, \cdots, n)$ be unital algebras over $\mathcal{R}$. Let ${}_iM_j$ be nonzero unital (A_i, A_j)-bimodules for $1 \leqslant i \leqslant j \leqslant n$ and ${}_iM_i = A_i$. We observe a family of (A_i, A_k)-bilinear homomorphisms

$$\eta^j_{i,k} : {}_iM_j \bigotimes_{A_j} {}_jM_k \longrightarrow {}_iM_k,$$

$$\eta^j_{i,j} : {}_iM_j \bigotimes_{A_j} A_j \cong {}_iM_j,$$

$$\eta^i_{i,j} : A_i \bigotimes_{A_i} {}_iM_j \cong {}_iM_j$$

and a family of diagrams

$$\begin{array}{ccc} {}_iM_j \underset{A_j}{\bigotimes} {}_jM_k \underset{A_k}{\bigotimes} {}_kM_l & \xrightarrow{I_{i,j}\otimes\eta^k_{j,l}} & {}_iM_j \underset{A_j}{\bigotimes} {}_jM_l\;, \\ \Big\downarrow{\scriptstyle \eta^j_{i,k}\otimes I_{k,l}} & & \Big\downarrow{\scriptstyle \eta^j_{i,l}} \\ {}_iM_k \underset{A_k}{\bigotimes} {}_kM_l & \xrightarrow{\eta^k_{i,l}} & {}_iM_l \end{array}$$

where $I_{i,j}$ and $I_{k,l}$ denote the identity mappings of ${}_iM_j$ and ${}_kM_l$, respectively. Let us consider the following set:

$$\mathcal{G}_n(A_i; {}_iM_j) = \left\{ \left[\begin{array}{ccccc} {}_1a_1 & {}_1m_2 & \cdots & {}_1m_{n-1} & {}_1m_n \\ {}_2m_1 & {}_2a_2 & \cdots & {}_2m_{n-1} & {}_2m_n \\ \vdots & \vdots & & \vdots & \vdots \\ {}_{n-1}m_1 & {}_{n-1}m_2 & \cdots & {}_{n-1}a_{n-1} & {}_{n-1}m_n \\ {}_nm_1 & {}_nm_2 & \cdots & {}_nm_{n-1} & {}_na_n \end{array} \right] \middle| \ {}_ia_i \in A_i, \ {}_im_j \in {}_iM_j \right\}.$$

Define the matrix-like addition and matrix-like multiplication on $\mathcal{G}_n(A_i; {}_iM_j)$ as below:

$$({}_im_j) \pm ({}_im'_j) = ({}_im_j \pm {}_im'_j),$$
$$({}_im_j) \cdot ({}_im'_j) = \left(\sum \eta^j_{i,k}({}_im_j \otimes {}_jm'_k)\right).$$

It is clear that this product is associative if and only if the family of diagrams above are commutative. One can check that $\mathcal{G}_n(A_i; {}_iM_j)$ is an $\mathcal{R}$-algebra under the matrix-like addition and the matrix-like multiplication. In this case, $\mathcal{G}_n(A_i; {}_iM_j)$ is said to be *Morita context ring* of order n associated with those bimodules ${}_iM_j (1 \leqslant i \leqslant j \leqslant n)$ and is usually written as

$$\mathcal{G}_n(A_i; {}_iM_j) = \left[\begin{array}{ccccc} A_1 & {}_1M_2 & \cdots & {}_1M_{n-1} & {}_1M_n \\ {}_2M_1 & A_2 & \cdots & {}_2M_{n-1} & {}_2M_n \\ \vdots & \vdots & & \vdots & \vdots \\ {}_{n-1}M_1 & {}_{n-1}M_2 & \cdots & A_{n-1} & {}_{n-1}M_n \\ {}_nM_1 & {}_nM_2 & \cdots & {}_nM_{n-1} & A_n \end{array} \right].$$

Up to isomorphism, arbitrary Morita context ring of order n is a Morita context ring of order 2. Indeed, if $\mathcal{G}_n(A_i; {}_iM_j)$ is a Morita context ring of order n, then there exist $\mathcal{R}$-algebras

$$A = \mathcal{G}_{n-1}(A_i; {}_iM_j) \quad (1 \leqslant i \leqslant j \leqslant n-1), \quad B = A_n,$$

a nonzero (A, B)-bimodule

$$M = \begin{bmatrix} {}_1M_n \\ {}_2M_n \\ \vdots \\ {}_{n-1}M_n \end{bmatrix} = \left\{ \begin{bmatrix} {}_1m_n \\ {}_2m_n \\ \vdots \\ {}_{n-1}m_n \end{bmatrix} \middle| \; {}_im_n \in {}_iM_n, \; 1 \leqslant i \leqslant n-1 \right\}$$

and a nonzero (B, A)-bimodule

$$\begin{aligned} N &= \begin{bmatrix} {}_nM_1 & {}_nM_2 & \cdots & {}_nM_{n-1} \end{bmatrix} \\ &= \left\{ \begin{bmatrix} {}_nm_1 & {}_nm_2 & \cdots & {}_nm_{n-1} \end{bmatrix} \middle| \; {}_nm_j \in {}_nM_j, \; 1 \leqslant j \leqslant n-1 \right\} \end{aligned}$$

such that

$$\mathcal{G}_n(A_i; {}_iM_j) \cong \begin{bmatrix} A & M \\ N & B \end{bmatrix}.$$

A special case of Morita context ring of order n is the case of *tensor generalized matrix algebras* of order n: the modules ${}_sM_r$ with $r - s \geqslant 2$ are tensor products ${}_sM_{s+1} \bigotimes_{A_{s+1}} \cdots \bigotimes_{A_{r-1}} {}_{r-1}M_r$. The role of the morphisms $\eta^j_{i,k}$ is played by the identity morphisms of ${}_iM_j \bigotimes_{A_j} {}_jM_k$, and the associativity of the product of $\mathcal{G}_n(A_i; {}_iM_j)$ results from the associativity of the tensor products. In view of the isomorphism relation between Morita context rings of order 2 and Morita context rings of order n and technical considerations, only Morita context rings of order 2 are studied in this book. Moreover, a Morita context ring is often called a *generalized matrix algebra* as well. In particular, if $A \neq 0$ and $B \neq 0$, then $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ is called nontrivial.

Proposition 1.1.1. *Every unital algebra with nontrivial idempotents is isomorphic to a nontrivial generalized matrix algebra.*

Proof. Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ be a unital algebra over $\mathcal{R}$. Suppose that there exists a nontrivial idempotent $e \in \mathcal{A}$. One can easily construct the following generalized matrix algebra:

$$\begin{aligned} \mathcal{G} &= \begin{bmatrix} e\mathcal{A}e & e\mathcal{A}(1-e) \\ (1-e)\mathcal{A}e & (1-e)\mathcal{A}(1-e) \end{bmatrix} \\ &= \left\{ \begin{bmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{bmatrix} \middle| \; a, b, c, d \in \mathcal{A} \right\}. \end{aligned}$$

According to the routine computation, we can verify the $\mathcal{R}$-linear mapping

$$\begin{aligned} \xi : \mathcal{A} &\longrightarrow \mathcal{G} \\ a &\longmapsto \begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix} \end{aligned}$$

is a homomorphism from $\mathcal{A}$ to $\mathcal{G}$. Moreover, if

$$\begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix} = \begin{bmatrix} ebe & eb(1-e) \\ (1-e)be & (1-e)b(1-e) \end{bmatrix},$$

then $eae = ebe$ and $ea(1-e) = eb(1-e)$. This leads to $ea = eb$. Likewise, we also have $(1-e)ae = (1-e)be$ and $(1-e)a(1-e) = (1-e)b(1-e)$. This gives that $(1-e)a = (1-e)b$. Thus $a = b$ and hence ξ is injective.

On the other hand, for any $\left[\begin{smallmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{smallmatrix}\right] \in \mathcal{G}$, there exists

$$eae + ec(1-e) + (1-e)de + (1-e)b(1-e) \in \mathcal{A}$$

such that

$$\xi(eae + ec(1-e) + (1-e)de + (1-e)b(1-e)) = \left[\begin{smallmatrix} eae & ec(1-e) \\ (1-e)de & (1-e)b(1-e) \end{smallmatrix}\right].$$

So ξ is surjective. Therefore ξ is an isomorphism from $\mathcal{A}$ to $\mathcal{G}$. □

Let $\mathcal{G} = (A, M, N, B)$ be a generalized matrix algebra. If M is faithful as left A-module and as right B-module, then we have the following two lemmas.

Lemma 1.1.2. *The centre of $\mathcal{G}$ is*

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \; am = mb, \; na = bn, \; \forall\, m \in M, \; \forall n \in N \right\}.$$

Proof. It follows from [61, Lemma 1] that the centre $\mathcal{Z}(\mathcal{G})$ consists of all diagonal matrices $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$, where $a \in \mathcal{Z}(A)$, $b \in \mathcal{Z}(B)$ and $am = mb$, $na = bn$ for all $m \in M$, $n \in N$. However, in our situation which M is faithful as a left A-module and also as a right B-module, the conditions that $a \in \mathcal{Z}(A)$ and $b \in \mathcal{Z}(B)$ become redundant and can be deleted. Indeed, if $am = mb$ for all $m \in M$, then for any $a' \in A$ we get

$$(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0.$$

The assumption that M is faithful as a left $\mathcal{A}$-module leads to $aa' - a'a = 0$ and hence $a \in \mathcal{Z}(A)$. Likewise, we also have $b \in \mathcal{Z}(B)$. □

Let us define two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{G} \to A$ and $\pi_B : \mathcal{G} \to B$ by

$$\pi_A : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \longmapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ n & b \end{bmatrix} \longmapsto b.$$

By Lemma 1.1.2 it is easy to see that $\pi_A(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(A)$ and that $\pi_B(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(B)$.

Lemma 1.1.3. *There exists a unique algebraic isomorphism*

$$\varphi : \pi_A(\mathcal{Z}(\mathcal{G})) \longrightarrow \pi_B(\mathcal{Z}(\mathcal{G}))$$

such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, $m \in M$, $n \in N$.

Proof. For a fixed $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, if $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right], \left[\begin{smallmatrix} a & 0 \\ 0 & b' \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$, we have $am = mb = mb'$ for any $m \in M$. Since M is faithful as a right B-module, $b = b'$. That means there exists a unique $b \in \pi_B(\mathcal{Z}(\mathcal{G}))$, which is denoted by $\varphi(a)$, such that $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$. Thus $\left[\begin{smallmatrix} a & 0 \\ 0 & \varphi(a) \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & m \\ n & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & m \\ n & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & 0 \\ 0 & \varphi(a) \end{smallmatrix}\right]$ for all $m \in M$, $n \in N$. So $am = m\varphi(a)$, $na = \varphi(a)n$ for any $m \in M$, $n \in N$. We easily observe that the mapping φ is also surjective. It remains to show that φ is an algebraic isomorphism.

For any $a, a' \in \pi_A(\mathcal{Z}(\mathcal{G}))$ and $r \in \mathcal{R}$, we have

$$(ra)m = r(am) = r(m\varphi(a)) = m(r\varphi(a)),$$

$$(a + a')m = m(\varphi(a) + \varphi(a'))$$

and

$$(aa')m = a(a'm) = (a'm)\varphi(a) = a'(m\varphi(a)) = m\varphi(a)\varphi(a').$$

Therefore $\varphi(ra) = r\varphi(a)$, $\varphi(a + a') = \varphi(a) + \varphi(a')$ and $\varphi(aa') = \varphi(a)\varphi(a')$, and these facts complete the proof of the lemma. □

Let 1_A (resp. 1_B) be the identity of the algebra A (resp. B), and let I be the identity of the Morita context ring $\mathcal{G}$. Sometimes, we use the following notations:

$$P = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = I - P = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$$

and

$$\mathcal{G}_{11} = P\mathcal{G}P, \ \ \mathcal{G}_{12} = P\mathcal{G}Q, \ \ \mathcal{G}_{21} = Q\mathcal{G}P, \ \ \mathcal{G}_{22} = Q\mathcal{G}Q.$$

Thus the Morita context ring $\mathcal{G}$ can be written as

$$\mathcal{G} = P\mathcal{G}P + P\mathcal{G}Q + Q\mathcal{G}P + Q\mathcal{G}Q = \mathcal{G}_{11} + \mathcal{G}_{12} + \mathcal{G}_{21} + \mathcal{G}_{22}.$$

$\mathcal{G}_{11}$ and $\mathcal{G}_{22}$ are subalgebras of $\mathcal{G}$ which are isomorphic to A and B, respectively. $\mathcal{G}_{12}$ is a $(\mathcal{G}_{11}, \mathcal{G}_{22})$-bimodule which is isomorphic to the (A, B)-bimodule M. $\mathcal{G}_{21}$ is a $(\mathcal{G}_{22}, \mathcal{G}_{11})$-bimodule which is isomorphic to the (B, A)-bimodule N.

1.2 Classical matrix algebras

In this section, we represent *full matrix algebras, upper (lower) triangular matrix algebra, block upper triangular matrix algebra* as Morita context rings.

1.2.1 Full matrix algebras

Let $\mathcal{R}$ be a commutative ring with identity, A be a unital $\mathcal{R}$-algebra and $M_n(A)$ be the algebra consisting of all $n \times n$ matrices over A ($n \geqslant 2$). Then the *full matrix algebra* $M_n(A)$ can be represented as a Morita context ring of the form

$$M_n(A) = \begin{bmatrix} A & M_{1\times(n-1)}(A) \\ M_{(n-1)\times 1}(A) & M_{n-1}(A) \end{bmatrix}.$$

1.2.2 Triangular matrix algebras

Denote the set of all $p \times q$ matrices over the $\mathcal{R}$-algebra A by $M_{p\times q}(A)$. Let us denote the set of all $n \times n$ upper triangular matrices over A and the set of all $n \times n$ lower triangular matrices over A by $T_n(A)$ and $T'_n(A)$, respectively. For $n \geqslant 2$ and each $1 \leqslant k \leqslant n-1$, the *upper triangular matrix algebra* $T_n(A)$ and *lower triangular matrix algebra* $T'_n(A)$ can be written as

$$T_n(A) = \begin{bmatrix} T_k(A) & M_{k\times(n-k)}(A) \\ O & T_{n-k}(A) \end{bmatrix}$$

and

$$T'_n(A) = \begin{bmatrix} T'_k(A) & O \\ M_{(n-k)\times k}(A) & T'_{n-k}(A) \end{bmatrix},$$

respectively.

1.2.3 Block upper triangular matrix algebras

Let $\mathbf{N}$ be the set of all positive integers and let $n \in \mathbf{N}$. For any positive integer m with $m \leqslant n$, we denote by $\bar{d} = (d_1, \cdots, d_i, \cdots, d_m) \in \mathbf{N}^m$ an ordered m-vector of positive integers such that $n = d_1 + \cdots + d_i + \cdots + d_m$. The *block upper triangular matrix algebra* $B_n^{\bar{d}}(A)$ is a subalgebra of $M_n(A)$ with form

$$B_n^{\bar{d}}(A) = \begin{bmatrix} M_{d_1}(A) & \cdots & M_{d_1 \times d_i}(A) & \cdots & M_{d_1 \times d_m}(A) \\ & \ddots & \vdots & & \vdots \\ & & M_{d_i}(A) & \cdots & M_{d_i \times d_m}(A) \\ & O & & \ddots & \vdots \\ & & & & M_{d_m}(A) \end{bmatrix}.$$

Likewise, the *block lower triangular matrix algebra* $B_n'^{\bar{d}}(A)$ is a subalgebra of $M_n(A)$ with form

$$B_n'^{\bar{d}}(A) = \begin{bmatrix} M_{d_1}(A) & & & & \\ \vdots & \ddots & & O & \\ M_{d_i \times d_1}(A) & \cdots & M_{d_i}(A) & & \\ \vdots & & \vdots & \ddots & \\ M_{d_m \times d_1}(A) & \dots & M_{d_m \times d_i}(A) & \dots & M_{d_m}(A) \end{bmatrix}.$$

Note that the full matrix algebra $M_n(A)$ of all $n \times n$ matrices over A and the upper(resp. lower) triangular matrix algebra $T_n(A)$ of all $n \times n$ upper triangular matrices over A are two special cases of block upper(resp. lower) triangular matrix algebras. If $n \geqslant 2$ and $B_n^{\bar{d}}(A) \neq M_n(A)$, then $B_n^{\bar{d}}(A)$ is an upper triangular algebra and can be written as

$$B_n^{\bar{d}}(A) = \begin{bmatrix} B_j^{\bar{d}_1}(A) & M_{j \times (n-j)}(A) \\ O_{(n-j) \times j} & B_{n-j}^{\bar{d}_2}(A) \end{bmatrix},$$

where $1 \leqslant j < m$ and $\bar{d}_1 \in \mathbf{N}^j, \bar{d}_2 \in \mathbf{N}^{m-j}$. Similarly, if $n \geqslant 2$ and $B_n'^{\bar{d}}(A) \neq M_n(A)$, then $B_n'^{\bar{d}}(A)$ is a lower triangular algebra and can be represented as

$$B_n'^{\bar{d}}(A) = \begin{bmatrix} B_j'^{\bar{d}_1}(A) & O_{j \times (n-j)} \\ M_{(n-j) \times j}(A) & B_{n-j}'^{\bar{d}_2}(A) \end{bmatrix},$$

where $1 \leqslant j < m$ and $\bar{d}_1 \in \mathbf{N}^j, \bar{d}_2 \in \mathbf{N}^{m-j}$.

Let $\mathbb{K}$ be a field of characteristic zero. The block upper (resp. lower) triangular matrix algebra $B_n^{\bar{d}}(\mathbb{K})$ (resp. $B_n'^{\bar{d}}(\mathbb{K})$) over $\mathbb{K}$ naturally arise in any finite-dimensional algebra and also implies that any finite-dimensional algebra contains sufficiently many subalgebras of the type $B_n^{\bar{d}}(\mathbb{K})$ and subalgebras of the type $B_n'^{\bar{d}}(\mathbb{K})$. $B_n^{\bar{d}}(\mathbb{K})$ and $B_n'^{\bar{d}}(\mathbb{K})$ are extensively applied in studying the exponent growth of various varieties of associative algebras over $\mathbb{K}$ (see [38, 39]).

1.2.4 Inflated algebras

Let A be a unital $\mathcal{R}$-algebra and V be an $\mathcal{R}$-linear space. Given an $\mathcal{R}$-bilinear form $\gamma : V \otimes_{\mathcal{R}} V \to A$, we define an associative algebra (not necessarily with identity) $B = B(A, V, \gamma)$ as follows: as an $\mathcal{R}$-linear space, B equals to $V \otimes_{\mathcal{R}} V \otimes_{\mathcal{R}} A$. The multiplication is defined as follows:

$$(a \otimes b \otimes x) \cdot (c \otimes d \otimes y) := a \otimes d \otimes x\gamma(b, c)y$$

for all $a, b, c, d \in V$ and any $x, y \in A$. This definition makes B become an associative $\mathcal{R}$-algebra and B is called an *inflated algebra* of A along V. The inflated algebras are closely connected with the cellular algebras which are extensively studied in representation theory. We refer the reader to [60] and the references therein for these algebras.

Let us assume that V is a non-zero linear space with a basis $\{v_1, \cdots, v_n\}$. Then the bilinear form γ can be characterized by an $n \times n$ matrix $\boldsymbol{M}$ over A, that is, $\boldsymbol{M} = (\gamma(v_i, v_j))$ for $1 \leqslant i, j \leqslant n$. Now we could define a new multiplication "$\circ$" on the full matrix algebra $M_n(A)$ by

$$\boldsymbol{X} \circ \boldsymbol{Y} := \boldsymbol{XMY} \quad \text{for all} \quad \boldsymbol{X}, \boldsymbol{Y} \in M_n(A).$$

Under the usual matrix addition and the new multiplication "$\circ$", $M_n(A)$ becomes a new associative algebra which is a generalized matrix algebra in the sense of Brown (see [19]). We denote this new algebra by $(M_n(A), \boldsymbol{M})$. It should be remarked that our current generalized matrix algebras contain all generalized matrix algebras defined by Brown in [19] as special cases. By [60, Lemma 4.1], the inflated algebra $B(A, V, \gamma)$ is isomorphic to $(M_n(A), \boldsymbol{M})$ and hence is a generalized matrix algebra in the sense of ours.