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郭柏灵院士

序

今年是恩师郭柏灵院士70寿辰，华南理工大学出版社决定出版《郭柏灵论文集》。郭老师的弟子，也就是我的师兄弟，推举我为文集作序。这使我深感荣幸。我于1985年考入北京应用物理与计算数学研究所，师从郭柏灵院士和周毓麟院士。研究生毕业后我留在研究所工作，继续跟随郭老师学习和研究偏微分方程理论。老师严谨的治学作风和对后学的精心培养与殷切期望，给我留下了深刻的印象，同时老师在科研上的刻苦精神也一直深深地印在我的脑海中。

郭老师1936年生于福建省龙岩市新罗区龙门镇，1953年从福建省龙岩市第一中学考入复旦大学数学系，毕业后留校工作。1963年，郭老师服从祖国的需要，从复旦大学调入北京应用物理与计算数学研究所，从事核武器研制中有关的数学、流体力学问题及其数值方法研究和数值计算工作。他全力以赴地做好了这项工作，为我国核武器的发展作出了积极的贡献。1978年改革开放以后，他又在非线性发展方程数学理论及其数值方法领域开展研究工作，现为该所研究员、博士生导师，中国科学院院士。迄今他共发表学术论文300余篇、专著9部，1987年获国家自然科学三等奖，1994年和1998年两度获得国防科工委科技进步一等奖，为我国的国防建设与人才培养作出了巨大贡献。

郭老师的研究方向涉及数学的多个领域，其中包括非线性发展方程的数学理论及其数值解、孤立子理论、无穷维动力系统等，其研究工作的主要特点是紧密联系数学物理中提出的各种重要问题。他对力学及物理学等应用学科中出现的许多重要的非线性发展方程进行了系统深入的研究，其中对Landau－Lifshitz方程和Benjamin－Ono方程的大初值的整体可解性、解的唯一性、正则性、渐近行为以及爆破现象等建立了系统而深刻的数学理论。在无穷维动力系统方面，郭老师研究了一批重要的无穷维动力系统，建立了有关整体吸引子、惯性流形和近似惯性流形的存在性和分形维数精细估计等理论，提出了一种证明强紧吸引子的新方法，并利用离散化等方法进行理论分析和数值计算，展示了吸引子的结构和图像。下面我从这几个方面介绍郭老师的一些学术成就。

Landau－Lifshitz方程（又称铁磁链方程）由于其结构的复杂性，特别是强耦合性和不带阻尼时的强退化性，在20世纪80年代之前国内外几乎没有从数学上进行理论研究的成果出现。最先进行研究的，当属周毓麟院士和郭老师，他们在1982年到1986年间，采用Leray－Schauder不动点定理、离散方法、Galerkin方法证明了从一维到高维的各种边值问题整体弱解的存在性，比国外在1992年才出现的同类结果早了将近10年。

20世纪90年代初期，周毓麟、郭柏灵和谭绍滨，郭柏灵和洪敏纯得到了两个在国内外至今影响很大的经典结果。第一，通过差分方法结合粘性消去法，利用十分巧妙的先验估计，证明了一维Landau－Lifshitz方程光滑解的存在唯

一性,对于一维问题给出了完整的答案,解决了长期悬而未决的难题。第二,系统分析了带阻尼的二维 Landau – Lifshitz 方程弱解的奇性,发现了 Landau – Lifshitz 方程与调和映照热流的联系,其弱解具有与调和映照热流完全相同的奇性。现在,国内外这方面的文章基本上都引用这个结果。调和映照的 Landau – Lifshitz 流的概念,即是源于此项结果。

20 世纪 90 年代中期,郭老师对于 Landau – Lifshitz 方程的长时间性态、Landau – Lifshitz 方程耦合 Maxwell 方程的弱解及光滑解的存在性问题进行了深入的研究,得到了一系列的成果。铁磁链方程的退化性以及缺少相应的线性化方程解的表达式,对研究解的长时间性态带来很大困难。郭老师的一系列成果克服了这些困难,证明了近似惯性流形的存在性、吸引子的存在性,给出了其 Hausdorff 和分形维数的上、下界的精细估计。此外,我们知道,与调和映照热流比较,高维铁磁链方程的研究至今还很不完善。其中最重要的是部分正则性问题,其难点在于单调不等式不成立,导致能量衰减估计方面的困难。另外一个是 Blow-up 解的存在性问题,至今没有解决;而对于调和映照热流来说,这样的问题的研究是比较成熟的。

对于高维问题,20 世纪 90 年代后期至今,郭老师和陈韵梅、丁时进、韩永前、杨干山一道,得出了许多成果,大大地推动了该领域的研究。首先,证明了二维问题的能量有限弱解的几乎光滑性及唯一性,这个结果类似于 Freire 关于调和映照热流的结果。第二,得到了高维 Landau – Lifshitz 方程初边值问题的奇点集合的 Hausdorff 维数和测度的估计。第三,得到了三维 Landau – Lifshitz – Maxwell 方程的奇点集合的 Hausdorff 维数和测度的估计。第四,得到了一些高维轴对称问题的整体光滑解和奇性解的精确表达式。郭老师还开创了一些新的研究领域。例如,关于一维非均匀铁磁链方程光滑解的存在唯一性结果后来被其他数学家引用并推广到一般流形上。其次,率先讨论了可压缩铁磁链方程测度值解的存在性。最近,在 Landau – Lifshitz 方程耦合非线性 Maxwell 方程方面,也取得了许多新的进展。

多年来,郭老师还对一大批非线性发展方程解的整体存在唯一性、有限时刻的爆破性、解的渐近性态等开展了广泛而深入的研究,受到国内外同行的广泛关注。研究的模型源于数学物理、水波、流体力学、相变等领域,如含磁场的 Schrödinger 方程组、Zakharov 方程、Schrödinger – Boussinesq 方程组、Schrödinger – KdV 方程组、长短波方程组、Maxwell 方程组、Davey – Stewartson 方程组、Klein – Gordon – Schrödinger 方程组、波动方程、广义 KdV 方程、Kadomtsev – Petviashvili(KP)方程、Benjamin – Ono 方程、Newton – Boussinesq 方程、Cahn – Hilliard 方程、Ginzburg – Landau 方程等。其中不少耦合方程组都是郭老师得到了第一个结果,开创了研究的先河,对国内外同行的研究产生了深远的影响。

郭老师在无穷维动力系统方面也开展了广泛的研究,取得了丰硕的成果。对耗散非线性发展方程所决定的无穷维动力系统,研究了整体吸引子的存在性、分形维数估计、惯性流形、近似惯性流形、指数吸引子等问题。特别是在研究无界域上耗散非线性发展方程的强紧整体吸引子存在性时所提出的化弱紧吸引子成为强紧吸引子的重要方法和技巧,颇受同行关注并广为利用。对五次非线性 Ginzburg-Landau 方程,郭老师利用空间离散化方法将无限维问题化为有限维问题,证明了该问题离散吸引子的存在性,并考虑五次 Ginzburg-Landau 方程的定态解、慢周期解、异宿轨道等的结构。利用有限维动力系统的理论和方法,结合数值计算得到具体的分形维数(不超过 4)和结构以及走向混沌、湍流的具体过程和图像,这是一种寻求整体吸引子细微结构新的探索和尝试,对其他方程的研究也是富有启发性的。1999 年以来,郭老师集中于近可积耗散的和 Hamilton 无穷维动力系统的结构性研究,利用孤立子理论、奇异摄动理论、Fenichel 纤维理论和无穷维 Melnikov 函数,对于具有小耗散的三次到五次非线性 Schrödinger 方程,证明了同宿轨道的不变性,并在有限维截断下证明了 Smale 马蹄的存在性,目前,正把这一方法应用于具小扰动的 Hamilton 系统的研究上。他对于非牛顿流无穷维动力系统也进行了系统深入的研究,建立了有关的数学理论,并把有关结果写成了专著。以上这些工作得到国际同行们的高度评价,被称为“有重大的国际影响”、“对无穷维动力系统理论有重要持久的贡献”。最近,郭老师及其合作者又证明了具耗散的 KdV 方程 L^2 整体吸引子的存在性,该结果也是引人注目的。

郭老师不仅自己辛勤地搞科研,还尽心尽力培养了大批的研究生(硕士生、博士生、博士后),据不完全统计,有 40 多人。他根据每个人不同的学习基础和特点,给予启发式的具体指导,其中的不少人已成为了该领域的学科带头人,有些人虽然开始时基础较差,经过培养,也得到了很大提高,成为该方向的业务骨干。

《郭柏灵论文集》按照郭老师在不同时期所从事的研究领域,分成多卷出版。文集中所搜集的都是郭老师正式发表过的学术成果。把这些成果整理成集出版,不仅系统地反映了他的科研成就,更重要的是对于从事这方面学习、研究的学者无疑大有裨益。这本文集的出版得到了多方面的帮助与支持,特别要感谢华南理工大学校长李元元教授、华南理工大学出版社范家巧社长和华南理工大学数学科学学院吴敏院长的支持。还要特别感谢华南理工大学的李用声教授、华南师范大学的丁时进教授、北京应用物理与计算数学研究所的苗长兴研究员等在论文的搜集、选择与校对等工作中付出了辛勤的劳动。感谢华南理工大学出版社的编辑对文集的精心编排。

谭绍滨

2005 年 8 月于厦门大学

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Diffusion Limit of Small Mean Free Path of Transfer Equation in $\mathbf{R}^3$ *

Guo Boling(郭柏灵) Han Yongqian(韩永前)

Abstract

This article is devoted to establish the well-posedness of solutions and diffusion limit of the small mean free path of the nonlinear transfer equations, which describes the spatial transport of radiation in a material medium. By using the comparison principle, we obtain the lower bound and upper bound of the solution, and then we prove the existence and uniqueness of the global solution. We show that the nonlinear transfer equation has a diffusion limit as the mean free path tends to zero. Our proof is based on asymptotic expansions. We show that the validity of these asymptotic expansions relies only on the smoothness of initial data, while two hypotheses, Fredholm alternative and centering condition, are removed.

Keywords transfer equation; asymptotic limit

1 Introduction

The spatial transport of radiation and its interaction with matter plays an important and sometimes dominant role in many physical systems (Bowers and Wilson, 1991; Larsen et al., 1983; Pomraning, 1973). Assume that the material is in local thermodynamic equilibrium and ignore material motion and thermal diffusion, the spatial transport of radiation in a material medium is described (Bowers and Wilson, 1991; Larsen et al., 1983; Pomraning, 1973) by the following transfer equation (Tr. Eq.)

$$\partial_t u + \frac{c}{\epsilon}\omega \cdot \nabla_x u + \frac{\sigma}{\epsilon^2} u = \left(\frac{\sigma}{\epsilon^2} - \sigma_a\right)\bar{u} + \sigma_a \beta v^4, \tag{1.1}$$

and the state equation of hydrodynamic

$$\partial_t v + \frac{\epsilon \sigma_a}{C_h}(\beta v^4 - \bar{u}) = 0. \tag{1.2}$$

Here $u(t, x, \omega)$ denotes the specific intensity at time $t \geqslant 0$, at location $x \in \mathbf{R}^3$ with the direction $\omega \in \mathbb{S}^2$ of travel of the photon, the scalar density defined by

$$\bar{u}(t, x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} u(t, x, \omega)\mathrm{d}\omega, \tag{1.3}$$

c is the vacuum speed of light, $\sigma = \sigma_s + \sigma_a$ is the transport coefficient, $\sigma_s > 0$ is the scattering

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* Transport Theory and Statistical Physics, 2011, 40:243 - 281.

coefficient, $\sigma_a > 0$ is the absorption coefficient, $\frac{\epsilon}{\sigma} > 0$ is the mean free path, $\beta > 0$ is called the Stefan-Boltzmann constant, material temperature denotes by v, $C_h > 0$ is the pseudo-heat capacity. In the absence of boundaries, equations (1.1) and (1.2) must be supplemented with initial conditions

$$u(0,x,\omega) = u^0(x,\omega), \quad v(0,x) = v^0(x), \quad x \in \mathbf{R}^3, \quad \omega \in \mathbb{S}^2 \subset \mathbf{R}^3. \qquad (1.4)$$

Here we assume that $u^0 \geqslant 0$ and $v^0 \geqslant 0$.

The asymptotic analysis (Larsen et al., 1983), approximate models, and computational methods of radiative transport equations (1.1) (1.2) are the topic of many recent articles (Adams and Larsen, 2002; Anistratov and Larsen, 2001; Bowers and Wilson, 1991; Larsen, 1982, 1988; Larsen et al., 1983; Li and Feng, 2008; Morel et al., 2006; Pomraning, 1973; Roberts and Anistratov, 2007) and therein references. In all these papers, no attempt has been made to be mathematically rigorous. Instead, physical intuition and physical arguments are emphasized.

This article is devoted to investigate the well-posedness of solutions and diffusion limit of small mean free path ϵ of the equations (1.1), (1.2), (1.4) and to get mathematically rigorous results. In order to establish the existence and uniqueness of the global solution of the equations (1.1), (1.2), (1.4), we need to estimate the lower bound and upper bound of the solution, which is fulfilled by using comparison principle. To obtain the diffusion limit of small mean free path ϵ of the equations (1.1), (1.2), (1.4), we first construct formal asymptotic expansions in the usual way (Bensoussan et al., 1979; Golse et al., 1999) and then verify the validity of these expansions.

The relative topic of the basic linear radiative transfer equation, which describes the propagation of the radiation field is extensively studied in Bardos et al. (1984), Bensoussan et al. (1979), Golse et al. (1999, 2003), Habetler and Matkowsky (1975), Larsen (1992), Larsen and Keller (1974), and Larsen and Miller (1980) and therein references. In Bensoussan et al. (1979), the validity of the asymptotic expansion rests on three general hypotheses: smoothness, Fredholm alternative, and centering condition. But to verify the validity of the asymptotic expansions of equations (1.1), (1.2), (1.4), we need to rely only on the smoothness of initial data u^0 and v^0. It is crucial to remove two hypotheses: Fredholm alternative and centering condition.

Throughout this article, different positive constants are all denoted by the same letter C. If necessary, we denote by $C(\cdot,\cdot)$ a constant depending only on the quantities appearing in parenthesis.

The outline of this article is as follows. In Section 2, we prove the existence and uniqueness of the global solution of equations (1.1), (1.2) and (1.4). In Section 3 and Section 4, we investigate the diffusion limit of small mean free path ϵ for equations (1.1), (1.2) and (1.4). A few concluding remarks are given in Section 5.

2 Existence and Uniqueness of Solution of Transfer Equation

We pass now to the question of existence. First we rewrite equations (1.1) and (1.2) as integral equations:

$$u(t,x,\omega) = u^0(x_{\omega,t},\omega)E(t) + \int_0^t E(t-s)\times\left[\sigma_a\beta v^4 + \left(\frac{\sigma}{\epsilon^2}-\sigma_a\right)\bar{u}\right](s,x_{\omega,s})\mathrm{d}s, \quad (2.1)$$

$$v(t,x) = v^0(x) - \frac{\epsilon\sigma_a}{C_h}\int_0^t \{\beta v^4(s,x) - \bar{u}(s,x)\}\mathrm{d}s, \quad (2.2)$$

where $x_{\omega,t} = x - \frac{ct}{\epsilon}\omega$ and $E(t) = \exp\left(-\frac{\sigma t}{\epsilon^2}\right)$. Here we assume that the mean free path $0 < \epsilon \leqslant 1$.

Theorem 2.1 (Existence and Uniqueness of Local Solution) Let $u^0 \in L^\infty(\mathbf{R}^3\times\mathbb{S}^2)$ and $v^0 \in L^\infty(\mathbf{R}^3)$. Then there exists $T_{\max} > 0$ such that the solution (u, v) of equations (1.1) and (1.2) with initial condition (1.4) is existent and unique. Moreover we have

$$u \in C([0,T_{\max}); L^\infty(\mathbf{R}^3\times\mathbb{S}^2)), \quad v \in C([0,T_{\max}); L^\infty(\mathbf{R}^3)).$$

If $T_{\max} < \infty$, then

$$\overline{\lim}_{t\uparrow T_{\max}}\{\|u(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{L^\infty}\} = \infty. \quad (2.3)$$

Proof Equations (2.1) and (2.2) can be solved by the principle of contraction mapping. Let $T > 0$ and set

$$B = \{(u,v) \mid u \in C([0,T]; L^\infty(\mathbf{R}^3\times\mathbb{S}^2)),\ v \in C([0,T]; L^\infty(\mathbf{R}^3)),$$
$$\max_{0\leqslant t\leqslant T}(\|u(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{L^\infty}) \leqslant 2(\|u^0\|_{L^\infty_{x,\omega}} + \|v^0\|_{L^\infty_x})\}. \quad (2.4)$$

We wish to find conditions on T which imply that the map Φ: $(u,v)\to(\Phi_1(u,v), \Phi_2(u,v))$, given by

$$\Phi_1(u,v)(t,x,\omega) = u^0(x_{\omega,t},\omega)E(t) + \int_0^t E(t-s)\times\left\{\sigma_a\beta v^4 + \left(\frac{\sigma}{\epsilon^2}-\sigma_a\right)\bar{u}\right\}(s,x_{\omega,s})\mathrm{d}s, \quad (2.5)$$

$$\Phi_2(u,v)(t,x) = v^0(x) - \frac{\epsilon\sigma_a}{C_h}\int_0^t\{\beta v^4(s,x) - \bar{u}(s,x)\}\mathrm{d}s, \quad (2.6)$$

is a strict contraction on B. For $0 \leqslant t \leqslant T$, using (2.5), we have

$$\|\Phi_1(u,v)(t,\cdot)\|_{L^\infty_{x,\omega}} \leqslant \|u^0\|_{L^\infty_{x,\omega}} + T\sigma_a(\beta\|v\|^4_{L^\infty_{t,x}} + \|u\|_{L^\infty_{t,x,\omega}}) + T\frac{\sigma}{\epsilon^2}\|u\|_{L^\infty_{t,x,\omega}}. \quad (2.7)$$

Employing (2.6), we obtain

$$\|\Phi_2(u,v)(t,\cdot)\|_{L^\infty_x} \leqslant \|v^0\|_{L^\infty_x} + \frac{T\sigma_a}{C_h}(\beta\|v\|^4_{L^\infty_{t,x}} + \|u\|_{L^\infty_{t,x,\omega}}), \quad \forall t \in [0,T]. \quad (2.8)$$

Let $T \leqslant \frac{1}{\sigma_a(1+1/C_h)(16\beta y_0^3+2)+2\sigma/\epsilon^2}$, where $y_0 = \|u^0\|_{L^\infty_{x,\omega}} + \|v^0\|_{L^\infty_x}$. Then $\Phi: B\to B$.

For any (u_1, v_1), $(u_2, u_2) \in B$, we have $(0 \leqslant t \leqslant T)$

$$\|\{\Phi_1(u_1,v_1)-\Phi_1(u_2,v_2)\}(t,\cdot)\|_{L^\infty_{x,\omega}}$$

$$\leqslant T\sigma_a(32\beta y_0^3\|v_1-v_2\|_{L^\infty_{t,x}}+\|u_1-u_2\|_{L^\infty_{t,x,\omega}})+T\frac{\sigma}{\epsilon^2}\|u_1-u_2\|_{L^\infty_{t,x,\omega}}, \tag{2.9}$$

$$\|\{\Phi_2(u_1,v_1)-\Phi_2(u_2,v_2)\}(t,\cdot)\|_{L^\infty_x}$$

$$\leqslant\frac{T\sigma_a}{C_h}(32\beta y_0^3\|v_1-v_2\|_{L^\infty_{t,x}}+\|u_1-u_2\|_{L^\infty_{t,x,\omega}}). \tag{2.10}$$

Let $T_0=\min\left\{\frac{1}{\sigma_a(1+1/C_h)(16\beta y_0^3+2)+2\sigma/\epsilon^2},\frac{1}{\sigma_a(1+1/C_h)(64\beta y_0^3+2)+2\sigma/\epsilon^2}\right\}$ and $T\leqslant T_0$. Then $\Phi:B\rightarrow B$ is a strict contraction. By the principle of contraction mapping, there exists a unique solution (u,v) of equations (1.1) and (1.2). Moreover, we have $u\in C([0,T_0];L^\infty(\mathbf{R}^3\times\mathbb{S}^2))$, $v\in C([0,T_0];L^\infty(\mathbf{R}^3))$, and

$$\max_{0\leqslant t\leqslant T_0}(\|u(t,\cdot)\|_{L^\infty}+\|v(t,\cdot)\|_{L^\infty})\leqslant 2(\|u^0\|_{L^\infty_{x,\omega}}+\|v^0\|_{L^\infty_x}). \tag{2.11}$$

Now we can solve equations (1.1) and (1.2) at the initial time moment $t=T_0$ instead of $t=0$ with initial data $(u(T_0,x,\omega),\ v(T_0,x))$ instead of (u^0,v^0) again. Then the time domain $[0,\ T_0]$ can be extended, which denotes by $[0,T_1]$, and the unique solution (u,v) of equations (1.1) and (1.2) is well defined for any $t\in[0,\ T_1]$. Here $T_1>T_0$. Repeating this procedure again and again, we can obtain a series $\{T_n\}$ and the existence of $T_{\max}=\sup T_n$.

If $T_{\max}<\infty$ and

$$\overline{\lim}_{t\uparrow T_{\max}}\{\|u(t,\cdot)\|_{L^\infty}+\|v(t,\cdot)\|_{L^\infty}\}<\infty,$$

then

$$K_0=\sup_{t\in[0,T_{\max})}\{\|u(t,\cdot)\|_{L^\infty}+\|v(t,\cdot)\|_{L^\infty}\}<\infty.$$

Let $T_\delta=\min\left\{\frac{1}{\sigma_a(1+1/C_h)(16\beta K_0^3+2)+2\sigma/\epsilon^2},\ \frac{1}{\sigma_a(1+1/C_h)(64\beta K_0^3+2)+2\sigma/\epsilon^2}\right\}$.

Now we solve equations (1.1) and (1.2) at the initial time moment $t=T_{\max}-\frac{1}{2}T_\delta$ instead of $t=0$ with initial data $\left(u\left(T_{\max}-\frac{1}{2}T_\delta,x,\omega\right),v\left(T_{\max}-\frac{1}{2}T_\delta,x\right)\right)$ instead of $(u^0,\ v^0)$. Then the unique solution (u,v) of equations (1.1) and (1.2) is well defined for any $t\in\left[0,T_{\max}+\frac{1}{2}T_\delta\right]$. This is contradictory.

Therefore (2.3) is proved.

If $u^0\geqslant 0$ and $v^0\geqslant 0$, we will prove that the solution of (1.1) and (1.2), which is constructed in Theorem 2.1, is global. First we prove the solution of (1.1) and (1.2) is non-negative.

Lemma 2.2 (Non-negative) Let $0\leqslant u^0\in L^\infty(\mathbf{R}^3\times\mathbb{S}^2)$ and $0\leqslant v^0\in L^\infty(\mathbf{R}^3)$. Then for the solutions (u,v) of (1.1), (1.2) and (1.4), the following estimate is valid:

$$u(t,x,\omega)\geqslant 0,\quad v(t,x)\geqslant\left\{\frac{3\epsilon\sigma_a\beta}{C_h}t+(v^0)^{-3}\right\}^{-1/3}\geqslant 0,$$

$$\forall t\in[0,T_{\max}),\quad x\in\mathbf{R}^3,\quad \omega\in\mathbb{S}^2. \tag{2.12}$$

Proof Let (u,v) be the solution of equations (1.1), (1.2) and (1.4) solved in Theorem 2.1.

Considering v as known function, we re-solve the equation (1.1). That is to solve the following equation

$$w(t,x,\omega)=u^0(x_{\omega,t},\omega)E(t)+\int_0^t E(t-s)\times\left[\sigma_a\beta v^4+\left(\frac{\sigma}{\epsilon^2}-\sigma_a\right)\overline{w}\right](s,x_{\omega,s})\mathrm{d}s, \tag{2.13}$$

where w is unknown function and v is known function. It is obvious that $w=u$ is the unique solution of equation (2.13), which the uniqueness follows as in the proof of Theorem 2.1 by the principle of contraction mapping.

On the other hand, we can resolve equation (2.13) by iteration. We let

$$w^{(1)}(t,x,\omega)=u^0(x_{\omega,t},\omega)E(t)+\int_0^t E(t-s)\sigma_a\beta v^4(s,x_{\omega,s})\mathrm{d}s, \tag{2.14}$$

$$w^{(n+1)}(t,x,\omega)=\int_0^t E(t-s)\left(\frac{\sigma}{\epsilon^2}-\sigma_a\right)\overline{w^{(n)}}(s,x_{\omega,s})\mathrm{d}s, \tag{2.15}$$

where $n=1,2,\cdots$. $u^0\geqslant 0$ implies that

$$w^{(n)}\geqslant 0,\quad n=1,2,\cdots. \tag{2.16}$$

For any $T\in(0,T_{\max})$, we find that

$$\|w^{(1)}(t,\cdot)\|_{L^\infty_{x,\omega}}\leqslant C(T),\quad \forall t\in[0,T] \tag{2.17}$$

and inductively

$$\|w^{(n)}(t,\cdot)\|_{L^\infty_{x,\omega}}\leqslant\frac{(C'(\epsilon)t)^n}{n!}C(T),\quad \forall t\in[0,T], \tag{2.18}$$

where $C(T)$ and $C'(\epsilon)$ are constants. Since the solution w^* of equation (2.13) is formally

$$w^*(t,x,\omega)=\sum_{n=1}^{\infty}w^{(n)}(t,x,\omega),\quad \forall t\in[0,T],$$

our estimates show that this series converges and $w^*(t,x,\omega)\geqslant 0$ exists for all $t\in[0,T]$.

By the uniqueness of solution of equation (2.13), we have $u(t,x,\omega)=w^*(t,x,\omega)\geqslant 0$ for all $t\in[0,T]$, $x\in\mathbf{R}^3$ and $\omega\in\mathbb{S}^2$. Since $T\in(0,T_{\max})$ is arbitrary, we can obtain that $u(t,x,\omega)\geqslant 0$ for all $t\in[0,T_{\max})$, $x\in\mathbf{R}^3$ and $\omega\in\mathbb{S}^2$.

Using equation (1.2) and $u\geqslant 0$, we have

$$\partial_t v+\frac{\epsilon\sigma_a\beta}{C_h}v^4\geqslant 0. \tag{2.19}$$

There exists a unique solution

$$w(t,x)=\left[\frac{3\epsilon\sigma_a\beta}{C_h}t+(v^0)^{-3}\right]^{-1/3}$$

of equations

$$\begin{cases}\partial_t w + \dfrac{\epsilon \sigma_a \beta}{C_h} w^4 = 0 \\ w|_{t=0} = v^0\end{cases}.$$

By comparison principle, we get $v(t,x) \geqslant w(t,x)$, $\forall t \in [0, T_{\max})$, $x \in \mathbf{R}^3$.

Let us construct a stationary state of equations (1.1) and (1.2) as follows:

$$u_s = \max\left\{ \| u^0 \|_{L^\infty_{x,\omega}}, \beta \| v^0 \|^4_{L^\infty} \right\}, \quad v_s = (u_s/\beta)^{1/4}. \tag{2.20}$$

We will prove that the solution of equations (1.1), (1.2) and (1.4) is bounded by (u_s, v_s).

Lemma 2.3 (Upper Bounded) Let $0 \leqslant u^0 \in L^\infty(\mathbf{R}^3 \times \mathbb{S}^2)$ and $0 \leqslant v^0 \in L^\infty(\mathbf{R}^3)$. Then for the solutions (u, v) of (1.1), (1.2) and (1.4), the following estimate is valid:

$$0 \leqslant u(t,x,\omega) \leqslant u_s, \quad 0 \leqslant v(t,x) \leqslant v_s, \quad \forall t \in [0, T_{\max}), x \in \mathbf{R}^3, \omega \in \mathbb{S}^2 \tag{2.21}$$

Proof Let us define

$$U(t,x,\omega) = u_s - u(t,x,\omega), \quad V(t,x) = v_s - v(t,x), \quad \forall t \in [0, T_{\max}) \tag{2.22}$$

where (u, v) is the solution of equations (1.1), (1.2) and (1.4) established in Theorem 2.1. Then (U, V) defined by (2.22) is the unique solution of the following equations

$$\partial_t U + \frac{c\omega}{\epsilon}\partial_x U + \frac{\sigma}{\epsilon^2} U = \left(\frac{\sigma}{\epsilon^2} - \sigma_a\right)\overline{U} + \sigma_a \beta f(v, v_s) V, \quad t \in [0, T_{\max}), \tag{2.23}$$

$$\partial_t V + \frac{\epsilon \sigma_a}{C_h}(\beta f(v, v_s) V - \overline{U}) = 0, \quad t \in [0, T_{\max}), \tag{2.24}$$

$$U(t,x,\omega)|_{t=0} = u_s - u^0 \geqslant 0, \quad V(t,x)|_{t=0} = v_s - v^0 \geqslant 0. \tag{2.25}$$

Here $f(v, v_s) = v^3 + v^2 v_s + v v_s^2 + v_s^3$. By the principle of contraction mapping, the uniqueness follows as in the proof of Theorem 2.1.

It implies (2.21) that U and V are non-negative. The following iteration argument leads directly to the desired results as we now show. Let

$$U^{(0)}(t,x,\omega) = (u_s - u^0)(x_{\omega,t}, \omega) E(t), \tag{2.26}$$

$$\begin{cases}\partial_t V^{(0)} + \dfrac{\epsilon \sigma_a}{C_h}\left[\beta f(v, v_s) V^{(0)} - \overline{U^{(0)}}\right] = 0, \\ V^{(0)}(t,x)|_{t=0} = v_s - v^0,\end{cases} \tag{2.27}$$

$$U^{(n+1)}(t,x,\omega) = \int_0^t E(t-s)\left[\sigma_a \beta f(v, v_s) V^{(n)} + \left(\frac{\sigma}{\epsilon^2} - \sigma_a\right)\overline{U^{(n)}}\right](s, x_{\omega,s}) ds, \tag{2.28}$$

$$\begin{cases}\partial_t V^{(n+1)} + \dfrac{\epsilon \sigma_a}{C_h}\left[\beta f(v, v_s) V^{(n+1)} - \overline{U^{(n)}}\right] = 0, \\ V^{(n+1)}(t,x)|_{t=0} = 0,\end{cases} \tag{2.29}$$

where $n = 0, 1, 2, \cdots$. $u_s - u^0 \geqslant 0$ implies that $U^{(0)} \geqslant 0$ and

$$\partial_t V^{(0)} + \frac{\epsilon \sigma_a \beta}{C_h} f(v, v_s) V^{(0)} \geqslant 0, \quad t \in [0, T_{\max}). \tag{2.30}$$

There exists a unique solution

$$w(t,x) = (v_s - v^0)\exp\left[-\frac{\epsilon\sigma_a\beta}{C_h}\int_0^t f(v(\tau),v_s)\mathrm{d}\tau\right], \quad t\in[0,T_{\max})$$

of equations

$$\begin{cases} \partial_t w + \dfrac{\epsilon\sigma_a\beta}{C_h} f(v,v_s) w = 0 \\ w|_{t=0} = v_s - v^0 \end{cases}.$$

By comparison principle, we get $V^{(0)}(t,x)\geqslant w(t,x)\geqslant 0$, $\forall t\in[0,T_{\max})$, $x\in\mathbf{R}^3$. By induction, we have

$$U^{(n)}(t,x,\omega)\geqslant 0, \quad V^{(n)}(t,x)\geqslant 0, \quad \forall t\in[0,T_{\max}), x\in\mathbf{R}^3, \omega\in\mathbb{S}^2, n\geqslant 1. \tag{2.31}$$

For any $T\in(0,T_{\max})$ and $t\in[0,T]$, we find that

$$\|U^{(0)}(t,\cdot)\|_{L^\infty_{x,\omega}}\leqslant u_s, \tag{2.32}$$

$$\|V^{(0)}(t,\cdot)\|_{L^\infty_x}\leqslant v_s + \frac{\epsilon^3\sigma_a}{C_h\sigma}u_s, \tag{2.33}$$

$$\|U^{(1)}(t,\cdot)\|_{L^\infty_{x,\omega}}\leqslant \frac{\sigma}{\epsilon^2}u_s t + \sigma_a\beta\|f\|_{L^\infty_{t,x,\omega}}\left(v_s + \frac{\epsilon^3\sigma_a}{C_h\sigma}u_s\right)t, \tag{2.34}$$

$$\|V^{(1)}(t,\cdot)\|_{L^\infty_x}\leqslant \frac{\epsilon\sigma_a}{C_h}u_s t, \tag{2.35}$$

and inductively

$$\|U^{(n)}(t,\cdot)\|_{L^\infty_{x,\omega}}\leqslant \frac{C_1^n t^n}{n!}\left(u_s + v_s + \frac{\epsilon^3\sigma_a}{C_h\sigma}u_s\right), \tag{2.36}$$

$$\|V^{(n+1)}(t,\cdot)\|_{L^\infty_x}\leqslant \frac{\epsilon\sigma_a}{C_h}\int_0^t\|U^{(n)}(s,\cdot)\|_{L^\infty_{x,\omega}}\mathrm{d}s, \tag{2.37}$$

where $n\geqslant 1$ and constant $C_1 = \frac{\sigma}{\epsilon^2} + \sigma_a\beta\|f\|_{L^\infty_{t,x,\omega}} + \frac{\epsilon\sigma_a}{C_h}$. Since the solution (U^*, V^*) of equations (2.23), (2.24) and (2.25) is formally

$$U^*(t,x,\omega) = \sum_{n=0}^{\infty}U^{(n)}(t,x,\omega), \quad \forall t\in[0, T],$$

$$V^*(t,x) = \sum_{n=0}^{\infty}V^{(n)}(t,x), \quad \forall t\in[0, T],$$

our estimates show that these two series converge and $(U^*(t,x,\omega), V^*(t,x))$ exists for all $t\in[0,T]$. Moreover, $U^*(t,x,\omega)\geqslant 0$ and $V^*(t,x)\geqslant 0$ for all $t\in[0,T]$, $x\in\mathbf{R}^3$ and $\omega\in\mathbb{S}^2$.

By the uniqueness of solution of equations (2.23) and (2.24), we have $U(t,x,\omega) = U^*(t,x,\omega)\geqslant 0$ and $V(t,x) = V^*(t,x)\geqslant 0$ for all $t\in[0,T]$, $x\in\mathbf{R}^3$ and $\omega\in\mathbb{S}^2$. Since $T\in(0,T_{\max})$ is arbitrary, we can obtain that $U(t,x,\omega)\geqslant 0$ and $V(t,x)\geqslant 0$ for all $t\in[0, T_{\max})$, $x\in\mathbf{R}^3$ and $\omega\in\mathbb{S}^2$. It implies (2.21) that U and V are non-negative.

As a consequence of (2.21), the solution of equations (1.1), (1.2) and (1.4) is global.

Theorem 2.4 (Global Solution) Let $0\leqslant u^0\in L^\infty(\mathbf{R}^3\times\mathbb{S}^2)$ and $0\leqslant v^0\in L^\infty(\mathbf{R}^3)$. Then the solution of equations (1.1), (1.2) and (1.4), constructed in Theorem 2.1, is global.

3 Diffusion Approximations Ⅰ: $C_h=O(\epsilon)$

We wish to describe the asymptotic behavior of the solution (u,v) of equations (1.1) and (1.2) as $\epsilon\to 0$. To this end, and under liberal smoothness assumptions, we first construct an asymptotic expansion for (u,v). Then we prove that the expansions are truly asymptotic.

In this section, let $C_h=O(\epsilon)$ and denote C_h by $C_h\epsilon$.

Let us introduce explicitly the variable $\tau=t/\epsilon^2$ and let us seek an expansion of the form

$$\begin{cases} u(t,x,\omega)=\sum_{k=0}^{\infty}\epsilon^k u_k(t,x,\omega)+\sum_{k=0}^{\infty}\epsilon^k u_k^{IL}(\tau,x,\omega) \\ v(t,x)=\sum_{k=0}^{\infty}\epsilon^k v_k(t,x)+\sum_{k=0}^{\infty}\epsilon^k v_k^{IL}(\tau,x) \end{cases}, \tag{3.1}$$

with the following initial conditions

$$\begin{cases} u_0(0,x,\omega)+u_0^{IL}(0,x,\omega)=u^0(x,\omega) \\ v_0(0,x)+v_0^{IL}(0,x)=v^0(x) \end{cases}. \tag{3.2}$$

Here the superscript IL stands for "initial layer." Inserting (3.1) into equations (1.1) and (1.2), we balance the terms order by order in ϵ.

First, by balancing the $O(1/\epsilon^2)$, we have

$$u_0(t,x,\omega)=\bar{u}_0(t,x)=\frac{1}{4\pi}\int_{\mathbb{S}^2}u_0(t,x,\omega)\mathrm{d}\omega, \tag{3.3}$$

$$\partial_\tau u_0^{IL}+\sigma u_0^{IL}=\sigma\bar{u}_0^{IL}, \tag{3.4}$$

$$\partial_\tau v_0^{IL}=0. \tag{3.5}$$

Integrating equation (3.4) respect to ω over $\mathbb{S}^2$, we get $\partial_\tau\bar{u}_0^{IL}=0$. Then

$$\bar{u}_0^{IL}(\tau,x)=\bar{u}_0^{IL}(0,x)=\bar{u}^0(x)-\bar{u}_0(0,x). \tag{3.6}$$

Since $\bar{u}_0^{IL}(\tau,x)\to 0$ as $\tau\to+\infty$, we have $\bar{u}_0^{IL}(\tau,x)\equiv 0$ for any $x\in\mathbf{R}^3$ and $\tau\geqslant 0$. Therefore,

$$u_0(0,x,\omega)=\bar{u}_0(0,x)=\bar{u}^0(x)=\frac{1}{4\pi}\int_{\mathbb{S}^2}u^0(x,\omega)\mathrm{d}\omega \tag{3.7}$$

and

$$\begin{cases} \partial_\tau u_0^{IL}+\sigma u_0^{IL}=0, \\ u_0^{IL}(0,x,\omega)=u^0(x,\omega)-\bar{u}^0(x). \end{cases} \tag{3.8}$$

Then there exists a unique solution

$$u_0^{IL}(\tau,x,\omega)=\{u^0(x,\omega)-\bar{u}^0(x)\}\mathrm{e}^{-\sigma\tau} \tag{3.9}$$

of equation (3.8). Since $v_0^{IL}(\tau,x)\to 0$ as $\tau\to+\infty$, we obtain $v_0^{IL}(\tau,x)\equiv 0$ for any $x\in\mathbf{R}^3$ and $\tau\geqslant 0$. Thus we have