CLASSICS IN MATHEMATICS

Thomas M. Liggett

Interacting Particle Systems

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To my family:
Chris, Tim, Amy

Preface

At what point in the development of a new field should a book be written about it? This question is seldom easy to answer. In the case of interacting particle systems, important progress continues to be made at a substantial pace. A number of problems which are nearly as old as the subject itself remain open, and new problem areas continue to arise and develop. Thus one might argue that the time is not yet ripe for a book on this subject. On the other hand, this field is now about fifteen years old. Many important problems have been solved and the analysis of several basic models is almost complete. The papers written on this subject number in the hundreds. It has become increasingly difficult for newcomers to master the proliferating literature, and for workers in allied areas to make effective use of it. Thus I have concluded that this is an appropriate time to pause and take stock of the progress made to date. It is my hope that this book will not only provide a useful account of much of this progress, but that it will also help stimulate the future vigorous development of this field.

My intention is that this book serve as a reference work on interacting particle systems, and that it be used as the basis for an advanced graduate course on this subject. The book should be of interest not only to mathematicians, but also to workers in related areas such as mathematical physics and mathematical biology. The prerequisites for reading it are solid one-year graduate courses in analysis and probability theory, at the level of Royden (1968) and Chung (1974), respectively. Material which is usually covered in these courses will be used without comment. In addition, a familiarity with a number of other types of stochastic processes will be helpful. However, references will be given when results from specialized parts of probability theory are used. No particular knowledge of statistical mechanics or mathematical biology is assumed. While this is the first book-length treatment of the subject of interacting particle systems, a number of surveys of parts of the field have appeared in recent years. Among these are Spitzer (1974a), Holley (1974a), Sullivan (1975b), Liggett (1977b), Stroock (1978), Griffeath (1979a, 1981), and Durrett (1981). These can serve as useful complements to the present work.

This book contains several new theorems, as well as many improvements on existing results. However, most of the material has appeared in one form or another in research papers. References to the relevant papers are given in the "Notes and References" section for each chapter. The bibliography contains not only the papers which are referred to in those sections, but also a fairly complete list of papers on this general subject. In order to encourage further work, I have listed a total of over sixty open problems at the end of the appropriate chapters. It should be understood that these problems are not all of comparable difficulty or importance. Undoubtedly, some will have been solved by the time this book is published.

The following remarks should help the reader orient himself to the book. Some of the most important models in the subject are described in the Introduction. The main questions involving them and a few of the most interesting results about them are discussed there as well. The treatment here is free of the technical details which become necessary later, so this is certainly the place to start reading the book.

The first chapter deals primarily with the problem of existence and uniqueness for interacting particle systems. In addition, it contains (in Section 4) several substantive results which follow from the construction and are rather insensitive to the precise nature of the interaction. From a logical point of view, the construction of the process must precede its analysis. However, the construction is more technical, and probably less interesting, than the material in the rest of the book. Thus it is important not to get bogged down in this first chapter. My suggestion is that, on the first reading, one concentrate on the first four sections of Chapter I, and perhaps not spend much time on the proofs there. Little will be lost if in later chapters one is willing to assume that the global dynamics of the process are uniquely determined by the informal infinitesimal description which is given. The martingale formulation which is presented following Section 4 has played an important role in the development of the subject, but will be used only occasionally in the remainder of this book.

Many of the tools which are used in the study of interacting particle systems are different from those used in other branches of probability theory, or if the same, they are often used differently. The second chapter is intended to introduce the reader to some of these tools, the most important of which are coupling and duality. In this chapter, the use of these techniques is illustrated almost exclusively in the context of countable state Markov chains, in order to facilitate their mastery. In addition, the opportunity is taken there to prove several nonstandard Markov chain results which are needed later in the book.

In Chapter III, the ideas and results of the first two chapters are applied to general spin systems—those in which only one coordinate changes at a time. It is here, for example, that the general theory of attractive systems is developed, and that duality and the graphical representation are introduced. Chapters IV–IX treat specific types of models: the stochastic Ising model, the voter model, the contact process, nearest-particle systems, the exclusion process, and processes with unbounded values. These chapters have been written so that they are largely independent of one another and may be read separately. A good first exposure to this book can be obtained by lightly reading the first four sections of Chapter I, reading the first half of Chapter II, Chapter III, and then any or all of Chapters IV, V, and VI.

While I have tried to incorporate many of the important ideas, techniques, results, and models which have been developed during the past fifteen years, this book is not an exhaustive account of the entire subject of interacting particle systems. For example, all models considered here have continuous time, in spite of the fact that a lot of work has been done on analogous discrete time systems, particularly in the Soviet Union. Not treated at all or barely touched on are important advances in the following closely related subjects: infinite systems of stochastic differential equations (see, for example, Holley and Stroock (1981), Shiga (1980a, b) and Shiga and Shimizu (1980)), measure-valued diffusions (see, for example, Dawson (1977) and Dawson and Hochberg (1979, 1982)), shape theory for finite interacting systems (see, for example, Richardson (1973), Bramson and Griffeath (1980c, 1981), Durrett and Liggett (1981), and Durrett and Griffeath (1982)), renormalization theory for interacting particle systems (see, for example, Bramson and Griffeath (1979b) and Holley and Stroock (1978b, 1979a)), cluster processes (see, for example, Kallenberg (1977), Fleischmann, Liemant, and Matthes (1982), and Matthes, Kerstan, and Mecke (1978)), and percolation theory (see, for example, Kesten (1982) and Smythe and Wierman (1978)).

The development of the theory of interacting particle systems is the result of the efforts and contributions of a large number of mathematicians. There are many who could be listed here, but if I tried to list them, I would not know where to stop. In any case, their names appear in the "Notes and References" sections, as well as in the Bibliography. I would particularly like to single out Rick Durrett, David Griffeath, Dick Holley, Ted Harris, and Frank Spitzer, both for their contributions to the subject and for the influence they have had on me. Enrique Andjel, Rick Durrett, David Griffeath, Dick Holley, Claude Kipnis, and Tokuzo Shiga have read parts of this book, and have made valuable comments and found errors in the original manuscript.

Since this is my first book, this is a good place to acknowledge the influence which Sam Goldberg at Oberlin College, and Kai Lai Chung and Sam Karlin at Stanford University had on my first years as a probabilist. I would like to thank Chuck Stone for his encouragement during the early years of my work on interacting particle systems, and in particular for handing me a preprint of Spitzer's 1970 paper with the comment that I would probably find something of interest in it. This book is proof that he was right.

More than anyone else, it was my wife, Chris, who convinced me that I should write this book. In addition to her moral support, she contributed greatly to the project through her excellent typing of the manuscript. Finally, I would like to acknowledge the financial support of the National Science Foundation, both during the many years I have spent working on this subject, and particularly during the past two years in which I have been heavily involved in this writing project.

Frequently Used Notation

S	A finite or countable set of sites.
Z^d	The d-dimensional integer lattice.
Y	The collection of finite subsets of S or $S \cup \{\infty\}$.
X	The state space of the process; usually $\{0, 1\}^S$.
$C(X)$	The continuous functions on X.
$D(X)$	The Lipschitz functions on X (see Section 3 of Chapter I).
$\mathcal{M}$	The increasing continuous functions on X.
$\mathcal{D}$	The functions on X which depend on finitely many coordinates.
$\mathcal{P}$	The probability measures on X.
$\mathcal{S}$	When $S = Z^d$, the translation invariant elements of $\mathcal{P}$.
$\mathcal{S}_e$	The extremal (or ergodic) elements of $\mathcal{S}$.
$\mathcal{I}$	The elements of $\mathcal{P}$ which are invariant for the process.
$\mathcal{I}_e$	The extremal elements of $\mathcal{I}$.
$\mathcal{R}$	The elements of $\mathcal{P}$ which are reversible for the process.
$\mathcal{G}$	The Gibbs measures corresponding to some potential.
$\mathcal{G}_e$	The extremal Gibbs measures.
δ_0, δ_1	The pointmasses on $\eta \equiv 0$ and $\eta \equiv 1$.
μ or ν	Typical elements of $\mathcal{P}$.
$\mu \leq \nu$	Stochastic monotonicity (see Definition 2.1 of Chapter II).
η_t or ζ_t	The Markov process which represents an interacting particle system.
$c(x, \eta)$	The flip rate at $x \in S$ when the configuration is $\eta \in X$.
$S(t)$	The semigroup corresponding to the process.
$\mu S(t)$	The distribution at time t when the initial distribution is $\mu \in \mathcal{P}$.
Ω	The generator or pregenerator of the process.
$\mathcal{D}(\Omega)$	The domain of Ω.
$\mathcal{R}(\Omega)$	The range of Ω.
Re	The real part of a complex number.
$p^{(n)}(x, y)$	The n-step transition probabilities for a discrete time Markov chain.
$p_t(x, y)$	The transition probabilities for a continuous time Markov chain.
$\mathcal{H}$	Harmonic functions; often with some additional constraints.

Contents

Introduction

The field of interacting particle systems began as a branch of probability theory in the late 1960's. Much of the original impetus came from the work of F. Spitzer in the United States and of R. L. Dobrushin in the Soviet Union. (For examples of their early work, see Spitzer (1969a, 1970) and Dobrushin (1971a, b).) During the decade and a half since then, this area has grown and developed rapidly, establishing unexpected connections with a number of other fields.

The original motivation for this field came from statistical mechanics. The objective was to describe and analyze stochastic models for the temporal evolution of systems whose equilibrium measures are the classical Gibbs states. In particular, it was hoped that this would lead to a better understanding of the phenomenon of phase transition. As time passed, it became clear that models with a very similar mathematical structure could be naturally formulated in other contexts—neural networks, tumor growth, spread of infection, and behavioral systems, for example.

From a more mathematical point of view, interacting particle systems represents a natural departure from the established theory of Markov processes. As such, with its different motivation, it has led to a large number of stimulating new types of problems. The solutions of many of these new problems has led in turn to the development of new tools, and to the exploitation on an entirely different level of tools which had earlier played only relatively minor roles in probability theory. A typical interacting particle system consists of finitely or infinitely many particles which, in the absence of the interaction, would evolve according to independent finite or countable state Markov chains. Superimposed on this underlying motion is some type of interaction. As a result of the interaction, the evolution of an individual particle is no longer Markovian. The system as a whole is of course Markovian. However, it is a large and complex Markov process which differs in many respects from the processes such as Brownian motion on Euclidean spaces which motivated much of the development of standard Markov process theory. Thus, while some connections with the Markovian universe are maintained, substantial departures from it occur as well.

Let us illustrate some of the differences between particle systems and the more standard Markov processes in a very simple context. Suppose that

$\{\eta_t(x), x \in S\}$ is a countable collection of independent irreducible continuous time Markov chains with state space $\{0, 1\}$. Of course, the analysis of this system is entirely elementary because of the independence assumption and the simple nature of the individual chains. Suppose however that the entire system η_t is viewed as a Markov process on the uncountable totally disconnected space $\{0, 1\}^S$. Then for any fixed initial configuration η_0, the distributions of η_t at different times are product measures which are mutually singular with respect to each other and with respect to the unique invariant measure for the process. This is of course vastly different from the much smoother behavior evidenced by Brownian motion or other more common Markov processes. If a simple interaction is superimposed on the underlying motion of this collection of two state Markov chains, these mutual singularity properties will in general remain, while the analysis based on independence is no longer available. Thus new techniques are required. The models to be treated in Chapters III–VII are obtained by superimposing various natural types of interactions on the simple systems described above. This is done by letting the flip rate of each coordinate depend on the values of other coordinates.

As might be expected, the behavior of an interacting particle system depends in a rather sensitive way on the precise nature of the interaction. Thus most of the research which has been done in this field has dealt with certain types of models in which the interaction is of a prescribed form. The unity of the subject comes not so much from the generality of the theorems which are proved, but rather from the nature of the processes which are studied, the types of problems which are posed about them, and the techniques which are used in their solution.

The main problems which have been treated involve the long-time behavior of the system. The first step in proving limit theorems is to describe the class of invariant measures for the process, since these are the possible limits as $t \to \infty$ of the distribution at time t. The next step is to determine to the extent possible the domain of attraction of each invariant measure. This means, to determine for each invariant measure, the class of all initial distributions for which the distribution at time t of the process converges to that measure as $t \to \infty$. In the case of the independent two state Markov chains, the answers to these questions are of course that there is a unique invariant measure for the process, which is the product of the stationary distributions for the individual two state chains. Its domain of attraction is the collection of all probability measures on $\{0, 1\}^S$.

In order to make the foregoing remarks more concrete, we will now describe informally some of the models which have received the most attention, and will specify the form which these problems take in each case. In the first three examples, only one coordinate of η_t changes at a time. In general, however, infinitely many coordinates will change in any interval of time. In the fourth example, two coordinates change at a time.

The Stochastic Ising Model. This is a model for magnetism which was introduced by Glauber (1963) and then first studied in some generality by Dobrushin (1971a, b). It is a Markov process with state space $\{-1, +1\}^{Z^d}$. The sites represent iron atoms, which are laid out on the d-dimensional integer lattice Z^d, while the value of ± 1 at a site represents the spin of the atom at that site. A configuration of spins η is then a point in $\{-1, +1\}^{Z^d}$. The dynamics of the evolution are specified by the requirement that a spin $\eta(x)$ at $x \in Z^d$ flips to $-\eta(x)$ at rate

$$\exp\left[-\beta \sum_{y:|y-x|=1} \eta(x)\eta(y)\right],$$

where β is a nonnegative parameter which represents the reciprocal of the temperature of the system. Note that the flip rate is higher when the spin at x is different from that at most of its neighbors than it is when it agrees with most of its neighbors. Thus the system "prefers" configurations in which the spins tend to be aligned with one another. In the language of statistical mechanics, this monotonicity is referred to as ferromagnetism. In the subject of interacting particle systems, such monotone systems are called "attractive." Of course when $\beta = 0$, the coordinates $\eta_t(x)$ are independent two-state Markov chains, so as observed earlier, the system has as its unique invariant measure the Bernoulli product measure ν on $\{-1, +1\}^{Z^d}$ with parameter $\frac{1}{2}$. Furthermore, for any initial distribution, the distribution at time t converges weakly as $t \to \infty$ to ν by the convergence theorem for finite-state irreducible Markov chains. Such a system, which has a unique invariant measure to which convergence occurs for any initial distribution, will be called ergodic. The first important problem to be resolved for the stochastic Ising model is to determine for which choices of β and d the process is ergodic. The first answer, as will be seen in Chapter IV, is that the process is ergodic for all β if $d = 1$. In fact, in one dimension the unique invariant measure is a stationary two-state Markov chain, which is regarded as a measure on $\{-1, 1\}^{Z^1}$. If $d \geq 2$, there is a critical $\beta_d > 0$ so that the process is ergodic if $\beta < \beta_d$ but not if $\beta > \beta_d$. If $d = 2$ and $\beta > \beta_2$, then there are exactly two extremal invariant measures. If $d \geq 3$ and β is sufficiently large, then there are infinitely many extremal invariant measures. Nonergodicity corresponds to the occurrence of phase transition, with distinct invariant measures corresponding to distinct phases.

The Voter Model. The voter model was introduced independently by Clifford and Sudbury (1973) and by Holley and Liggett (1975). Here the state space is $\{0, 1\}^{Z^d}$ and the evolution mechanism is described by saying that $\eta(x)$ changes to $1 - \eta(x)$ at rate

$$\frac{1}{2d} \sum_{y:|y-x|=1} 1_{\{\eta(y) \neq \eta(x)\}}.$$

In the voter interpretation of Holley and Liggett, sites in Z^d represent voters who can hold either of two political positions, which are denoted by zero and one. A voter waits an exponential time with parameter one, and then adopts the position of a neighbor chosen at random. In the invasion interpretation of Clifford and Sudbury, $\{x \in Z^d: \eta(x)=0\}$ and $\{x \in Z^d: \eta(x)=1\}$ represent territory held by each of two competing populations. A site is invaded at a rate proportional to the number of neighboring sites controlled by the opposing population. The voter model has two trivial invariant measures: the pointmasses at $\eta \equiv 0$ and $\eta \equiv 1$ respectively. Thus the voter model is not ergodic. The first main question in this case is whether there are any other extremal invariant measures. As will be seen in Chapter V, there are no others if $d \leq 2$. On the other hand, if $d \geq 3$, there is a one-parameter family $\{\mu_\rho, 0 \leq \rho \leq 1\}$ of extremal invariant measures, where μ_ρ is translation invariant and ergodic, and $\mu_\rho\{\eta: \eta(x)=1\}=\rho$. This dichotomy is closely related to the fact that a simple random walk on Z^d is recurrent if $d \leq 2$ and transient if $d \geq 3$. In terms of the voter interpretation, one can describe the result by saying that a consensus is approached as $t \to \infty$ if $d \leq 2$, but that disagreements persist indefinitely if $d \geq 3$.

The Contact Process. This process was introduced and first studied by Harris (1974). It again has state space $\{0, 1\}^{Z^d}$. The dynamics are specified by the following transition rates: at site x,

$$1 \to 0 \quad \text{at rate } 1,$$

and

$$0 \to 1 \quad \text{at rate } \lambda \sum_{y:|y-x|=1} \eta(y),$$

where λ is a positive parameter which is interpreted as the infection rate. With this interpretation, sites at which $\eta(x)=1$ are regarded as infected, while sites at which $\eta(x)=0$ are regarded as healthy. Infected individuals become healthy after an exponential time with parameter one, independently of the configuration. Healthy individuals become infected at a rate which is proportional to the number of infected neighbors. The contact process has a trivial invariant measure: the pointmass at $\eta \equiv 0$. The first important question is whether or not there are others. As will be seen in Chapter VI, there is a critical λ_d for $d \geq 1$ so that the process is ergodic for $\lambda < \lambda_d$, but has at least one nontrivial invariant measure if $\lambda > \lambda_d$. The value of λ_d is not known exactly. Bounds on λ_d are available, however. For example,

$$\frac{1}{2d-1} \leq \lambda_d \leq \frac{2}{d}$$

for all $d \geq 1$. Good convergence theorems are known when $d = 1$. However,