天元基金 影印系列丛书

Ib Madsen and Jørgen Tornehave 著

从微积分到上同调

From Calculus to Cohomology

De Rham cohomology and characteristic classes

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PREFACE

This text offers a self-contained exposition of the cohomology of differential forms, de Rham cohomology, and of its application to characteristic classes defined in terms of the curvature tensor. The only formal prerequisites are knowledge of standard calculus and linear algebra, but for the later part of the book some prior knowledge of the geometry of surfaces, Gaussian curvature, will not hurt the reader.

The first seven chapters present the cohomology of open sets in Euclidean spaces and give the standard applications usually covered in a first course in algebraic topology, such as Brouwer's fixed point theorem, the topological invariance of domains and the Jordan–Brouwer separation theorem. The next four chapters extend the definition of cohomology to smooth manifolds, present Stokes' theorem and give a treatment of degree and index of vector fields, from both the cohomological and geometric point of view.

The last ten chapters give the more advanced part of cohomology: the Poincaré–Hopf theorem, Poincaré duality, Chern classes, the Euler class, and finally the general Gauss–Bonnet formula. As a novel point we prove the so called splitting principles for both complex and real oriented vector bundles.

The text grew out of numerous versions of lecture notes for the beginning course in topology at Aarhus University. The inspiration to use de Rham cohomology as a first introduction to topology comes in part from a course given by G. Segal at Oxford many years ago, and the first few chapters owe a lot to his presentation of the subject. It is our hope that the text can also serve as an introduction to the modern theory of smooth four-manifolds and gauge theory.

The text has been used for third and fourth year students with no prior exposure to the concepts of homology or algebraic topology. We have striven to present all arguments and constructions in detail. Finally we sincerely thank the many students who have been subjected to earlier versions of this book. Their comments have substantially changed the presentation in many places.

Aarhus, January 1996

Contents

1. INTRODUCTION

It is well-known that a continuous real function, that is defined on an open set of $\mathbf{R}$ has a primitive function. How about multivariable functions? For the sake of simplicity we restrict ourselves to smooth (or C^∞-) functions, i.e. functions that have continuous partial derivatives of all orders.
We begin with functions of two variables. Let $f: U \to \mathbf{R}^2$ be a smooth function defined on an open set of $\mathbf{R}^2$.

Question 1.1 Is there a smooth function $F: U \to \mathbf{R}$, such that

$$\frac{\partial F}{\partial x_1} = f_1 \quad \text{and} \quad \frac{\partial F}{\partial x_2} = f_2, \quad \text{where } f = (f_1, f_2)? \tag{1}$$

Since

$$\frac{\partial^2 F}{\partial x_2 \partial x_1} = \frac{\partial^2 F}{\partial x_1 \partial x_2}$$

we must have

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}. \tag{2}$$

The correct question is therefore whether F exists, assuming $f = (f_1, f_2)$ satisfies (2). Is condition (2) also sufficient?

Example 1.2 Consider the function $f: \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

It is easy to show that (2) is satisfied. However, there is no function $F: \mathbf{R}^2 - \{0\} \to \mathbf{R}$ that satisfies (1). Assume there were; then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta) d\theta = F(1,0) - F(1,0) = 0.$$

On the other hand the chain rule gives

$$\begin{aligned} \frac{d}{d\theta} F(\cos\theta, \sin\theta) &= \frac{dF}{dx} \cdot (-\sin\theta) + \frac{\partial F}{\partial y} \cdot \cos\theta \\ &= -f_1(\cos\theta, \sin\theta) \cdot \sin\theta + f_2(\cos\theta, \sin\theta) \cdot \cos\theta = 1. \end{aligned}$$

This contradiction can only be explained by the non-existence of F.

Definition 1.3 A subset $X \subset \mathbf{R}^n$ is said to be *star-shaped* with respect to the point $x_0 \in X$ if the line segment $\{tx_0 + (1-t)x | t \in [0,1]\}$ is contained in X for all $x \in X$.

Theorem 1.4 *Let $U \subset \mathbf{R}^2$ be an open star-shaped set. For any smooth function $(f_1, f_2): U \to \mathbf{R}^2$ that satisfies* (2), *Question* 1.1 *has a solution.*

Proof. For the sake of simplicity we assume that $x_0 = 0 \in \mathbf{R}^2$. Consider the function $F : U \to \mathbf{R}$,

$$F(x_1, x_2) = \int_0^1 [x_1 f_1(tx_1, tx_2) + x_2 f_2(tx_1, tx_2)]dt.$$

Then one has

$$\frac{\partial F}{\partial x_1}(x_1, x_2) = \int_0^1 \left[f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_2}{\partial x_1}(tx_1, tx_2) \right] dt$$

and

$$\frac{d}{dt} t f_1(tx_1, tx_2) = f_1(tx_1, tx_2) + tx_1 \frac{\partial f_1}{\partial x_1}(tx_1, tx_2) + tx_2 \frac{\partial f_1}{\partial x_2}(tx_1, tx_2).$$

Substituting this result into the formula, we get

$$\begin{aligned}\frac{\partial F}{\partial x_1}(x_1, x_2) &= \int_0^1 \left[\frac{d}{dt} t f_1(tx_1, tx_2) + tx_2 \left(\frac{\partial f_2}{\partial x_1}(tx_1, tx_2) - \frac{\partial f_1}{\partial x_2}(tx_1, tx_2) \right) \right] dt \\ &= [t f_1(tx_1, tx_2)]_{t=0}^1 = f_1(x_1, x_2).\end{aligned}$$

Analogously, $\frac{\partial F}{\partial x_2} = f_2(x_1, x_2)$.

Example 1.2 and Theorem 1.4 suggest that the answer to Question 1.1 depends on the "shape" or "topology" of U. Instead of searching for further examples or counterexamples of sets U and functions f, we define an *invariant* of U, which tells us whether or not the question has an affirmative answer (for all f), assuming the necessary condition (2).

Given the open set $U \subseteq \mathbf{R}^2$, let $C^\infty(U, \mathbf{R}^k)$ denote the set of smooth functions $\phi: U \to \mathbf{R}^k$. This is a vector space. If $k = 2$ one may consider $\phi: U \to \mathbf{R}^k$ as a vector field on U by plotting $\phi(u)$ from the point u. We define the *gradient* and the *rotation*

$$\text{grad}: C^\infty(U, \mathbf{R}) \to C^\infty(U, \mathbf{R}^2), \qquad \text{rot}: C^\infty(U, \mathbf{R}^2) \to C^\infty(U, \mathbf{R})$$

by

$$\text{grad}(\phi) = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right), \qquad \text{rot}(\phi_1, \phi_2) = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}$$

Note that rot $\circ$ grad $= 0$. Hence the kernel of rot contains the image of grad,

$$\text{Ker(rot)} = \text{Kernel of rot}$$
$$\text{Im(grad)} = \text{Image of grad}$$

Since both rot and grad are linear operators, Im(grad) is a subspace of Ker(rot). Therefore we can consider the quotient vector space, i.e. the vector space of cosets $\alpha + \text{Im(grad)}$ where $\alpha \in \text{Ker(rot)}$:

(3) $$H^1(U) = \text{Ker(rot)}/\text{Im(grad)}.$$

Both Ker(rot) and Im(grad) are infinite-dimensional vector spaces. It is remarkable that the quotient space $H^1(U)$ is usually finite-dimensional.
We can now reformulate Theorem 1.4 as

(4) $$H^1(U) = 0 \quad \text{whenever } U \subseteq \mathbf{R}^2 \text{ is star-shaped.}$$

On the other hand, Example 1.2 tells us that $H^1(\mathbf{R}^2 - \{0\}) \neq 0$. Later on we shall see that $H^1(\mathbf{R}^2 - \{0\})$ is 1-dimensional, and that $H^1(\mathbf{R}^2 - \bigcup_{i=1}^k \{x_i\}) \cong \mathbf{R}^k$. The dimension of $H^1(U)$ is the number of "holes" in U.
In analogy with (3) we introduce

(5) $$H^0(U) = \text{Ker(grad)}.$$

This definition works for open sets U of $\mathbf{R}^k$ with $k \geq 1$, when we define

$$\text{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Theorem 1.5 *An open set $U \subseteq \mathbf{R}^k$ is connected if and only if $H^0(U) = \mathbf{R}$.*

Proof. Assume that $\text{grad}(f) = 0$. Then f is locally constant: each $x_0 \in U$ has a neighborhood $V(x_0)$ with $f(x) = f(x_0)$ when $x \in V(x_0)$. If U is connected, then every locally constant function is constant. Indeed, for $x_0 \in U$ the set

$$\{x \in U | f(x) = f(x_0)\} = f^{-1}(f(x_0)).$$

is closed because f is continuous, and open since f is locally constant. Hence it is equal to U, and $H^0(U) = \mathbf{R}$. Conversely, if U is not connected, then there exists a smooth, surjective function $f: U \to \{0, 1\}$. Such a function is locally constant, so $\text{grad}(f) = 0$. It follows that $\dim H^0(U) > 1$. □

The reader may easily extend the proof of Theorem 1.5 to show that $\dim H^0(U)$ is precisely the number of connected components of U.

We next consider functions of three variables. Let $U \subseteq \mathbf{R}^3$ be an open set. A real function on U has three partial derivatives and (2) is replaced by three equations. We introduce the notation

$$\begin{aligned} \text{grad}\colon C^\infty(U,\mathbf{R}) &\to C^\infty(U,\mathbf{R}^3) \\ \text{rot}\colon C^\infty(U,\mathbf{R}^3) &\to C^\infty(U,\mathbf{R}^3) \\ \text{div}\colon C^\infty(U,\mathbf{R}^3) &\to C^\infty(U,\mathbf{R}) \end{aligned}$$

for the linear operators defined by

$$\begin{aligned} \text{grad}\,(f) &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \\ \text{rot}\,(f_1, f_2, f_3) &= \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\right) \\ \text{div}\,(f_1, f_2, f_3) &= \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}. \end{aligned}$$

Note that rot $\circ$ grad $= 0$ and div $\circ$ rot $= 0$. We define $H^0(U)$ and set $H^1(U)$ as in Equations (3) and (5) and

$$H^2(U) = \text{Ker(div)}/\text{Im(rot)}. \tag{6}$$

Theorem 1.6 *For an open star-shaped set in* $\mathbf{R}^3$ *we have that* $H^0(U) = \mathbf{R}$, $H^1(U) = 0$ *and* $H^2(U) = 0$.

Proof. The values of $H^0(U)$ and $H^1(U)$ are obtained as above, so we shall restrict ourselves to showing that $H^2(U) = 0$. It is convenient to assume that U is star-shaped with respect to 0. Consider a function $F : U \to \mathbf{R}^3$ with $\text{div}\, F = 0$, and define $G : U \to \mathbf{R}^3$ by

$$G(\mathbf{x}) = \int_0^1 (F(t\mathbf{x}) \times t\mathbf{x})dt$$

where $\times$ denotes the cross product,

$$\begin{aligned} (f_1, f_2, f_3) \times (x_1, x_2, x_3) &= \begin{vmatrix} e_1 & f_1 & x_1 \\ e_2 & f_2 & x_2 \\ e_3 & f_3 & x_3 \end{vmatrix} \\ &= (f_2x_3 - f_3x_2, f_3x_1 - f_1x_3, f_1x_2 - f_2x_1). \end{aligned}$$

Straightforward calculations give

$$\text{rot}(F(t\mathbf{x}) \times t\mathbf{x}) = \frac{d}{dt}(t^2 F(t\mathbf{x})).$$

Hence

$$\text{rot}G(\mathbf{x}) = \int_0^1 \frac{d}{dt}(t^2 F(t\mathbf{x}))dt = F(\mathbf{x}).$$ □

If $U \subseteq \mathbf{R}^3$ is not star-shaped both $H^1(U)$ and $H^2(U)$ may be non-zero.

Example 1.7 Let $S = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | x_1^2 + x_2^2 = 1, x_3 = 0\}$ be the unit circle in the (x_1, x_2)-plane. Consider the function

$$f(x_1, x_2, x_3) = \left(\frac{-2x_1x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{-2x_2x_3}{x_3^2 + (x_1^2 + x_2^2 - 1)^2}, \frac{x_1^2 + x_2^2 - 1}{x_3^2 + (x_1^2 + x_2^2 - 1)^2} \right)$$

on the open set $U = \mathbf{R}^3 - S$.

One finds that $\mathrm{rot}(f) = 0$. Hence f defines an element $[f] \in H^1(U)$. By integration along a curve γ in U, which is linked to S (as two links in a chain), we shall show that $[f] \neq 0$. The curve in question is

$$\gamma(t) = \left(\sqrt{1 + \cos t}, 0, \sin t \right), \quad -\pi \leq t \leq \pi.$$

Assume $\mathrm{grad}(F) = f$ as a function on U. We can determine the integral of $\frac{d}{dt}F(\gamma(t))$ in two ways. On the one hand we have

$$\int_{-\pi+\epsilon}^{\pi-\epsilon} \frac{d}{dt}F(\gamma(t))dt = F(\gamma(\pi - \epsilon)) - F(\gamma(-\pi + \epsilon)) \to 0 \quad \text{for } \epsilon \to 0$$

and on the other hand the chain rule gives

$$\begin{aligned} \frac{d}{dt}F(\gamma(t)) &= f_1(\gamma(t)) \cdot \gamma_1'(t) + f_2(\gamma(t)) \cdot \gamma_2'(t) + f_3(\gamma(t)) \cdot \gamma_3'(t) \\ &= \sin^2 t + 0 + \cos^2 t = 1. \end{aligned}$$

Therefore the integral also converges to 2π, which is a contradiction.

Example 1.8 Let U be an open set in $\mathbf{R}^k$ and $X: U \to \mathbf{R}^k$ a smooth function (a smooth vector field). Recall that the *energy* $A_\gamma(X)$, of X along a smooth curve $\gamma: [a, b] \to U$ is defined by the integral

$$A_\gamma(X) = \int_a^b \langle X \circ \gamma(t) \, , \, \gamma'(t) \rangle dt$$

where $\langle \, , \rangle$ denotes the standard inner product. If $X = \mathrm{grad}(\Phi)$ and $\Phi\gamma(a) = \Phi\gamma(b)$, then the energy is zero, since

$$\langle X \circ \gamma(t) \, , \, \gamma'(t) \rangle = \frac{d}{dt} \Phi(\gamma(t))$$

by the chain rule; compare Example 1.2.

2. THE ALTERNATING ALGEBRA

Let V be a vector space over $\mathbf{R}$. A map

$$f : \underbrace{V \times V \times \cdots \times V}_{k \text{ times}} \to \mathbf{R}$$

is called k-linear (or multilinear), if f is linear in each factor.

Definition 2.1 A k-linear map $\omega\colon V^k \to \mathbf{R}$ is said to be alternating if $\omega(\xi_1, \dots, \xi_k) = 0$ whenever $\xi_i = \xi_j$ for some pair $i \neq j$. The vector space of alternating, k-linear maps is denoted by $\mathrm{Alt}^k(V)$.

We immediately note that $\mathrm{Alt}^k(V) = 0$ if $k > \dim V$. Indeed, let $e_1, \dots, e_n$ be a basis of V, and let $\omega \in \mathrm{Alt}^k(V)$. Using multilinearity,

$$\omega(\xi_1, \dots, \xi_k) = \omega\Big(\sum \lambda_{i,1} e_i, \dots, \sum \lambda_{i,k} e_i\Big) = \sum \lambda_J \omega(e_{j_1}, \dots, e_{j_k})$$

with $\lambda_J = \lambda_{j_1,1} \dots \lambda_{j_k,k}$. Since $k > n$, there must be at least one repetition among the elements $e_{j_1}, \dots, e_{j_k}$. Hence $\omega(e_{j_1}, \dots, e_{j_k}) = 0$.

The symmetric group of permutations of the set $\{1, \dots, k\}$ is denoted by $S(k)$. We remind the reader that any permutation can be written as a composition of transpositions. The transposition that interchanges i and j will be denoted by (i, j). Furthermore, and this fact will be used below, any permutation can be written as a composition of transpositions of the type $(i, i{+}1)$, $(i, i{+}1) \circ (i{+}1, i{+}2) \circ (i, i{+}1) = (i, i + 2)$ and so forth. The sign of a permutation:

$$\text{sign}\colon S(k) \to \{\pm 1\}, \tag{1}$$

is a homomorphism, $\text{sign}(\sigma \circ \tau) = \text{sign}(\sigma) \circ \text{sign}(\tau)$, which maps every transposition to -1. Thus the sign of $\sigma \in S(k)$ is -1 precisely if σ decomposes into a product consisting of an odd number of transpositions.

Lemma 2.2 *If $\omega \in \mathrm{Alt}^k(V)$ and $\sigma \in S(k)$, then*

$$\omega\big(\xi_{\sigma(1)}, \dots, \xi_{\sigma(k)}\big) = \text{sign}(\sigma)\omega(\xi_1, \dots, \xi_k).$$

Proof. It is sufficient to prove the formula when $\sigma = (i, j)$. Let

$$\omega_{i,j}(\xi, \xi') = \omega(\xi_1, \dots, \xi, \dots, \xi', \dots, \xi_k),$$

with ξ and ξ' occurring at positions i and j respectively. The remaining $\xi_\nu \in V$ are arbitrary but fixed vectors. From the definition it follows that $\omega_{i,j} \in \mathrm{Alt}^2(V)$. Hence $\omega_{i,j}(\xi_i + \xi_j, \xi_i + \xi_j) = 0$. Bilinearity yields that $\omega_{i,j}(\xi_i, \xi_j) + \omega_{i,j}(\xi_j, \xi_i) = 0$. □

Example 2.3 Let $V = \mathbf{R}^k$ and $\xi_i = (\xi_{i1}, \dots, \xi_{ik})$. The function $\omega(\xi_1, \dots, \xi_k) = \det((\xi_{ij}))$ is alternating, by the calculational rules for determinants.

We want to define the exterior product

$$\wedge \colon \mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V) \to \mathrm{Alt}^{p+q}(V).$$

When $p = q = 1$ it is given by $(\omega_1 \wedge \omega_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_2(\xi_1)\omega_1(\xi_2)$.

Definition 2.4 A (p, q)-shuffle σ is a permutation of $\{1, \dots, p+q\}$ satisfying

$$\sigma(1) < \dots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \dots < \sigma(p+q).$$

The set of all such permutations is denoted by $S(p, q)$. Since a (p, q)-shuffle is uniquely determined by the set $\{\sigma(1), \dots, \sigma(p)\}$, the cardinality of $S(p, q)$ is $\binom{p+q}{p}$.

Definition 2.5 (Exterior product) For $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$, we define

$$\begin{aligned} &(\omega_1 \wedge \omega_2)(\xi_1, \dots, \xi_{p+q}) \\ &= \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}) \cdot \omega_2(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}). \end{aligned}$$

It is obvious that $\omega_1 \wedge \omega_2$ is a $(p+q)$-linear map, but moreover:

Lemma 2.6 *If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$ then $\omega_1 \wedge \omega_2 \in \mathrm{Alt}^{p+q}(V)$.*

Proof. We first show that $(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \dots, \xi_{p+q}) = 0$ when $\xi_1 = \xi_2$. We let

(i) $S_{12} = \{\sigma \in S(p,q) \mid \sigma(1) = 1, \sigma(p+1) = 2\}$
(ii) $S_{21} = \{\sigma \in S(p,q) \mid \sigma(1) = 2, \sigma(p+1) = 1\}$
(iii) $S_0 = S(p,q) - (S_{12} \cup S_{21})$.

If $\sigma \in S_0$ then either $\omega_1(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})$ or $\omega_2(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)})$ is zero, since $\xi_{\sigma(1)} = \xi_{\sigma(2)}$ or $\xi_{\sigma(p+1)} = \xi_{\sigma(p+2)}$. Left composition with the transposition $\tau = (1, 2)$ is a bijection $S_{12} \to S_{21}$. We therefore have

$$\begin{aligned} (\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \dots, \xi_{p+q}) = \\ &\sum_{\sigma \in S_{12}} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}) \\ &- \sum_{\sigma \in S_{12}} \mathrm{sign}(\sigma)\omega_1(\xi_{\tau\sigma(1)}, \dots, \xi_{\tau\sigma(p)}) \cdot \omega_2(\xi_{\tau\sigma(p+1)}, \dots \xi_{\tau\sigma(p+q)}). \end{aligned}$$

Since $\sigma(1) = 1$ and $\sigma(p+1) = 2$, while $\tau\sigma(1) = 2$ and $\tau\sigma(p+1) = 1$, we see that $\tau\sigma(i) = \sigma(i)$ whenever $i \neq 1, p+1$. But $\xi_1 = \xi_2$ so the terms in the two sums cancel. The case $\xi_i = \xi_{i+1}$ is similar. Now $\omega_1 \wedge \omega_2$ will be alternating according to Lemma 2.7 below. □

Lemma 2.7 *A k-linear map ω is alternating if $\omega(\xi_1, \dots, \xi_k) = 0$ for all k-tuples with $\xi_i = \xi_{i+1}$ for some $1 \le i \le k-1$.*

Proof. $S(k)$ is generated by the transpositions $(i, i+1)$, and by the argument of Lemma 2.2,

$$\omega(\xi_1, \dots, \xi_i, \xi_{i+1}, \dots, \xi_k) = -\omega(\xi_1, \dots, \xi_{i+1}, \xi_i, \dots, \xi_k).$$

Hence Lemma 2.2 holds for all $\sigma \in S(k)$, and ω is alternating. □

It is clear from the definition that

$$\begin{aligned}(\omega_1 + \omega_1') \wedge \omega_2 &= \omega_1 \wedge \omega_2 + \omega_1' \wedge w_2 \\ (\lambda\omega_1) \wedge \omega_2 &= \lambda(\omega_1 \wedge \omega_2) = \omega_1 \wedge \lambda\omega_2 \\ \omega_1 \wedge (\omega_2 + \omega_2') &= \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_2'\end{aligned}$$

for $\omega_1, \omega_1' \in \mathrm{Alt}^p(V)$ and $\omega_2, \omega_2' \in \mathrm{Alt}^q(V)$.

Lemma 2.8 *If $\omega_1 \in \mathrm{Alt}^p(V)$ and $\omega_2 \in \mathrm{Alt}^q(V)$ then $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$.*

Proof. Let $\tau \in S(p+q)$ be the element with

$$\begin{aligned}\tau(1) = p+1, \tau(2) = p+2, \dots, \tau(q) = p+q \\ \tau(q+1) = 1, \tau(q+2) = 2, \dots, \tau(p+q) = p.\end{aligned}$$

We have $\mathrm{sign}(\tau) = (-1)^{pq}$. Composition with τ defines a bijection

$$S(p,q) \xrightarrow{\cong} S(q,p); \quad \sigma \mapsto \sigma \circ \tau.$$

Note that

$$\begin{aligned}\omega_2(\xi_{\sigma\tau(1)}, \dots, \xi_{\sigma\tau(q)}) &= \omega_2(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}) \\ \omega_1(\xi_{\sigma\tau(q+1)}, \dots, \xi_{\sigma\tau(p+q)}) &= \omega_1(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}).\end{aligned}$$

Hence

$$\begin{aligned}&\omega_2 \wedge \omega_1(\xi_1, \dots, \xi_{p+q}) \\ &= \sum_{\sigma \in S(q,p)} \mathrm{sign}(\sigma)\omega_2(\xi_{\sigma(1)}, \dots, \xi_{\sigma(q)})\omega_1(\xi_{\sigma(q+1)}, \dots, \xi_{\sigma(p+q)}) \\ &= \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma\tau)\omega_2(\xi_{\sigma\tau(1)}, \dots, \xi_{\sigma\tau(q)})\omega_1(\xi_{\sigma\tau(q+1)}, \dots, \xi_{\sigma\tau(p+q)}) \\ &= (-1)^{pq} \sum_{\sigma \in S(p,q)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(p+q)}) \\ &= (-1)^{pq}\omega_1 \wedge \omega_2(\xi_1, \dots, \xi_{p+q}).\end{aligned}$$

□

Lemma 2.9 *If* $\omega_1 \in \mathrm{Alt}^p(V)$, $\omega_2 \in \mathrm{Alt}^q(V)$ *and* $\omega_3 \in \mathrm{Alt}^r(V)$ *then*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3.$$

Proof. Let $S(p,q,r) \subset S(p+q+r)$ consist of the permutations σ with

$$\begin{aligned} \sigma(1) < \ldots < \sigma(p), \\ \sigma(p+1) < \ldots < \sigma(p+q), \\ \sigma(p+q+1) < \ldots < \sigma(p+q+r). \end{aligned}$$

We will also need the subsets $S(\bar{p},q,r)$ and $S(p,q,\bar{r})$ of $S(p,q,r)$ given by

$$\begin{aligned} &\sigma \in S(\bar{p},q,r) \Longleftrightarrow \sigma \text{ is the identity on } \{1,\ldots,p\} \text{ and } \sigma \in S(p,q,r) \\ &\sigma \in S(p,q,\bar{r}) \Longleftrightarrow \sigma \text{ is the identity on } \{p+q+1,\ldots,p+q+r\} \\ &\qquad\qquad \text{and } \sigma \in S(p,q,r). \end{aligned}$$

There are bijections

$$\begin{aligned} &S(p,q+r) \times S(\bar{p},q,r) \overset{\cong}{\to} S(p,q,r); (\sigma,\tau) \to \sigma \circ \tau \\ &S(p+q,r) \times S(p,q,\bar{r}) \overset{\cong}{\to} S(p,q,r); (\sigma,\tau) \to \sigma \circ \tau. \end{aligned} \tag{2}$$

With these notations we have

$$\begin{aligned} &[\omega_1 \wedge (\omega_2 \wedge \omega_3)](\xi_1,\ldots,\xi_{p+q+r}) \\ &= \sum_{\sigma \in S(p,q+r)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})(\omega_2 \wedge \omega_3)(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q+r)}) \\ &= \sum_{\sigma \in S(p,q+r)} \mathrm{sign}(\sigma) \sum_{\tau \in S(\bar{p},q,r)} \mathrm{sign}(\tau)\big[\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)}) \\ &\qquad \omega_2(\xi_{\sigma\tau(p+1)},\ldots,\xi_{\sigma\tau(p+q)})\omega_3(\xi_{\sigma\tau(p+q+1)},\ldots,\xi_{\sigma\tau(p+q+r)})\big] \\ &= \sum_{u \in S(p,q,r)} \big[\mathrm{sign}(u)\omega_1(\xi_{u(1)},\ldots,\xi_{u(p)})\omega_2(\xi_{u(p+1)},\ldots,\xi_{u(p+q)}) \\ &\qquad \omega_3(\xi_{u(p+q+1)},\ldots,\xi_{u(p+q+r)})\big] \end{aligned}$$

where the last equality follows from the first equation in (2). Quite analogously one can calculate $[(\omega_1 \wedge \omega_2) \wedge \omega_3](\xi_1,\ldots,\xi_{p+q+r})$, employing the second equation in (2). □

Remark 2.10 In other textbooks on alternating functions one can often see the definition

$$\begin{aligned} &\omega_1 \bar{\wedge} \omega_2(\xi_1,\ldots,\xi_{p+q}) \\ &= \frac{1}{p!q!} \sum_{\sigma \in S(p+q)} \mathrm{sign}(\sigma)\omega_1(\xi_{\sigma(1)},\ldots,\xi_{\sigma(p)})\omega_2(\xi_{\sigma(p+1)},\ldots,\xi_{\sigma(p+q)}). \end{aligned}$$

Note that in this formula $\{\sigma(1),\ldots,\sigma(p)\}$ and $\{\sigma(p+1),\ldots,\sigma(p+q)\}$ are not ordered. There are exactly $S(p)\times S(q)$ ways to come from an ordered set to the arbitrary sequence above; this causes the factor $\frac{1}{p!q!}$, so $\omega_1\bar{\wedge}\omega_2=\omega_1\wedge\omega_2$.

An **R**-algebra A consists of a vector space over **R** and a bilinear map $\mu\colon A\times A\to A$ which is associative, $\mu(a,\mu(b,c))=\mu(\mu(a,b),c)$ for every $a,b,c\in A$. The algebra is called *unitary* if there exists a unit element for μ, $\mu(1,a)=\mu(a,1)=a$ for all $a\in A$.

Definition 2.11

(i) A graded **R**-algebra A_* is a sequence of vector spaces $A_k, k=0,1\ldots$, and bilinear maps $\mu\colon A_k\times A_l\to A_{k+l}$ which are associative.
(ii) The algebra A_* is called connected if there exists a unit element $1\in A_0$ and if $\epsilon\colon \mathbf{R}\to A_0$, given by $\epsilon(r)=r\cdot 1$, is an isomorphism.
(iii) The algebra A_* is called (graded) commutative (or anti-commutative), if $\mu(a,b)=(-1)^{kl}\mu(b,a)$ for $a\in A_k$ and $b\in A_l$.

The elements in A_k are said to have degree k. The set $\mathrm{Alt}^k(V)$ is a vector space over **R** in the usual manner:

$$(\omega_1+\omega_2)(\xi_1,\ldots,\xi_k)=\omega_1(\xi_1,\ldots,\xi_k)+\omega_2(\xi_1,\ldots,\xi_k)$$
$$(\lambda\omega)(\xi_1,\ldots,\xi_k)=\lambda\omega(\xi_1,\ldots,\xi_k),\quad \lambda\in\mathbf{R}.$$

The product from Definition 2.5 is a bilinear map from $\mathrm{Alt}^p(V)\times\mathrm{Alt}^q(V)$ to $\mathrm{Alt}^{p+q}(V)$. We set $\mathrm{Alt}^0(V)=\mathbf{R}$ and expand the product to $\mathrm{Alt}^0(V)\times\mathrm{Alt}^p(V)$ by using the vector space structure. The basic formal properties of the alternating forms can now be summarized in

Theorem 2.12 $\mathrm{Alt}^*(V)$ *is an anti-commutative and connected graded algebra.* □

$\mathrm{Alt}^*(V)$ is called the exterior or alternating algebra associated to V.

Lemma 2.13 *For 1-forms* $\omega_1,\ldots,\omega_p\in\mathrm{Alt}^1(V)$,

$$(\omega_1\wedge\ldots\wedge\omega_p)(\xi_1,\ldots,\xi_p)=\det\begin{pmatrix}\omega_1(\xi_1)&\omega_1(\xi_2)&\cdots&\omega_1(\xi_p)\\ \omega_2(\xi_1)&\omega_2(\xi_2)&\cdots&\omega_2(\xi_p)\\ \vdots&\vdots&&\vdots\\ \omega_p(\xi_1)&\omega_p(\xi_2)&\cdots&\omega_p(\xi_p)\end{pmatrix}$$

Proof. The case $p=2$ is obvious. We proceed by induction on p. According to Definition 2.5,

$$\omega_1 \wedge (\omega_2 \wedge \ldots \wedge \omega_p)(\xi_1, \ldots, \xi_p)$$
$$= \sum_{j=1}^{p} (-1)^{j+1} \omega_1(\xi_j)(\omega_2 \wedge \ldots \wedge \omega_p)\Big(\xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_p\Big)$$

where $(\xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_p)$ denotes the $(p-1)$-tuple where ξ_j has been omitted. The lemma follows by expanding the determinant by the first row. □

Note, from Lemma 2.13, that if the 1-forms $\omega_1, \ldots, \omega_p \in \mathrm{Alt}^1(V)$ are linearly independent then $\omega_1 \wedge \ldots \wedge \omega_p \neq 0$. Indeed, we can choose elements $\xi_i \in V$ with $\omega_i(\xi_j) = 0$ for $i \neq j$ and $\omega_j(\xi_j) = 1$, so that $\det(\omega_i(\xi_j)) = 1$. Conversely, if $\omega_1, \ldots, \omega_p$ are linearly dependent, we can express one of them, say ω_p, as a linear combination of the others. If $\omega_p = \sum_{i=1}^{p-1} r_i \omega_i$, then

$$\omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_p = \sum_{i=1}^{p-1} r_i \omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_i = 0,$$

as the determinant in Lemma 2.13 has two equal rows. We have proved

Lemma 2.14 *For 1-forms $\omega_1, \ldots, \omega_p$ on V, $\omega_1 \wedge \ldots \wedge \omega_p \neq 0$ if and only if they are linearly independent.* □

Theorem 2.15 *Let $e_1, \ldots, e_n$ be a basis of V and $\epsilon_1, \ldots, \epsilon_n$ the dual basis of* $\mathrm{Alt}^1(V)$. *Then*

$$\left\{\epsilon_{\sigma(1)} \wedge \epsilon_{\sigma(2)} \wedge \ldots \wedge \epsilon_{\sigma(p)}\right\}_{\sigma \in S(p,n-p)}$$

is a basis of $\mathrm{Alt}^p(V)$. *In particular*

$$\dim \mathrm{Alt}^p(V) = \binom{\dim V}{p}.$$

Proof. Since $\epsilon_i(e_j) = 0$ when $i \neq j$, and $\epsilon_i(e_i) = 1$, Lemma 2.13 gives

$$(3) \qquad \epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_p}(e_{j_1}, \ldots, e_{j_p}) = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_p\} \neq \{j_1, \ldots, j_p\} \\ \mathrm{sign}(\sigma) & \text{if } \{i_1, \ldots, i_p\} = \{j_1, \ldots, j_p\} \end{cases}$$

Here σ is the permutation $\sigma(i_k) = j_k$. From Lemma 2.2 and (3) we get

$$\omega = \sum_{\sigma \in S(p,n-p)} \omega\big(e_{\sigma(1)}, \ldots, e_{\sigma(p)}\big) \epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)}$$

for any alternating p-form. Thus $\epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)}$ generates the vector space $\mathrm{Alt}^p(V)$. Linear independence follows from (3), since a relation

$$\sum_{\sigma \in S(p,n-p)} \lambda_\sigma \epsilon_{\sigma(1)} \wedge \ldots \wedge \epsilon_{\sigma(p)} = 0, \quad \lambda_\sigma \in \mathbf{R}$$

evaluated on $(e_{\sigma(1)}, \ldots, e_{\sigma(p)})$ gives $\lambda_\sigma = 0$. □