

NONLINEAR
PHYSICAL
SCIENCE

Abdul-Majid Wazwaz

Partial Differential Equations and Solitary Waves Theory

偏微分方程与孤波理论



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With 14 figures



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*This book is dedicated to my wife, our son,
and our three daughters for supporting me in
all my endeavors*

Preface

Partial Differential Equations and Solitary Waves Theory is designed to serve as a text and a reference. The book is designed to be accessible to advanced undergraduate and beginning graduate students as well as research monograph to researchers in applied mathematics, science and engineering. This text is different from other texts in that it explains classical methods in a non abstract way and it introduces and explains how the newly developed methods provide more concise methods to provide efficient results.

Partial Differential Equations and Solitary Waves Theory is designed to focus readers' attentions on these recently developed valuable techniques that have proven their effectiveness and reliability over existing classical methods. Moreover, this text also explains the necessary classical methods because the aim is that new methods would complement the traditional methods in order to improve the understanding of the material.

The book avoids approaching the subject through the compact and classical methods that make the material impossible to be grasped, especially by students who do not have the background in these abstract concepts. Compact theorems and abstract handling of the material are not presented in this text.

The book was developed as a result of many years of experience in teaching partial differential equations and conducting research work in this field. The author has taken account on his teaching experience, research work as well as valuable suggestions received from students and scholars from a wide variety of audience. Numerous examples and exercises, ranging in level from easy to difficult, but consistent with the material, are given in each section to give the reader the knowledge, practice and skill in partial differential equations and solitary waves theory. There is plenty of material in this text to be covered in two semesters for senior undergraduates and beginning graduates of Mathematics, Science, and Engineering.

The content of the book is divided into two distinct parts, each is a self-contained and practical part. Part I contains eleven chapters that handle the partial differential equations by using the newly developed methods, namely, *Adomian decomposition method* and *Variational Iteration Method*. Some of the traditional methods are used in this part. With a diverse readership and interdisciplinary audience of applied

mathematics, science, and engineering, attempts are made so that part I presents both analytical and numerical approaches in a clear and systematic fashion to make this book accessible to many who work in this field.

Part II contains seven chapters devoted to thoroughly examine solitary waves theory. Since the discovery of solitons in 1965, mathematicians, engineers, and physicists have been intrigued by the rich mathematical structure of solitons. Solitons play a prevalent role in propagation of light in fibers, surface waves in nonlinear dielectrics, optical bistability, optical switching in slab wave guides, and many other phenomena in plasma and fluid dynamics.

Chapter 1 provides the basic definitions and introductory concepts. Initial value problems and boundary value problems are discussed. In Chapter 2, the first order partial differential equations are handled by the newly developed methods, namely, the *Adomian decomposition method* (ADM) and the *variational iteration method* (VIM). The method of characteristics is introduced and explained in detail. Chapter 3 deals with the one-dimensional heat flow where homogeneous and inhomogeneous initial-boundary value problems are approached by using the decomposition method, the variational iteration method and the method of separation of variables. Chapter 4 is entirely devoted to the two-dimensional and three-dimensional heat flow. Chapter 5 provides the reader with a comprehensive discussion of the literature related to the one-dimensional wave equation. The decomposition method and the variational iteration method are used in handling the wave equations in a bounded and an unbounded domain. Moreover, the method of separation of variables and the D'Alembert method are also used. Chapter 6 presents a comprehensive study on wave equations in two-dimensional and three-dimensional spaces. Chapter 7 is devoted to the Laplace's equation in two- and three-dimensional rectangular coordinates and in polar coordinates. Moreover, the Laplace's equation in annulus form is also investigated by using the decomposition method and the separation of variables method. Chapter 8 introduces a comprehensive study on nonlinear partial differential equations. Even though the subject is considered difficult and mostly addressed in distinct books independent of linear PDEs, but it will be handled successfully and elegantly by using the newly developed decomposition method and the variational iteration method. Chapter 9 provides the reader with a variety of linear and nonlinear applications selected from mathematical physics, population growth models and evolution concepts. The useful concept of solitons and the recently developed concept of *Compactons* are thoroughly examined by using both traditional and new methods. Chapter 10 is concerned with the numerical techniques. Emphasis in this chapter will be on combining the decomposition series solution, the variational iteration method, and the Padé approximants to provide a promising tool that can be applied for further applications. Chapter 11 is concerned with the concepts of solitons and compactons. In this chapter, the solitons and compactons are determined by using prescribed conditions, a necessary condition for the applicability of the decomposition method.

Part II of this book gives a self-contained, practical and realistic approach to solitary wave theory. The dissipation and the dispersion effects are thoroughly investigated. Solitons play a prevalent role in many scientific and engineering phenomena.

The newly discovered *compactons*: solitons with a compact support are also studied. Part II of this book is devoted to use mainly the Hirota's bilinear method, combined with simplified version developed by Hereman and the tanh-coth method. Chapter 12 presents discussions about the dissipation and dispersion effects, analytic and nonanalytic solutions, conservation laws and multiple-soliton solutions, tanh-coth method, and Hirota's bilinear method combined with the Hereman's simplified form of the Hirota's method. In Chapter 13, the family of the KdV equations is studied. Multi-soliton solutions are obtained for only completely integrable equations of this family. Compactons solutions are also examined. Chapter 14 is concerned with KdV and mKdV equations of higher orders. The single solitons and the multiple-soliton solutions for completely integrable equations are addressed by using the Hirota's bilinear method. In addition, the Hirota-Satsuma equations and the generalized short wave equations were investigated for multiple-soliton solutions.

Chapter 15 investigates many KdV-type of equations where soliton solutions and multi-soliton solutions are obtained by using tanh-coth method and Hirota's method respectively. Chapter 16 is entirely devoted to study a family of well-known physical models for solitons and multi-soliton solutions as well. Some of these equations are Boussinesq equation, Klein-Gordon equation, Liouville equation, sine-Gordon equation, DBM equation, and others. Chapter 17 provides the reader with a comprehensive discussion of the literature related to Burgers, Fisher, Huxley, FitzHugh-Nagumo equations and related equations. Most of these equations are characterized by the dissipation phenomena that give kinks solutions. Chapter 18 presents a comprehensive study on two distinct types of equations that appear in solitary wave theory. The family of Camassa-Holm equations is examined to obtain the nonanalytic solution of peakons. On the other hand, the Schrodinger and Ginzburg-Landau equations of different orders are studied in this chapter.

The book concludes with six useful appendices. Moreover, the book introduces the traditional methods in the same amount of concern to provide the reader with the knowledge needed to make a comparison.

I deeply acknowledge Professor Louis Pennisi who made very valuable suggestions that helped a great deal in directing this book towards its main goal. I also deeply acknowledge Professor Masaaki Ito and Professor Willy Hereman for many helpful discussions and useful remarks. I owe them my deepest thanks.

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The author would highly appreciate any note concerning any constructive suggestion.

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Contents

Part I Partial Differential Equations

1	Basic Concepts	3
1.1	Introduction	3
1.2	Definitions	4
1.2.1	Definition of a PDE	4
1.2.2	Order of a PDE	5
1.2.3	Linear and Nonlinear PDEs	6
1.2.4	Some Linear Partial Differential Equations	7
1.2.5	Some Nonlinear Partial Differential Equations	7
1.2.6	Homogeneous and Inhomogeneous PDEs	9
1.2.7	Solution of a PDE	9
1.2.8	Boundary Conditions	11
1.2.9	Initial Conditions	12
1.2.10	Well-posed PDEs	12
1.3	Classifications of a Second-order PDE	14
	References	17
2	First-order Partial Differential Equations	19
2.1	Introduction	19
2.2	Adomian Decomposition Method	19
2.3	The Noise Terms Phenomenon	36
2.4	The Modified Decomposition Method	41
2.5	The Variational Iteration Method	47
2.6	Method of Characteristics	54
2.7	Systems of Linear PDEs by Adomian Method	59
2.8	Systems of Linear PDEs by Variational Iteration Method	63
	References	68

3	One Dimensional Heat Flow	69
3.1	Introduction	69
3.2	The Adomian Decomposition Method	70
3.2.1	Homogeneous Heat Equations	73
3.2.2	Inhomogeneous Heat Equations	80
3.3	The Variational Iteration Method	83
3.3.1	Homogeneous Heat Equations	84
3.3.2	Inhomogeneous Heat Equations	87
3.4	Method of Separation of Variables	89
3.4.1	Analysis of the Method	89
3.4.2	Inhomogeneous Boundary Conditions	99
3.4.3	Equations with Lateral Heat Loss	102
	References	106
4	Higher Dimensional Heat Flow	107
4.1	Introduction	107
4.2	Adomian Decomposition Method	108
4.2.1	Two Dimensional Heat Flow	108
4.2.2	Three Dimensional Heat Flow	116
4.3	Method of Separation of Variables	124
4.3.1	Two Dimensional Heat Flow	124
4.3.2	Three Dimensional Heat Flow	134
	References	140
5	One Dimensional Wave Equation	143
5.1	Introduction	143
5.2	Adomian Decomposition Method	144
5.2.1	Homogeneous Wave Equations	146
5.2.2	Inhomogeneous Wave Equations	152
5.2.3	Wave Equation in an Infinite Domain	157
5.3	The Variational Iteration Method	162
5.3.1	Homogeneous Wave Equations	162
5.3.2	Inhomogeneous Wave Equations	168
5.3.3	Wave Equation in an Infinite Domain	170
5.4	Method of Separation of Variables	174
5.4.1	Analysis of the Method	174
5.4.2	Inhomogeneous Boundary Conditions	184
5.5	Wave Equation in an Infinite Domain: D'Alembert Solution	190
	References	194
6	Higher Dimensional Wave Equation	195
6.1	Introduction	195
6.2	Adomian Decomposition Method	195
6.2.1	Two Dimensional Wave Equation	196
6.2.2	Three Dimensional Wave Equation	210

6.3	Method of Separation of Variables	220
6.3.1	Two Dimensional Wave Equation	221
6.3.2	Three Dimensional Wave Equation	231
	References	236
7	Laplace's Equation	237
7.1	Introduction	237
7.2	Adomian Decomposition Method	238
7.2.1	Two Dimensional Laplace's Equation	238
7.3	The Variational Iteration Method	247
7.4	Method of Separation of Variables	251
7.4.1	Laplace's Equation in Two Dimensions	251
7.4.2	Laplace's Equation in Three Dimensions	259
7.5	Laplace's Equation in Polar Coordinates	267
7.5.1	Laplace's Equation for a Disc	268
7.5.2	Laplace's Equation for an Annulus	275
	References	283
8	Nonlinear Partial Differential Equations	285
8.1	Introduction	285
8.2	Adomian Decomposition Method	287
8.2.1	Calculation of Adomian Polynomials	288
8.2.2	Alternative Algorithm for Calculating Adomian Polynomials	292
8.3	Nonlinear ODEs by Adomian Method	301
8.4	Nonlinear ODEs by VIM	312
8.5	Nonlinear PDEs by Adomian Method	319
8.6	Nonlinear PDEs by VIM	334
8.7	Nonlinear PDEs Systems by Adomian Method	341
8.8	Systems of Nonlinear PDEs by VIM	347
	References	351
9	Linear and Nonlinear Physical Models	353
9.1	Introduction	353
9.2	The Nonlinear Advection Problem	354
9.3	The Goursat Problem	360
9.4	The Klein-Gordon Equation	370
9.4.1	Linear Klein-Gordon Equation	371
9.4.2	Nonlinear Klein-Gordon Equation	375
9.4.3	The Sine-Gordon Equation	378
9.5	The Burgers Equation	381
9.6	The Telegraph Equation	388
9.7	Schrodinger Equation	394
9.7.1	The Linear Schrodinger Equation	394
9.7.2	The Nonlinear Schrodinger Equation	397

9.8	Korteweg-deVries Equation	401
9.9	Fourth-order Parabolic Equation	405
9.9.1	Equations with Constant Coefficients	405
9.9.2	Equations with Variable Coefficients	408
	References	413
10	Numerical Applications and Padé Approximants	415
10.1	Introduction	415
10.2	Ordinary Differential Equations	416
10.2.1	Perturbation Problems	416
10.2.2	Nonperturbed Problems	421
10.3	Partial Differential Equations	427
10.4	The Padé Approximants	430
10.5	Padé Approximants and Boundary Value Problems	439
	References	455
11	Solitons and Compactons	457
11.1	Introduction	457
11.2	Solitons	459
11.2.1	The KdV Equation	460
11.2.2	The Modified KdV Equation	462
11.2.3	The Generalized KdV Equation	464
11.2.4	The Sine-Gordon Equation	464
11.2.5	The Boussinesq Equation	465
11.2.6	The Kadomtsev-Petviashvili Equation	467
11.3	Compactons	469
11.4	The Defocusing Branch of $K(n, n)$	474
	References	475
Part II Solitary Waves Theory		
12	Solitary Waves Theory	479
12.1	Introduction	479
12.2	Definitions	480
12.2.1	Dispersion and Dissipation	482
12.2.2	Types of Travelling Wave Solutions	484
12.2.3	Nonanalytic Solitary Wave Solutions	490
12.3	Analysis of the Methods	491
12.3.1	The Tanh-coth Method	491
12.3.2	The Sine-cosine Method	493
12.3.3	Hirota's Bilinear Method	494
12.4	Conservation Laws	496
	References	502

13	The Family of the KdV Equations	503
13.1	Introduction	503
13.2	The Family of the KdV Equations	505
13.2.1	Third-order KdV Equations	505
13.2.2	The $K(n,n)$ Equation	507
13.3	The KdV Equation	507
13.3.1	Using the Tanh-coth Method	508
13.3.2	Using the Sine-cosine Method	510
13.3.3	Multiple-soliton Solutions of the KdV Equation	510
13.4	The Modified KdV Equation	518
13.4.1	Using the Tanh-coth Method	519
13.4.2	Using the Sine-cosine Method	520
13.4.3	Multiple-soliton Solutions of the mKdV Equation	521
13.5	Singular Soliton Solutions	523
13.6	The Generalized KdV Equation	526
13.6.1	Using the Tanh-coth Method	526
13.6.2	Using the Sine-cosine Method	528
13.7	The Potential KdV Equation	528
13.7.1	Using the Tanh-coth Method	529
13.7.2	Multiple-soliton Solutions of the Potential KdV Equation	531
13.8	The Gardner Equation	533
13.8.1	The Kink Solution	533
13.8.2	The Soliton Solution	534
13.8.3	N -soliton Solutions of the Positive Gardner Equation	535
13.8.4	Singular Soliton Solutions	537
13.9	Generalized KdV Equation with Two Power Nonlinearities	542
13.9.1	Using the Tanh Method	543
13.9.2	Using the Sine-cosine Method	544
13.10	Compactons: Solitons with Compact Support	544
13.10.1	The $K(n,n)$ Equation	546
13.11	Variants of the $K(n,n)$ Equation	547
13.11.1	First Variant	548
13.11.2	Second Variant	549
13.11.3	Third Variant	551
13.12	Compacton-like Solutions	553
13.12.1	The Modified KdV Equation	553
13.12.2	The Gardner Equation	554
13.12.3	The Modified Equal Width Equation	554
	References	555

14	KdV and mKdV Equations of Higher-orders	557
14.1	Introduction	557
14.2	Family of Higher-order KdV Equations	558
14.2.1	Fifth-order KdV Equations	558
14.2.2	Seventh-order KdV Equations	561
14.2.3	Ninth-order KdV Equations	562
14.3	Fifth-order KdV Equations	562
14.3.1	Using the Tanh-coth Method	563
14.3.2	The First Condition	564
14.3.3	The Second Condition	566
14.3.4	N -soliton Solutions of the Fifth-order KdV Equations	567
14.4	Seventh-order KdV Equations	576
14.4.1	Using the Tanh-coth Method	576
14.4.2	N -soliton Solutions of the Seventh-order KdV Equations	578
14.5	Ninth-order KdV Equations	582
14.5.1	Using the Tanh-coth Method	583
14.5.2	The Soliton Solutions	584
14.6	Family of Higher-order mKdV Equations	585
14.6.1	N -soliton Solutions for Fifth-order mKdV Equation	586
14.6.2	Singular Soliton Solutions for Fifth-order mKdV Equation	587
14.6.3	N -soliton Solutions for the Seventh-order mKdV Equation	589
14.7	Complex Solution for the Seventh-order mKdV Equations	591
14.8	The Hirota-Satsuma Equations	592
14.8.1	Using the Tanh-coth Method	593
14.8.2	N -soliton Solutions of the Hirota-Satsuma System	594
14.8.3	N -soliton Solutions by an Alternative Method	596
14.9	Generalized Short Wave Equation	597
	References	602
15	Family of KdV-type Equations	605
15.1	Introduction	605
15.2	The Complex Modified KdV Equation	606
15.2.1	Using the Sine-cosine Method	607
15.2.2	Using the Tanh-coth Method	608
15.3	The Benjamin-Bona-Mahony Equation	612
15.3.1	Using the Sine-cosine Method	612
15.3.2	Using the Tanh-coth Method	613
15.4	The Medium Equal Width (MEW) Equation	615
15.4.1	Using the Sine-cosine Method	615
15.4.2	Using the Tanh-coth Method	616
15.5	The Kawahara and the Modified Kawahara Equations	617
15.5.1	The Kawahara Equation	618
15.5.2	The Modified Kawahara Equation	619

15.6	The Kadomtsev-Petviashvili (KP) Equation	620
15.6.1	Using the Tanh-coth Method	621
15.6.2	Multiple-soliton Solutions of the KP Equation	622
15.7	The Zakharov-Kuznetsov (ZK) Equation	626
15.8	The Benjamin-Ono Equation	629
15.9	The KdV-Burgers Equation	630
15.10	Seventh-order KdV Equation	632
15.10.1	The Sech Method	632
15.11	Ninth-order KdV Equation	634
15.11.1	The Sech Method	634
	References	637
16	Boussinesq, Klein-Gordon and Liouville Equations	639
16.1	Introduction	639
16.2	The Boussinesq Equation	641
16.2.1	Using the Tanh-coth Method	641
16.2.2	Multiple-soliton Solutions of the Boussinesq Equation	643
16.3	The Improved Boussinesq Equation	646
16.4	The Klein-Gordon Equation	648
16.5	The Liouville Equation	649
16.6	The Sine-Gordon Equation	651
16.6.1	Using the Tanh-coth Method	651
16.6.2	Using the Bäcklund Transformation	654
16.6.3	Multiple-soliton Solutions for Sine-Gordon Equation	655
16.7	The Sinh-Gordon Equation	657
16.8	The Dodd-Bullough-Mikhailov Equation	658
16.9	The Tzitzeica-Dodd-Bullough Equation	659
16.10	The Zhiber-Shabat Equation	661
	References	662
17	Burgers, Fisher and Related Equations	665
17.1	Introduction	665
17.2	The Burgers Equation	666
17.2.1	Using the Tanh-coth Method	667
17.2.2	Using the Cole-Hopf Transformation	668
17.3	The Fisher Equation	670
17.4	The Huxley Equation	671
17.5	The Burgers-Fisher Equation	673
17.6	The Burgers-Huxley Equation	673
17.7	The FitzHugh-Nagumo Equation	675
17.8	Parabolic Equation with Exponential Nonlinearity	676
17.9	The Coupled Burgers Equation	678
17.10	The Kuramoto-Sivashinsky (KS) Equation	680
	References	681

18	Families of Camassa-Holm and Schrodinger Equations	683
18.1	Introduction	683
18.2	The Family of Camassa-Holm Equations	686
18.2.1	Using the Tanh-coth Method	686
18.2.2	Using an Exponential Algorithm	688
18.3	Schrodinger Equation of Cubic Nonlinearity	689
18.4	Schrodinger Equation with Power Law Nonlinearity	690
18.5	The Ginzburg-Landau Equation	692
18.5.1	The Cubic Ginzburg-Landau Equation	693
18.5.2	The Generalized Cubic Ginzburg-Landau Equation	694
18.5.3	The Generalized Quintic Ginzburg-Landau Equation	695
	References	696
	Appendix	699
A	Indefinite Integrals	699
A.1	Fundamental Forms	699
A.2	Trigonometric Forms	700
A.3	Inverse Trigonometric Forms	700
A.4	Exponential and Logarithmic Forms	701
A.5	Hyperbolic Forms	701
A.6	Other Forms	702
B	Series	703
B.1	Exponential Functions	703
B.2	Trigonometric Functions	703
B.3	Inverse Trigonometric Functions	704
B.4	Hyperbolic Functions	704
B.5	Inverse Hyperbolic Functions	704
C	Exact Solutions of Burgers' Equation	705
D	Padé Approximants for Well-Known Functions	707
D.1	Exponential Functions	707
D.2	Trigonometric Functions	707
D.3	Hyperbolic Functions	709
D.4	Logarithmic Functions	709
E	The Error and Gamma Functions	711
E.1	The Error function	711
E.2	The Gamma function $\Gamma(x)$	711