

NONLINEAR  
PHYSICAL  
SCIENCE

Albert C. J. Luo

# Discontinuous Dynamical Systems on Time-varying Domains

动态域上的不连续动力学系统



高等教育出版社  
HIGHER EDUCATION PRESS

Albert C.J. Luo

# Discontinuous Dynamical Systems on Time-varying Domains

动态域上的不连续动力学系统

With 96 figures, 4 of them in color



高等教育出版社  
HIGHER EDUCATION PRESS

Author

Albert C.J. Luo

Department of Mechanical and Industrial Engineering

Southern Illinois University Edwardsville

Edwardsville, IL 62026-1805, USA

Email: aluo@siue.edu

© 2009 Higher Education Press, 4 Dewai Dajie, 100120, Beijing, P.R.China

### 图书在版编目(CIP)数据

动态域上的不连续动力学系统=Discontinuous Dynamical Systems on Time-varying Domains: 英文 / 罗朝俊著. —北京: 高等教育出版社, 2009.5

(非线性物理科学 / 罗朝俊, (瑞典) 伊布拉基莫夫主编)

ISBN 978-7-04-025759-5

I. 动… II. 罗… III. 系统动力学-研究-英文 IV. N941.3

中国版本图书馆 CIP 数据核字(2009)第 025635 号

策划编辑 王丽萍 责任编辑 刘占伟 封面设计 杨立新 责任绘图 尹 莉  
版式设计 陆瑞红 责任校对 姜国萍 责任印制 陈伟光

---

出版发行	高等教育出版社	购书热线	010-58581118
社 址	北京市西城区德外大街 4 号	免费咨询	400-810-0598
邮政编码	100120	网 址	<a href="http://www.hep.edu.cn">http://www.hep.edu.cn</a>
总 机	010-58581000		<a href="http://www.hep.com.cn">http://www.hep.com.cn</a>
		网上订购	<a href="http://www.landaco.com">http://www.landaco.com</a>
经 销	蓝色畅想图书发行有限公司		<a href="http://www.landaco.com.cn">http://www.landaco.com.cn</a>
印 刷	涿州市星河印刷有限公司	畅想教育	<a href="http://www.widedu.com">http://www.widedu.com</a>
开 本	787 × 1092 1/16		
印 张	14.75	版 次	2009 年 5 月第 1 版
字 数	250 000	印 次	2009 年 5 月第 1 次印刷
插 页	2	定 价	46.00 元

---

本书如有缺页、倒页、脱页等质量问题, 请到所购图书销售部门联系调换。

版权所有 侵权必究

物料号 25759-00

Sales only inside the mainland of China

{仅限中国大陆地区销售}



**Earthquake and Heartbreak**

*that shocks one's passion and spirit,  
and recalls everyone to a sense of duty.*

## Preface

This book is about discontinuous dynamical systems on time-varying domains. I had not planned to write this book originally. As a scientist working on dynamics and vibration, the 5.12 earthquake of Wenchuan (Sichuan province, China) shocked my heart and made me feel guilty because my research cannot make any direct contributions to help them. Therefore, I would like to write two words “Zhenhan” in Chinese calligraphy on the dedication page to express my passion. The meaning of “Zhenhan” is “Earthquake and Heartbreak” that shocks one’s passion and spirit, and recalls everyone to a sense of duty. Herein, I would like to accumulate recent research developments of discontinuous dynamical systems on time-varying domains. One likes to use continuous models for discontinuous dynamical systems. However, sometimes such continuous modeling cannot provide adequate descriptions of discontinuous dynamical systems. Recently, researchers have gradually realized that discontinuous modeling may provide an adequate and acceptable prediction of engineering systems. Currently, most research still focuses on discontinuous dynamical systems on time-invariant domains. To better describe practical problems, some research on discontinuous systems on time-varying domains is scattered here and there but without a systematic theory. The purpose of this book is to systematically present a theory of discontinuous dynamical systems on time-varying domains for university students and researchers.

This book mainly focuses on the switchability of discontinuous dynamical systems on time-varying domains. Based on such concepts, principles of dynamical system interactions without any connections are presented. This book consists of seven chapters. Chapter 1 discusses two examples to show where discontinuous dynamical systems exist. Chapter 2 presents a basic theory for the switchability of a flow to the separation boundary in discontinuous dynamical systems, and switching bifurcations are also addressed. In Chapter 3, transversality and sliding phenomena for a controlled dynamical system to an inclined line boundary of control logic are presented to show how to apply such a new theory. In Chapter 4, dynamics of a frictional oscillator on a traveling belt with time-varying speeds is presented, which is a simple example of discontinuous dynamical systems on the time-varying domains. Chapter 5 presents the dynamics mechanism of impacting chatter and stick

phenomena in two dynamical systems with impact laws. In Chapter 6, dynamical behaviors of two systems connected with friction are presented. In Chapters 3–6, the similar writing styles and formats are adopted to show how to apply the new theory to different problems. In Chapter 7, a generalized theory for the interaction principle of dynamical systems is developed from discontinuous dynamical systems on time-varying domains. I hope the information presented in this book may stimulate more research in the area of discontinuous dynamical systems.

Finally, I would like to appreciate my students (Brandon M. Rapp, Brandon C. Gegg, Dennis O'Connor and Sagun Thapa) for applying the new concepts to mechanical systems and completing all numerical computations. This book is dedicated to the people who died during the earthquake of Wenchuan (Sichuan province, China) on May 12, 2008. Finally, I would like to thank my wife (Sherry X. Huang) and my children (Yanyi Luo, Robin Ruo-Bing Luo, and Robert Zong-Yuan Luo) for their tolerance, patience, understanding and support.

Albert C. J. Luo  
Edwardsville, Illinois  
August, 2008

# Contents

<b>1</b>	<b>Introduction</b> .....	1
1.1	Discontinuous systems .....	1
1.2	Book layout .....	5
	References .....	6
<b>2</b>	<b>Flow Switchability</b> .....	9
2.1	Discontinuous dynamic systems .....	9
2.2	$G$ -functions .....	11
2.3	Passable flows .....	15
2.4	Non-passable flows .....	19
2.5	Tangential flows .....	28
2.6	Switching bifurcations .....	39
	References .....	52
<b>3</b>	<b>Transversality and Sliding Phenomena</b> .....	55
3.1	A controlled system .....	55
3.2	Transversality conditions .....	57
3.3	Mappings and predictions .....	60
3.4	Periodic and chaotic motions .....	67
	References .....	75
<b>4</b>	<b>A Frictional Oscillator on Time-varying Belt</b> .....	77
4.1	Mechanical model .....	77
4.2	Analytical conditions .....	80
4.2.1	Equations of motion .....	80
4.2.2	Passable flows to boundary .....	83
4.2.3	Sliding flows on boundary .....	85
4.2.4	Grazing flows to boundary .....	89
4.3	Generic mappings and force product criteria .....	91
4.3.1	Generic mappings .....	91

4.3.2	Sliding flows and fragmentation . . . . .	93
4.3.3	Grazing flows . . . . .	96
4.4	Periodic motions . . . . .	98
4.4.1	Mapping structures . . . . .	98
4.4.2	Illustrations . . . . .	100
4.5	Numerical simulations . . . . .	112
	References . . . . .	113
<b>5</b>	<b>Two Oscillators with Impacts and Stick . . . . .</b>	<b>115</b>
5.1	Physical problem . . . . .	115
5.1.1	Introduction to problem . . . . .	115
5.1.2	Equations of motion . . . . .	117
5.2	Domains and vector fields . . . . .	119
5.2.1	Absolute motion description . . . . .	119
5.2.2	Relative motion description . . . . .	125
5.3	Mechanism of stick and grazing . . . . .	127
5.3.1	Analytical conditions . . . . .	128
5.3.2	Physical interpretation . . . . .	133
5.4	Mapping structures and motions . . . . .	134
5.4.1	Switching sets and basic mappings . . . . .	134
5.4.2	Mapping equations . . . . .	137
5.4.3	Mapping structures . . . . .	141
5.4.4	Bifurcation scenario . . . . .	144
5.5	Periodic motion prediction . . . . .	144
5.5.1	Approach . . . . .	145
5.5.2	Impacting chatter . . . . .	147
5.5.3	Impacting chatter with stick . . . . .	150
5.5.4	Parameter maps . . . . .	152
5.6	Numerical illustrations . . . . .	153
5.6.1	Impacting chatter . . . . .	153
5.6.2	Impacting chatter with stick . . . . .	155
5.6.3	Further illustrations . . . . .	158
	References . . . . .	160
<b>6</b>	<b>Dynamical Systems with Frictions . . . . .</b>	<b>161</b>
6.1	Problem statement . . . . .	161
6.2	Switching and stick motions . . . . .	164
6.2.1	Equations of motion . . . . .	164
6.2.2	Analytical conditions . . . . .	166
6.3	Periodic motions . . . . .	172
6.3.1	Switching planes and mappings . . . . .	172
6.3.2	Mapping structures and motions . . . . .	175
6.3.3	Bifurcation scenario . . . . .	178
6.4	Numerical illustrations . . . . .	181
6.4.1	Periodic motion without stick . . . . .	181



6.4.2	Periodic motion with stick	184
6.4.3	Periodic motion with stick only	189
	References	189
<b>7</b>	<b>Principles for System Interactions</b>	191
7.1	Two dynamical systems	191
7.1.1	Dynamical systems with interactions	191
7.1.2	Discontinuous description	195
7.1.3	Resultant dynamical systems	196
7.2	Fundamental interactions	199
7.3	Interactions with singularity	206
7.4	Interactions with corner singularity	210
	References	215
	<b>Appendix</b>	217
A.1	Basic solution	217
A.2	Stability and bifurcation	219
	<b>Index</b>	221

# Chapter 1

## Introduction

In engineering, discontinuous dynamical systems exist everywhere. One usually uses continuous models to describe discontinuous dynamical systems. However such continuous models cannot provide suitable predictions of discontinuous dynamical systems. To better understand discontinuous systems, one should realize that discontinuous models will provide an adequate and real prediction of engineering systems. Thus, one considers a global discontinuous system consisting of many continuous sub-systems in different domains. For each continuous subsystem, it possesses dynamical properties different from the adjacent continuous subsystems. Because of such difference between two adjacent subsystems, the switch ability and/or transport laws on their boundaries should be addressed. The investigation on such discontinuous systems mainly focused on the time-independent boundary between two dynamical systems. In fact, the boundary relative to time is more popular. In this book, discontinuous dynamical systems on time-varying domains will be of great interest. A brief survey will be given through two practical examples. Finally, the book layout will be presented, and the summarization of all chapters of the main body of this book will be given.

### 1.1 Discontinuous systems

In mechanical engineering, there are two common and important contacts between two dynamical systems, i.e., impact and friction. For example, gear transmission systems possess both impact and friction. Such gear transmission systems are used to transmit power between parallel shafts or to change direction. During the power transmission, a pair of two gears forms a resultant dynamical system. Each gear has its own dynamical system connected with shafts and bearings. Because two subsystems are not connected directly, the power transmission is completed through the impact and friction. Because both of sub-systems are independent each other except for impacting and sliding together, such two dynamical systems have a common

time-varying boundary for impacting, which causes domains for the two dynamical systems to be time-varying.

In the early investigation, a piecewise stiffness model was used to investigate dynamics of gear transmission systems. Although such a dynamical system is discontinuous, the corresponding domains for vector fields of the dynamical system are time-independent. For instance, den Hartog and Mikina (1932) used a piecewise linear system without damping to model gear transmission systems, and the symmetric periodic motion in such a system was investigated. For low-speed gear systems, such a linear model gave a reasonable prediction of gear-tooth vibrations. With increasing rotation speed in gear transmission systems, vibrations and noise become serious. Ozguven and Houser (1988) gave a survey on the mathematical models of gear transmission systems. The piecewise linear model and the impact model were two of the main mechanical models to investigate the origin of vibration and noise in gear transmission systems. Natsiavas (1998) investigated a piecewise linear system with a symmetric tri-linear spring, and the stability and bifurcation of periodic motions in such a system were investigated through the variation of initial conditions. Based on a piecewise linear model, the dynamics of gear transmission systems was investigated by Comparin and Singh (1989), and Theodossiades and Natsiavas (2000). Pfeiffer (1984) presented an impact model of gear transmissions, and the theoretical and experimental investigations on regular and chaotic motions in the gear box were later carried out by Karagiannis and Pfeiffer (1991).

To model vibrations in gear transmission systems, Luo and Chen (2005) gave an analytical prediction of the simplest periodic motion through a piecewise linear impacting system. In addition, the corresponding grazing of periodic motions was observed, and chaotic motions were simulated numerically through such a piecewise linear system. From the local singularity theory in Luo (2005), the grazing mechanism of the strange fragmentation of the piecewise linear system was discussed by Luo and Chen (2006). Luo and Chen (2007) used the mapping structure technique to analytically predict arbitrary periodic motions of such a piecewise linear system. In that piecewise linear model, it was assumed that impact locations were fixed, and the perfectly plastic impact was considered. The separation of the two gears occurred at the same location as the gears impact. Compared with the existing models, this model can give a better prediction of periodic motions in gear transmission systems, but the relative assumptions may not be realistic to practical transmission systems because all the aforementioned investigations are based on a time-independent boundary or a given motion boundary. To consider the dynamical systems with the time-varying boundary, Luo and O'Connor (2007a,b) proposed a mechanical model to determine the mechanism of the impacting chatter and stick in gear transmission systems. The corresponding analytical conditions for such impacting chatter and stick were developed.

In mechanical engineering, the friction contact between surfaces of two systems is important for motion transmissions (e.g., clutch systems, brake systems, etc.). In addition, the two systems are independent except for the friction contact. Such problem will have time-varying boundary and domains. For such a friction problem, it should return back to the 1930's. den Hartog (1931) investigated the peri-

odic motion of a forced, damped, linear oscillator contacting a surface with friction. Levitan (1961) investigated the existence of periodic motions in a friction oscillator with a periodically driven base. Filippov (1964) investigated the motion existence of a Coulomb friction oscillator, and presented a differential equation theory with discontinuous right-hand sides. The differential inclusion was introduced via the set-valued analysis for the sliding motion along the discontinuous boundary. The investigations of discontinuous differential equations were summarized by Filippov (1988). However, the Filippov's theory mainly focused on the existence and uniqueness of solutions for non-smooth dynamical systems. Such a differential equation theory with discontinuity is difficult to apply to practical problems. Luo (2005a) developed a general theory to handle the local singularity of discontinuous dynamical systems. To determine the sliding and source motions in discontinuous dynamical systems, the imaginary, sink and source flows were introduced by Luo (2005b). The detailed discussions can be referred to Luo (2006).

On the other hand, Hundal (1979) used a periodic, continuous function to investigate the frequency-amplitude response of a friction oscillator. Shaw (1986) investigated non-stick, periodic motions of a friction oscillator through Poincaré mapping. Feeny (1992) analytically investigated the non-smooth of the Coulomb friction oscillator. To verify the analytic results, Feeny and Moon (1994) investigated chaotic dynamics of a dry-friction oscillator experimentally and numerically. Feeny (1996) gave a systematic discussion of the nonlinear dynamical mechanism of stick-slip motion of friction oscillators. Hinrichs, Oestreich and Popp (1997) investigated the nonlinear phenomena in an impact and friction oscillator under external excitations (also see, Hinrichs, Oestreich and Popp (1998)). Natsiavas (1998) developed an algorithm to numerically determine the periodic motion and the corresponding stability of piecewise linear oscillators with viscous and dry friction damping (also see, Natsiavas and Verros (1999)). Ko, Taponat and Pfaifer (2001) investigated the friction-induced vibrations with and without external excitations. Andreaus and Casini (2002) gave a closed form solution of a Coulomb friction-impact model without external excitations. Thomsen and Fidlin (2003) gave an approximate estimation of response amplitude for the stick-slip motion in a nonlinear friction oscillator. Kim and Perkins (2003) investigated stick-slip motions in a friction oscillator via the harmonic balance or Galerkin method. Li and Feng (2004) investigated the bifurcation and chaos in a friction-induced oscillator with a nonlinear friction model. Pilipchuk and Tan (2004) investigated the dynamical behaviors of a 2DOF mass-damper-spring system contacting on a decelerating rigid strip with friction. Awrejcewicz and Pyryev (2004) gave an investigation on frictional periodic processes by accelerating or braking a shaft-pad system. In 2007, Hetzler, Schwarzer and Seemann (2007) considered a nonlinear friction model to analytically investigate the Hopf-bifurcation in a sliding friction oscillator with applications to the low frequency disk brake noise.

In the aforementioned investigations, the conditions for motion switchability to the discontinuous boundary were not considered enough. Luo and Gegg (2005a) used the local singularity theory of Luo (2005a, 2006) to develop the force criteria for motion switchability on the velocity boundary in a harmonically driven linear

oscillator with dry-friction (also see, Luo and Gegg (2005b)). Through such an investigation, the traditional eigenvalue analysis may not be useful for motion switching at the discontinuous boundary. Lu (2007) mathematically proved the existence of such a periodic motion in a friction oscillator. Luo and Gegg(2006a,b) investigated the dynamics of a friction-induced oscillator contacting on time-varying belts with friction. Recently, to model the disk brake system, many researchers still considered the mechanical model as in Hetzler, Schwarzer and Seemann (2007). Luo and Thapa (2007) proposed a new method to model the brake system which consists of two oscillators, and the two oscillators are connected through a contacting surface with friction. Based on this model, the nonlinear dynamical behaviors of a brake system under a periodical excitation were investigated.

The other developments on non-smooth dynamical systems in recent decades will be addressed as well. Feigin (1970) investigated the C-bifurcation in piecewise-continuous systems via the Floquet theory of mappings, and the motion complexity was classified by the eigenvalues of mappings, which can be referred to recent publications (e.g., Feigin 1995; di Bernnado et al 1999). The C-bifurcation is also termed the grazing bifurcation by many researchers. Nordmark (1991) used “grazing” terminology to describe the grazing phenomena in a simple impact oscillator. No strict mathematical description was given, but the grazing condition (i.e., the velocity  $dx/dt = 0$  for displacement  $x$ ) in such an impact oscillator was obtained. From Luo (2005a, 2006), such a grazing condition is a necessary condition only. The grazing is the tangency between an  $n$ -D flow of dynamical systems and the discontinuous boundary surface. From differential geometry points of view, Luo (2005a) gave the strict mathematic definition of the “grazing”, and the necessary and sufficient conditions of the general discontinuous boundary were presented (also see, Luo 2006). Nordmark’s result is a special case. Nusse and Yorke (1992) used the simple discrete mapping from Nordmark’s impact oscillator and showed the bifurcation phenomena numerically. Based on the numerical observation, the sudden change bifurcation in the numerical simulation is called the *border-collision bifurcation*. So, the similar discrete mappings in discontinuous dynamical system were further developed. Especially, Nordmark and Dankowicz (1999) developed a discontinuous mapping from a general way to investigate the grazing bifurcation, and the discontinuous mapping is based on the Taylor series expansion in the neighborhood of the discontinuous boundary. Following the same idea, di Bernardo et al (2001a,b; 2002) developed a normal form to describe the grazing bifurcation. In addition, di Bernardo et al (2001c) used the normal form to obtain the discontinuous mapping and numerically observed such a border-collision bifurcation through a discontinuous mapping. From the discontinuous mapping and its normal form, the aforementioned bifurcation theory structure was developed for the so-called, codimension 1 dynamical system.

The discontinuous mapping and normal forms on the discontinuous boundary were developed from the Taylor series expansion in the neighborhood of the discontinuous boundary. However, the normal form requires the vector field with the  $C^r$ -continuity and the corresponding convergence, where the order  $r$  is the highest order of the total power numbers in each term of normal form. For piecewise lin-

ear and nonlinear systems, the  $C^1$ -continuity of the vector field cannot provide an enough mathematical base to develop the normal form. The normal form also cannot be used to investigate global periodic motions in the discontinuous system. Leine et al (2000) used the Filippov theory to investigate bifurcations in nonlinear discontinuous systems. However, the discontinuous mapping techniques were employed to determine the bifurcation via the Floquet multiplier. The more discussion about the traditional analysis of bifurcation in non-smooth dynamical systems can be referred to Zhusubaliyev and Mosekilde (2003). Based on the recent research, the Floquet multiplier also may not be adequate for periodic motions involved with the grazing and sliding motions in non-smooth dynamical systems. Therefore, Luo (2005a) proposed a general theory for the local singularity of non-smooth dynamical systems on connectable domains (also see, Luo (2006)). To resolve the difficulty, Luo (2007b) developed a general theory for the switching possibilities of flow on the boundary from the passable to non-passable one, and so on. In this book, from recent developments in Luo (2007a; 2008a,b), a generalized theory for discontinuous systems on time-varying domains will be presented.

## 1.2 Book layout

To help readers easily read this book, the main contents in this book are summarized as follows.

In Chapter 2, from Luo (2008a,b), a switchability theory for a flow to the boundary in discontinuous dynamical systems will be presented. The  $G$ -functions for discontinuous dynamical systems will be introduced to investigate singularity in discontinuous dynamical systems. Based on the new  $G$ -function, the full and half sink and source, non-passable flows to the separation boundary in discontinuous dynamical systems will be discussed, and the switchability of a flow from a domain to an adjacent one will be addressed. Finally, the switching bifurcations between the passable and non-passable flows will be presented.

In Chapter 3, the switchability of a flow from one domain into another in discontinuous dynamical systems will be presented through a periodically forced, discontinuous dynamical system with a time-invariant boundary. The normal vector-field for flow switching on the separation boundary will be introduced and the passability condition of a flow to the separation boundary will be given through such normal vector fields. The sliding and grazing conditions to the separation boundary will be presented as well. This investigation may help one better understand the sliding mode control.

In Chapter 4, an oscillator moving on the periodically traveling belt with dry friction is investigated as a dynamical system with a time-varying boundary. The conditions of stick and non-stick motions in such an frictional oscillator will be developed, and the periodic motions in the oscillator will be investigated as well. The grazing and stick (or sliding) bifurcations will be investigated. The significance

of this investigation is to show how to control motions in such friction-induced oscillators in industry.

In Chapter 5, impact and stick motions of two oscillators at the time-varying boundary and domains will be presented under an impact law. The dynamics mechanism of the impacting chatter with stick at the moving boundary will be investigated from the local singularity theory of discontinuous dynamical systems. The analytical conditions for the onset and vanishing of stick motions will be obtained, and the condition for maintaining stick motion is achieved as well. This chapter will show how two dynamical systems interact.

In Chapter 6, two dynamical systems connected with friction will be presented, and the motion switchability on the moving discontinuous boundary will be discussed through the theory of discontinuous dynamical systems. The onset and vanishing of motions will be discussed through the bifurcation and grazing analyses. It will be observed that two different systems can stick together forever.

In Chapter 7, a theory for two dynamical systems with a general interaction will be presented. Such an interaction occurs at a time-varying boundary. From the discontinuous theory in Chapter 2, a general methodology for two system interactions will be presented in order to determine the complex motion of the two systems, which is caused by the interaction between two systems.

## References

- Andreas, U. and Casini, P. (2002), Friction oscillator excited by moving base and colliding with a rigid or deformable obstacle, *International Journal of Non-Linear Mechanics*, **37**, pp.117-133.
- Awrejcewicz, J. and Pyryev, Y. (2004), Tribological periodical processes exhibited by acceleration or braking of a shaft-pad system, *Communications in Nonlinear Science and Numerical Simulation*, **9**, pp.603-614.
- Comparin, R. J. and Singh, R. (1989), Nonlinear frequency response characteristics of an impact pair, *Journal of Sound and Vibration*, **134**, pp.259-290.
- Dankowicz, H. and Nordmark A. B. (2000), On the origin and bifurcations of stick-slip oscillations, *Physica D*, **136**, pp.280-302.
- den Hartog, J. P. (1931), Forced vibrations with Coulomb and viscous damping, *Transactions of the American Society of Mechanical Engineers*, **53**, pp.107-115.
- den Hartog, J. P. and Mikina, S. J. (1932), Forced vibrations with non-linear spring constants, *ASME Journal of Applied Mechanics*, **58**, pp.157-164.
- di Bernardo, M., Budd, C. J. and Champneys, A. R. (2001a), Grazing and border-collision in piecewise-smooth systems: a unified analytical framework, *Physical Review Letters*, **86**, pp.2553-2556.
- di Bernardo, M., Budd, C. J. and Champneys, A. R. (2001b), Normal form maps for grazing bifurcation in  $n$ -dimensional piecewise-smooth dynamical systems, *Physica D*, **160**, pp.222-254.
- di Bernardo, M., Budd, C. J. and Champneys, A. R. (2001c), Corner-collision implies border-collision bifurcation, *Physica D*, **154**, pp.171-194.
- di Bernardo, M., Feigin, M. I., Hogan, S. J. and Homer, M. E. (1999), Local analysis of C-bifurcations in  $n$ -dimensional piecewise-smooth dynamical systems, *Chaos, Solitons & Fractals*, **10**, pp.1881-1908.
- di Bernardo, M., Kowalczyk, P. and Nordmark, A. B. (2002), Bifurcation of dynamical systems with sliding: derivation of normal form mappings, *Physica D*, **170**, pp.175-205.

- Feeny, B. F. (1992), A non-smooth Coulomb friction oscillator, *Physics D*, **59**, pp.25-38.
- Feeny, B. F.(1994), The nonlinear dynamics of oscillators with stick-slip friction, in: A. Guran, F. Pfeiffer and K. Popp (eds), *Dynamics with Friction*, River Edge: World Scientific, pp.36-92.
- Feeny, B. F. and Moon, F. C. (1994), Chaos in a forced dry-friction oscillator: experiments and numerical modeling, *Journal of Sound and Vibration*, **170**, pp.303-323.
- Feigin, M. I. (1970), Doubling of the oscillation period with C-bifurcation in piecewise-continuous systems, *PMM*, **34**, pp.861-869.
- Feigin, M. I. (1995), The increasingly complex structure of the bifurcation tree of a piecewise-smooth system, *Journal of Applied Mathematics and Mechanics*, **59**, pp.853-863.
- Filippov, A. F. (1964), Differential equations with discontinuous right-hand side, *American Mathematical Society Translations, Series 2*, **42**, pp.199-231.
- Filippov, A. F. (1988), *Differential Equations with Discontinuous Righthand Sides*, Dordrecht: Kluwer Academic Publishers.
- Hetzler, H., Schwarzer, D. and Seemann, W. (2007), Analytical investigation of steady-state stability and Hopf-bifurcation occurring in sliding friction oscillators with application to low-frequency disc brake noise, *Communications in Nonlinear Science and Numerical Simulation*, **12**, pp.83-99.
- Hinrichs, N., Oestreich, M. and Popp, K.(1997), Dynamics of oscillators with impact and friction, *Chaos, Solitons and Fractals*, **8**, pp. 535-558.
- Hinrichs, N., Oestreich, M. and Popp, K. (1998), On the modeling of friction oscillators, *Journal of Sound and Vibration*, **216**, pp.435-459.
- Hundal, M. S.(1979), Response of a base excited system with Coulomb and viscous friction, *Journal of Sound and Vibration*, **64**, pp.371-378.
- Karagiannis, K. and Pfeiffer, F. (1991), Theoretical and experimental investigations of gear box, *Nonlinear Dynamics*, **2**, pp.367-387.
- Kim, W. J. and Perkins, N. C. (2003), Harmonic balance/Galerkin method for non-smooth dynamical system, *Journal of Sound and Vibration*, **261**, pp.213-224.
- Ko, P. L., Taponat, M. C. and Pfäifer, R. (2001), Friction-induced vibration-with and without external disturbance, *Tribology International*, **34**, pp.7-24.
- Leine, R. I., van Campen, D. H. and van de Vrande (2000), Bifurcations in nonlinear discontinuous systems, *Nonlinear Dynamics*, **23**, pp.105-164.
- Levitan, E. S. (1960), Forced oscillation of a spring-mass system having combined Coulomb and viscous damping, *Journal of the Acoustical Society of America*, **32**, pp.1265-1269.
- Li, Y. and Feng, Z. C. (2004), Bifurcation and chaos in friction-induced vibration, *Communications in Nonlinear Science and Numerical Simulation*, **9**, pp.633-647.
- Lu, C. (2007), Existence of slip and stick periodic motions in a non-smooth dynamical system, *Chaos, Solitons and Fractals*, **35**, pp.949-959.
- Luo, A. C. J. (2005a), A theory for non-smooth dynamical systems on connectable domains, *Communication in Nonlinear Science and Numerical Simulation*, **10**, pp.1-55.
- Luo, A. C. J. (2005b), Imaginary, sink and source flows in the vicinity of the separatrix of non-smooth dynamic system, *Journal of Sound and Vibration*, **285**, pp.443-456.
- Luo, A. C. J.( 2006), *Singularity and Dynamics on Discontinuous Vector Fields*, Amsterdam: Elsevier.
- Luo, A. C. J. (2007a), Differential geometry of flows in nonlinear dynamical systems, *Proceedings of IDECT'07*, ASME International Design Engineering Technical Conferences, September 4-7, 2007, Las Vegas, Nevada, USA. DETC2007-84754.
- Luo, A. C. J. (2007b), On flow switching bifurcations in discontinuous dynamical system, *Communications in Nonlinear Science and Numerical Simulation*, **12**, pp.100-116.
- Luo, A. C. J. (2008a), A theory for flow swtchability in discontinuous dynamical systems, *Nonlinear Analysis: Hybrid Systems*, **2**, pp.1030-1061.
- Luo, A. C. J. (2008b), *Global Transversality, Resonance and Chaotic dynamics*, Singapore: World Scientific.
- Luo, A. C. J., and Chen, L. D. (2005), Periodic motion and grazing in a harmonically forced, piecewise linear, oscillator with impacts, *Chaos, Solitons and Fractals*, **24**, pp.567-578.



- Luo, A. C. J. and Chen, L. D. (2006), The grazing mechanism of the strange attractor fragmentation of a harmonically forced, piecewise, linear oscillator with impacts, *IMechE Part K: Journal of Multi-body Dynamics*, **220**, pp.35-51.
- Luo, A. C. J. and Chen, L. D. (2007), Arbitrary periodic motions and grazing switching of a forced piecewise-linear, impacting oscillator, *ASME Journal of Vibration and Acoustics*, **129**, pp.276-284.
- Luo, A. C. J. and Gegg, B. C. (2005a), On the mechanism of stick and non-stick periodic motion in a forced oscillator including dry-friction, *ASME Journal of Vibration and Acoustics*, **128**, pp.97-105.
- Luo, A. C. J. and Gegg, B. C. (2005b), Stick and non-stick periodic motions in a periodically forced, linear oscillator with dry friction, *Journal of Sound and Vibration*, **291**, pp.132-168.
- Luo, A. C. J. and Gegg, B. C. (2006a), Periodic motions in a periodically forced oscillator moving on an oscillating belt with dry friction, *ASME Journal of Computational and Nonlinear Dynamics*, **1**, pp.212-220.
- Luo, A. C. J. and Gegg, B. C. (2006b), Dynamics of a periodically excited oscillator with dry friction on a sinusoidally time-varying, traveling surface, *International Journal of Bifurcation and Chaos*, **16**, pp.3539-3566.
- Luo, A. C. J. and O'Connor, D. (2007a), Nonlinear dynamics of a gear transmission system, Part I: mechanism of impacting chatter with stick, *Proceedings of IDETC'07*, 2007 ASME International Design Engineering Conferences and Exposition, September 4-7, 2007, Las Vegas, Nevada. IDETC2007-34881.
- Luo, A. C. J. and O'Connor, D. (2007b), Nonlinear dynamics of a gear transmission system, Part II: periodic impacting chatter and stick, *Proceedings of IMECE'07*, 2007 ASME International Mechanical Engineering Congress and Exposition, November 10-16, 2007, Seattle, Washington. IMECE2007-43192.
- Luo, A. C. J. and Thapa, S. (2007), On nonlinear dynamics of simplified brake dynamical systems, *Proceedings of IMECE2007*, 2007 ASME International Mechanical Engineering Congress and Exposition, November 5-10, 2007, Seattle, Washington, USA. IMECE2007-42349.
- Natsiavas, S. (1998), Stability of piecewise linear oscillators with viscous and dry friction damping, *Journal of Sound and Vibration*, **217**, pp.507-522.
- Natsiavas, S. and Verros, G. (1999), Dynamics of oscillators with strongly nonlinear asymmetric damping, *Nonlinear Dynamics*, **20**, pp.221-246.
- Nordmark, A. B. (1991), Non-periodic motion caused by grazing incidence in an impact oscillator, *Journal of Sound and Vibration*, **145**, pp.279-297.
- Nusse, H. E. and Yorke J. A. (1992), Border-collision bifurcations including "period two to period three" for piecewise smooth systems, *Physica D*, 1992, **57**, pp.39-57.
- Ozguven, H. N. and Houser, D. R. (1988), Mathematical models used in gear dynamics—a review, *Journal of Sound and Vibration*, **121**, pp.383-411.
- Pfeiffer, F. (1984), Mechanische Systeme mit unstetigen Ubergangen, *Ingenieur-Archiv*, **54**, pp.232-240.
- Pilipchuk, V. N. and Tan, C. A. (2004), Creep-slip capture as a possible source of squeal during decelerating sliding, *Nonlinear Dynamics* **35**, pp.258-285.
- Shaw, S. W. (1986), On the dynamic response of a system with dry-friction, *Journal of Sound and Vibration*, **108**, pp.305-325.
- Theodossiades, S. and Natsiavas, S. (2000), Non-linear dynamics of gear-pair systems with periodic stiffness and backlash, *Journal of Sound and Vibration*, **229**, pp.287-310.
- Thomsen, J. J. and Fidin, A. (2003), Analytical approximations for stick-slip vibration amplitudes, *International Journal of Non-Linear Mechanics*, **38**, pp.389-403.
- Zhusubaliyev, Z. and Mosekilde, E. (2003), *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems*, Singapore: World Scientific.