丘成桐中学数学奖

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恒隆数学奖获奖论文集

The Collection of Hang Lung Mathematics Awards Papers

丘成桐 主编

高等教育出版社
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图书在版编目(CIP)数据

恒隆数学奖获奖论文集：英文 / 丘成桐主编. —北京：高等教育出版社，2009.6
ISBN 978-7-04-025482-2

Ⅰ. 恒… Ⅱ. 丘… Ⅲ. 数学－文集－英文 Ⅳ. 01-53

中国版本图书馆CIP数据核字（2009）第062420号

策划编辑	王　超	**责任编辑**	王　超	**封面设计**	王凌波
责任绘图	尹　莉	**责任校对**	杨雪莲	**责任印制**	朱学忠

出版发行	高等教育出版社	**购书热线**	010-58581118
社　　址	北京市西城区德外大街 4 号	**免费咨询**	800-810-0598
邮政编码	100120	**网　　址**	http://www.hep.edu.cn
总　　机	010-58581000		http://www.hep.com.cn
		网上订购	http://www.landraco.com
经　　销	蓝色畅想图书发行有限公司		http://www.landraco.com.cn
印　　刷	保定市中画美凯印刷有限公司	**畅想教育**	http://www.widedu.com
开　　本	890×1240 1/32	**版　　次**	2009年 6 月第 1 版
印　　张	6.75	**印　　次**	2009年 6 月第1次印刷
字　　数	160 000	**定　　价**	15.00 元

物料号　25482-00

Contents

MARKED RULER AS A TOOL FOR GEOMETRIC CONSTRUCTIONS— FROM ANGLE TRISECTION TO N-SIDED POLYGON

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1 Introduction

Trisecting an arbitrary angle with a compass and straightedge was one of the famous ancient Greeks unsolved construction problems. Together with duplicating the cube, these problems have been pending to be resolved for more than 2000 years.

Plato (427–347 BC) defined clearly the rules of ruler and compass construction, which implies that the marks or scales in the ruler should not be relevant to the geometric construction. Many learned people tried employing different tools and methods to tackle the problem, in particular, the interesting and simple construction algorithm proposed by Archimedes (287–212 BC) who had employed a marked ruler and compass to solve the trisecting problem, which was very close to the Platos rules.

In 19th Century, Pierre L. Wantzel (1814–1848) proved in 1837 that based upon Plato's criteria, it is possible to trisect an arbitrary angle. The problem became even more interesting after it was proved to be possible because of the "magic" marked ruler, which have opened a new area for the study of geometric constructions with the marked ruler and a compass.

Theorem 1.1 *If we have a marked ruler and a compass, then it is possible to trisect an arbitrary angle.*

Proof. Archimedes proved this theorem by giving a construction algorithm. As shown in Fig. 1 below, let $\angle AOB$ be the angle being trisected and the lengths $|OA| = |OB| = 1$, which is the distance between the two marks on the ruler. Draw a semicircle centered at O from B through A. If we mark C and D such that C is on the semi-circle and D is the intersection of the lines OB and AC with $|CD| = 1$, then $\angle AOB = 3\angle ADB$.

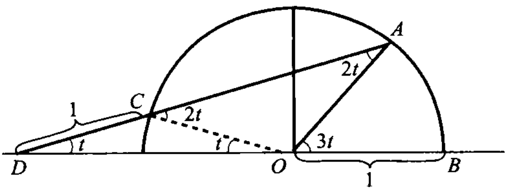

Fig. 1

Let $\angle ADB = t$. Then $\angle COD = t$ and $\angle OCA = \angle OAC = 2t$ (base angles of isosceles triangle).

By the interior angle sum of triangle, $\angle AOC = \pi - 4t$. Hence,

$$\angle AOB = \angle COD - \angle AOC = \pi - t - (\pi - 4t) = 3t = 3\angle ADB.$$

□

In Theorem 1.1, Archimedes made use of the so-called marked ruler instead of a typical straight edge in the construction. Some people criticized that Archimedes did not respect the conventional definition of the ruler and his approach was not strict enough hence it was not commonly accepted.

In spite of this, it was quite natural, when compared with using conics, trisectrix or some other strange curves to give a solution to an angle trisection, the marked ruler was an easy available tool in real life. One should appreciate why adding two marks on a ruler makes the impossible to be possible. In this project, we try to give some terminology of the marked ruler and clarify which types of geometric constructions are possible by using the marked ruler and a compass.

Definition 1.2 *A ruler or more precisely a straight edge with two notches on it is called a marked ruler. Without loss of generality, the distance between the two notches is taken to be* 1.

From definition 1.2, we notice that the marked ruler introduces the concept of unit length into the system of geometric construction. It allows us to cut off equal distance on a straight line in particular, and we will show that the marked ruler is much more useful with the help of compass in the subsequent sections. In order to study geometric construction algebraically, we will introduce a rectangular coordinate system on the two dimensional Euclidean Space. Moreover, by the end of this report, we will study how the problem of constructing regular n-sided polygon is related to the construction by using marked ruler and compass.

Now, let us define the meaning of constructible points and constructible curves.

Definition 1.3 *A constructible curve is a curve constructed from given quantities such as points, lengths, etc, which are provided by given points and constructible points. A constructible point is a point of intersection of two constructible curves.*

Our task is getting clearer that we treat construction as drawing the constructible curves. If we know what curves marked ruler and compass can draw, we will know the properties of the constructible points. Before going deep into our main goal, lets take a brief review on the general construction.

2 Classification of Construction

Up to this moment, our understanding on the term "construction" is too vague for a mathematical theory to build on. In the following sections, we will give a clear definition of "construction", then classify different types of construction and their relative field of extension.

Definition 2.1 *A construction C is defined to be a finite set of constructible points $\{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_n\}$, where $0 = (0,0)$, $\mathbf{u} = (1,0)$ and $A_0 = (0,1)$, such that A_{n+1} is a point of intersection of any two of the constructible curves γ_i constructed from the points in the sub-construction $C_k = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_k\}$ where $k = 1, 2, \ldots, n$ under specific construction rules.*

Definition 2.2 *Let $C = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_n\}$ be a construction of n steps and $C_k = \{0, \mathbf{u}, A_0, A_1, A_2, \ldots, A_k\}$ where $k \leqslant n$ be a sub-construction of C. Also, let $z_1, z_2, \ldots, z_n$ be the complex numbers that represent the points $A_1, A_2, \ldots, A_n$ respectively. Then, $\mathrm{K}[C] = \mathbb{Q}(\mathbf{i}, z_1, z_2, \ldots, z_n)$ is defined to* *be the field of extension of $\mathbb{Q}$ by construction $\mathcal{C}$. Note that* $\mathrm{K}[\mathcal{C}]$ *is the smallest field that contains* $\mathbf{i}$, $z_1, \ldots, z_n$ *and we have*

$$\mathrm{K}[\mathcal{C}_k] = \mathrm{K}[\mathcal{C}_{k-1}](z_k) \text{ for } k = 1, 2, \ldots, n$$

Remark: Since $0 = (0, 0)$, $\mathbf{u} = (1, 0)$ and $A_0 = (0, 1)$, it is easy to see that $\mathrm{K}[\mathcal{C}_0] = \mathbb{Q}(\mathbf{i})$.

Definition 2.3 *A construction is called plane if it can be solved by using ruler and compass only.*
A construction is called solid if it can be solved by using conic sections only.
A construction is called higher dimensional if it is not plane or solid.

Remark: This classification was introduced by Pappus, but I replace the term "linear" by "higher dimensional" since it will be more appropriate.

Definition 2.4 *For plane constructions, a constructible straight line is a line, which passes through two constructible points; and a constructible circle is a circle centered at a constructible point, which passes through another constructible point.*

Before we state the well-known theorem for ruler and compass construction, we give a lemma, which is used to prove this theorem.

Lemma 2.5 *If two circles intersect or a circle and a straight line intersects, where the coefficients of the equations of the circles and straight lines are in field* K*, then the coordinates of the point of intersection lie in a field of quadratic extension over* K.

Proof. Firstly, let $y = mx + c$ and $x^2 + y^2 + dx + ey + f = 0$ be the equations of a straight line and a circle with m, c, d, e, $f \in \mathrm{K}$ respectively. Then, by solving the two equations, we have

$$x^2 + (mx + c)^2 + dx + e(mx + c) + f = 0$$
$$\Rightarrow (1 + m^2)x^2 + (2mc + d + me)x + c^2 + ce + f = 0.$$

Note that the coefficients of the above equation are in K, its roots lie in a field of quadratic extension of K. Also from $y = mx + c$, y is linear to x and so the coordinates of the points of intersection lie in a field of quadratic extension over K.

Secondly, let $x^2 + y^2 + d_1x + e_1y + f_1 = 0$ and $x^2 + y^2 + d_2x + e_2y + f_2 = 0$ be the equations of two distinct circles with $d_1, d_2, e_1, e_2, f_1, f_2 \in \mathrm{K}$. Then by subtracting the two equations, it yields a straight line $(d_2 - d_1)x + (e_2 - e_1)y + (f_2 - f_1) = 0$ with its coefficients in K. Hence, by the above argument, the coordinates of the points of intersection lie in a field of quadratic extension over K. □

Theorem 2.6 *A point* (x, y) *has a plane construction if and only if* $x + y\mathbf{i} \in \mathbb{C}$ *lies in a sub-field* K *of* $\mathbb{C}$ *such that* $\exists K_i$, $i = 0, 1, \ldots, n$, *satisfying that*

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 1$ *or* 2 *for* $j = 1, 2, \ldots, n$.

Proof. Let $\mathcal{C} = \{0, \mathbf{u}, A_0, A_1, \ldots, A_{n-1}\}$ be a plane construction with $A_{n-1} = (x, y)$. It is clear that when two constructible lines intersect, no extension of field is needed. Also note that the extension $\mathbb{Q}(\mathbf{i})$ is of degree 2. Then, by lemma 2.5, we must have $[\mathrm{K}[\mathcal{C}_{k+1}] : \mathrm{K}[\mathcal{C}_k]] = 1$ or 2, where $k = 1, 2, \ldots, n - 2$.

Hence, by letting $K_{j+1} = K[\mathcal{C}_j]$, we have

$$\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = K[\mathcal{C}] = K$$

and the index $[K_j : K_{j-1}] = 1,\ 2$ for $j = 1, 2, \ldots, n$.

Conversely, given a tower of fields

$$\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = K$$

where the index $[K_j : K_{j-1}] = 1$ or 2 for $j = 1, 2, \ldots, n$. It suffices to verify that there is a plane construction associated to each step of field extension. If $[K_j : K_{j-1}] = 1$, then $K_j = mathrm\, K_{j-1}$ and the result is trivial. If $[K_j : K_{j-1}] = 2$, let $z_j = x_j + y_j\mathbf{i}$ such that $K_j = K_{j-1}[z_j]$ where $z_j \notin K_{j-1}$ for $j = 1, 2, \ldots, n$. Since the degree of extension is 2, both x_j and y_j are roots of certain quadratic equations with coefficients in K_{j-1}, say, $x^2 + ax + b = 0$ and $y^2 + py + q = 0$ respectively. Then by constructing the circle $x^2 + y^2 + ax + b = 0$ and the line $y = 0$, we solve x_j as the x-coordinate. Similarly, y_j can be obtained as the y-coordinate of the intersection of the circle $x^2 + y^2 + px + q = 0$ and the line $x = 0$. Hence $z_j = x_j + y_j\mathbf{i}$ has a plane construction. □

Remark: In fact, the index $[K_j : K_{j-1}] = 1$ actually means that $K_j = K_{j-1}$, so we can omit 1 in the preceeding theorem.

Next, we want to show is that every point with solid construction lies in a 2-3-tower over $\mathbb{Q}$. So we should first show that every points are closed under quartic equation and on the other hand every equation of degree at most 4 is solvable by solid construction.

Lemma 2.7 *Every points of intersection of any two conic sections are roots of an equation of degree at most four.*

Proof. Let $A_1x^2 + B_1xy + C_1y^2 + D_1x + E_1y + F_1 = 0$ and $A_2x^2 + B_2xy + C_2y^2 + D_2x + E_2y + F_2 = 0$ be two distinct conic sections. Then by Bezout's Theorem, which states that two algebraic plane curves with degree m and n respectively and with no common component have exactly mn points of intersection counting multiplicity and points at infinity, it follows that there are at most mn points of intersection for any two algebraic curves with degree m and n respectively. The points of intersection of the two conic sections, each of degree 2, when solving together, are therefore roots of quartic equations for irreducible cases, and thus yielding four distinct intersections. For reducible cases, the degree of the equation will be even lower. So every points of intersection are roots of an equation of degree at most four.

Practically, one may transform the first equation in the form

$$y^2 = -\frac{A_1}{C_1}x^2 - \frac{B_1}{C_1}xy - \frac{D_1}{C_1}x - \frac{E_1}{C_1}y - \frac{F_1}{C_1} \quad \text{for } C_1 \neq 0$$

and it is used to reduce the degree of the second equation in y. Eventually, a quartic equation maybe reducible or irreducible is yielded. If $C_1 = C_2 = 0$, then we can eliminate y without much difficulty and also an equation of degree not exceeding four is yielded. □

Lemma 2.8 *Trisecting an arbitrary angle has a solid construction.*

Proof. Whenever we can construct an angle α, it is equivalent to say that we can construct the length $\cos\alpha$, since one can readily construct a right angle triangle with hypotenuse 1 and one side $\cos\alpha$ where the included angle is α from either one condition. Suppose 3θ be the angle to be trisected, then $\cos 3\theta$ is constructible and we aim at showing that $\cos\theta$ is also constructible.

Consider the two conic sections

$$\begin{cases} y = x^2, \\ xy - 3x - 2\cos 3\theta = 0. \end{cases}$$

Since both of them have their coefficients in $\mathbb{Q}(\cos 3\theta)$, they are both constructible. Solving them together yields a cubic equation

$$x^3 - 3x - 2\cos 3\theta = 0$$

It is not too difficult to show that $2\cos\theta$, $2\cos\left(\theta + \frac{2}{3}\pi\right)$ and $2\cos\left(\theta + \frac{4}{3}\theta\right)$ are the roots of it. Therefore $\cos\theta$ is constructible and trisecting an arbitrary angle has a solid construction. □

Lemma 2.9 *Find the cube root of arbitrary length has a solid construction.*

Proof. Suppose the length l is constructible. We are aiming to show that $\sqrt[3]{l}$ is also constructible.

Consider the two conic sections,

$$\begin{cases} y = x^2, \\ xy = l, \end{cases}$$

Both of them have their coefficients in $\mathbb{Q}(l)$, so they are constructible. Then by solving them together, we yield

$$x^3 = l,$$

and so $\sqrt[3]{l}$ is the only real root that satisfies this equation. Hence the cube root of arbitrary length has a solid construction. □

Theorem 2.10 *All equations of degree at most 4 can be solved if and only if one can trisect an angle and find that cube root for arbitrary length in addition to the use of ruler and compass.*

Proof. [$\Rightarrow$] Suppose all equations of degree at most 4 can be solved. Then the equation $x^3 - a = 0$ can be solved and we can find the cube root. As shown in lemma 2.8, arbitrary angle 3α is trisectable if and only if $\cos\alpha$ is constructible length. Since $\cos 3\alpha = 4\cos 3\alpha - 3\cos\alpha$, replacing with $x = \cos\alpha$, it can be written as

$$4x^3 - 3x - \cos 3\alpha = 0,$$

which is solvable, hence we can trisect an arbitrary angle.

[$\Leftarrow$] Now, suppose we can trisect an angle and find the cube root, then both equations

$$\begin{cases} x^3 - k = 0 & (2.1) \\ 4x^3 - 3x - \cos 3\alpha = 0 & (2.2) \end{cases}$$

is solvable.

Case 1: For linear and quadratic equation, we can trivially solve them by ruler and compass only.

Case 2: For a general cubic equation $x^3 + ax^2 + bx + c = 0$, by a suitable change of variable $\left(x = y - \frac{a}{3}\right)$, we always give a principal cubic equation $y^3 + py + q = 0$.

When $p = 0$, by equation (2.1), we can solve it.

When $p \neq 0$, from Cardano Formulas, the solutions of $y^3 + py + q = 0$ are,

$$A + B, A\xi + B\xi^2, A\xi^2 + B\xi, \text{ where } \xi = e^{2/3\pi \mathrm{i}} \text{ and}$$

$$A = \sqrt[3]{-1/2q + \sqrt{(1/2q)^2 + (1/3p)^3}},$$

$$B = \sqrt[3]{-1/2q - \sqrt{(1/2q)^2 + (1/3p)^3}}$$

Since $(1/2q)^2+(1/3p)^3$ is constructible, if $(1/2q)^2+(1/3p)^3 \geqslant 0$, A^3 and B^3 are also constructible. Hence, also by equation (2.1), A and B are constructible, and thus $y^3+py+q=0$ can be solved.

Now suppose $(1/2q)^2+(1/3p)^3<0$, iff $p^3<27/4q^2 \leqslant 0$, then $p<0$ and

$$\left|-1/2q\sqrt{-27/p^3}\right|<1,$$

hence, there exists α such that

$$\cos 3\alpha = -1/2q\sqrt{-27/p^3}.$$

By a suitable substitution $y = 2\sqrt{1/3p}$, we have

$$4t^3-3t-\cos 3\alpha = 0.$$

Then, by equation (2.2), $y^3+py+q=0$ is solvable. Therefore, all cubic equations can be solved.

Case 3: Consider a general quartic equation $x^4+ax^3+bx^2+cx+d=0$.

By substituting $x = y - a/4$, we obtain the depressed quartic

$$y^4+py^2+qy+r=0. \tag{2.3}$$

If $q=0$, we solve the quartic by solving the quadratic equation in y^2.

If $q \neq 0$, we rewrite (2.3) as

$$y^4 = -py^2-qy-r. \tag{2.4}$$

By adding $2zy^2+z^2$ to both sides of (2.4), we have

$$(y^2+z)^2 = (2z-p)y^2-qy+(z^2-r)$$

Since z is arbitrary depending on our choice, we wish to find z such that

$$(2z-p)y^2 - qy + (z^2 - r) = (gy+h)^2 \tag{2.5}$$

for some constants g, h. Then, we solve (2.3) by $y^2 + z^2 = \pm(gy+h)$ and solve two resulting quadratic equations.

But this situation occurs iff $(2z-p)y^2 - qy + (z^2 - r) = 0$ has a double root, and thus iff

$$q^2 - 4(2z-p)(z^2-r) = 0. \tag{2.6}$$

Rewrite (2.6) as

$$8z^3 - 4pz^2 - 8rz + 4pr - q^2 = 0 \tag{2.7}$$

which is a cubic equation in z.

Now, from case 2, we can solve (2.7), and then by case 1, we can solve (2.5). Hence (2.3) is solvable.

Therefore, we can solve all equations with degree at most four. □

Theorem 2.11 *A point (x, y) has a solid construction if and only if $x + y\mathbf{i} \in \mathbb{C}$ lies in a sub-field K of $\mathbb{C}$ such that there exists $\mathrm{K}_i, i = 0, 1, \ldots, n$, which satisfies*

$$\mathbb{Q} = \mathrm{K}_0 \subset \mathrm{K}_1 \subset \mathrm{K}_2 \subset \cdots \subset \mathrm{K}_n = \mathrm{K}$$

and the index $[\mathrm{K}_j : \mathrm{K}_{j-1}] = 2$ or 3 for $j = 1, 2, \ldots, n$.

3 Construction with Marked Ruler and Compass

After classifying different types of construction and their relative field of extension, we should now analyze the constructions made by marked ruler and compass, and thus to