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郭柏灵论文集

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郭柏灵 著

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序

今年是恩师郭柏灵院士70寿辰，华南理工大学出版社决定出版《郭柏灵论文集》。郭老师的弟子，也就是我的师兄弟，推举我为文集作序。这使我深感荣幸。我于1985年考入北京应用物理与计算数学研究所，师从郭柏灵院士和周毓麟院士。研究生毕业后我留在研究所工作，继续跟随郭老师学习和研究偏微分方程理论。老师严谨的治学作风和对后学的精心培养与殷切期望，给我留下了深刻的印象，同时老师在科研上的刻苦精神也一直深深地印在我的脑海中。

郭老师1936年生于福建省龙岩市新罗区龙门镇，1953年从福建省龙岩市第一中学考入复旦大学数学系，毕业后留校工作。1963年，郭老师服从祖国的需要，从复旦大学调入北京应用物理与计算数学研究所，从事核武器研制中有关的数学、流体力学问题及其数值方法研究和数值计算工作。他全力以赴地做好了这项工作，为我国核武器的发展做出了积极的贡献。1978年改革开放以后，他又在非线性发展方程数学理论及其数值方法领域开展研究工作，现为该所研究员、博士生导师，中国科学院院士。迄今他共发表学术论文300余篇、专著9部，1987年获国家自然科学三等奖，1994年和1998年两度获得国防科工委科技进步一等奖，为我国的国防建设与人才培养作出了巨大贡献。

郭老师的研究方向涉及数学的多个领域，其中包括非线性发展方程的数学理论及其数值解、孤立子理论、无穷维动力系统等，其研究工作的主要特点是紧密联系数学物理中提出的各种重要问题。他对力学及物理学等应用学科中出现的许多重要的非线性发展方程进行了系统深入的研究，其中对Landau-Lifshitz方程和Benjamin-Ono方程的大初值的整体可解性、解的唯一性、正则性、渐近行为以及爆破现象等建立了系统而深刻的数学理论。在无穷维动力系统方面，郭老师研究了一批重要的无穷维动力系统，建立了有关整体吸引子、惯性流形和近似惯性流形的存在性和分形维数精细估计等理论，提出了一种证明强紧吸引子的新方法，并利用离散化等方法进行理论分析和数值计算，展示了吸引子的结构和图像。下面我从这几个方面介绍郭老师的一些学术成就。

Landau-Lifshitz方程（又称铁磁链方程）由于其结构的复杂性，特别是强耦合性和不带阻尼时的强退化性，在20世纪80年代之前国内外几乎没有从数学上进行理论研究的成果出现。最先进行研究的，当属周毓麟院士和郭老师，他们在1982年到1986年间，采用Leray-Schauder不动点定理、离散方法、Galerkin方法证明了从一维到高维的各种边值问题整体弱解的存在性，比国外在1992年才出现的同类结果早了将近10年。

20世纪90年代初期，周毓麟、郭柏灵和谭绍滨，郭柏灵和洪敏纯得到了两个在国内外至今影响很大的经典结果。第一，通过差分方法结合粘性消去法，利用十分巧妙的先验估计，证明了一维Landau-Lifshitz方程光滑解的存在唯一

性,对于一维问题给出了完整的答案,解决了长期悬而未决的难题。第二,系统分析了带阻尼的二维 Landau - Lifshitz 方程弱解的奇性,发现了 Landau - Lifshitz 方程与调和映照热流的联系,其弱解具有与调和映照热流完全相同的奇性。现在,国内外这方面的文章基本上都引用这个结果。调和映照的 Landau - Lifshitz 流的概念,即是源于此项结果。

20 世纪 90 年代中期,郭老师对于 Landau - Lifshitz 方程的长时间性态、Landau - Lifshitz 方程耦合 Maxwell 方程的弱解及光滑解的存在性问题进行了深入的研究,得到了一系列的成果。铁磁链方程的退化性以及缺少相应的线性化方程解的表达式,对研究解的长时间性态带来很大困难。郭老师的一系列成果克服了这些困难,证明了近似惯性流形的存在性、吸引子的存在性,给出了其 Hausdorff 和分形维数的上、下界的精细估计。此外,我们知道,与调和映照热流比较,高维铁磁链方程的研究至今还很不完善。其中最重要的是部分正则性问题,其难点在于单调不等式不成立,导致能量衰减估计方面的困难。另外一个是 Blow-up 解的存在性问题,至今没有解决;而对于调和映照热流来说,这样的问题的研究是比较成熟的。

对于高维问题,20 世纪 90 年代后期至今,郭老师和陈韵梅、丁时进、韩永前、杨干山一道,得出了许多成果,大大地推动了该领域的研究。首先,证明了二维问题的能量有限弱解的几乎光滑性及唯一性,这个结果类似于 Freire 关于调和映照热流的结果。第二,得到了高维 Landau - Lifshitz 方程初边值问题的奇点集合的 Hausdorff 维数和测度的估计。第三,得到了三维 Landau - Lifshitz - Maxwell 方程的奇点集合的 Hausdorff 维数和测度的估计。第四,得到了一些高维轴对称问题的整体光滑解和奇性解的精确表达式。郭老师还开创了一些新的研究领域。例如,关于一维非均匀铁磁链方程光滑解的存在唯一性结果后来被其他数学家引用并推广到一般流形上。其次,率先讨论了可压缩铁磁链方程测度值解的存在性。最近,在 Landau - Lifshitz 方程耦合非线性 Maxwell 方程方面,也取得了许多新的进展。

多年来,郭老师还对一大批非线性发展方程解的整体存在唯一性、有限时刻的爆破性、解的渐近性态等开展了广泛而深入的研究,受到国内外同行的广泛关注。研究的模型源于数学物理、水波、流体力学、相变等领域,如含磁场的 Schrödinger 方程组、Zakharov 方程、Schrödinger - Boussinesq 方程组、Schrödinger - KdV 方程组、长短波方程组、Maxwell 方程组、Davey - Stewartson 方程组、Klein - Gordon - Schrödinger 方程组、波动方程、广义 KdV 方程、Kadomtsev - Petviashvili(KP)方程、Benjamin - Ono 方程、Newton - Boussinesq 方程、Cahn - Hilliard 方程、Ginzburg - Landau 方程等。其中不少耦合方程组都是郭老师得到了第一个结果,开创了研究的先河,对国内外同行的研究产生了深远的影响。

郭老师在无穷维动力系统方面也开展了广泛的研究,取得了丰硕的成果。

对耗散非线性发展方程所决定的无穷维动力系统,研究了整体吸引子的存在性、分形维数估计、惯性流形、近似惯性流形、指数吸引子等问题。特别是在研究无界域上耗散非线性发展方程的强紧整体吸引子存在性时所提出的化弱紧吸引子成为强紧吸引子的重要方法和技巧,颇受同行关注并广为利用。对五次非线性 Ginzburg - Landau 方程,郭老师利用空间离散化方法将无限维问题化为有限维问题,证明了该问题离散吸引子的存在性,并考虑五次 Ginzburg - Landau 方程的定态解、慢周期解、异宿轨道等的结构。利用有限维动力系统的理论和方法,结合数值计算得到具体的分形维数(不超过 4)和结构以及走向混沌、湍流的具体过程和图像,这是一种寻求整体吸引子细微结构新的探索和尝试,对其他方程的研究也是富有启发性的。1999 年以来,郭老师集中于近可积耗散的和 Hamilton 无穷维动力系统的结构性研究,利用孤立子理论、奇异摄动理论、Fenichel 纤维理论和无穷维 Melnikov 函数,对于具有小耗散的三次到五次非线性 Schrödinger 方程,证明了同宿轨道的不变性,并在有限维截断下证明了 Smale 马蹄的存在性,目前,正把这一方法应用于具小扰动的 Hamilton 系统的研究上。他对于非牛顿流无穷维动力系统也进行了系统深入的研究,建立了有关的数学理论,并把有关结果写成了专著。以上这些工作得到国际同行们的高度评价,被称为"有重大的国际影响"、"对无穷维动力系统理论有重要持久的贡献"。最近,郭老师及其合作者又证明了具耗散的 KdV 方程 L^2 整体吸引子的存在性,该结果也是引人注目的。

郭老师不仅自己辛勤地搞科研,还尽心尽力培养了大批的研究生(硕士生、博士生、博士后),据不完全统计,有 40 多人。他根据每个人不同的学习基础和特点,给予启发式的具体指导,其中的不少人已成为了该领域的学科带头人,有些人虽然开始时基础较差,经过培养,也得到了很大提高,成为该方向的业务骨干。

《郭柏灵论文集》按照郭老师在不同时期所从事的研究领域,分成多卷出版。文集中所搜集的都是郭老师正式发表过的学术成果。把这些成果整理成集出版,不仅系统地反映了他的科研成就,更重要的是对于从事这方面学习、研究的学者无疑大有裨益。这本文集的出版得到了多方面的帮助与支持,特别要感谢华南理工大学校长李元元教授、华南理工大学出版社范家巧社长和华南理工大学数学科学学院吴敏院长的支持。还要特别感谢华南理工大学的李用声教授、华南师范大学的丁时进教授、北京应用物理与计算数学研究所的苗长兴研究员等人在论文的搜集、选择与校对等工作中付出了辛勤的劳动。感谢华南理工大学出版社的编辑对文集的精心编排工作。

谭绍滨

2005 年 8 月于厦门大学

目　　录

1998 年

1999 年

2000 年

2001 年

Partial Regularity for Two Dimensional Landau-Lifshitz Equations *

Chen Yunmei(陈韵梅) Ding Shijin(丁时进) Guo Boling(郭柏灵)

Abstract

It is proved that any weak solution to the initial value problem of two dimensional Landau-Lifshitz equation is unique and is smooth with the exception of at most finitely many points, provided that the weak solution has finite energy.

Keywords Landau-Lifshitz equation; nonlinear parabolic equations; uniqueness; regularity; Hodge decomposition

1 Introduction

Let M be an m-dimensional compact Riemannian manifold without boundary. The Landau-Lifshitz equation for the ferromagnetic spin chain with Gilbert damping term (without the external field) takes the form (see [1])

$$\partial_t u = -\alpha_1 u \times (u \times \Delta_M u) + \alpha_2 u \times \Delta_M u, \tag{1.1}$$

where "$\times$" denotes the vector cross product in $\mathbf{R}^3$, $u = (u^1, u^2, u^3): M \times \mathbf{R}^+ \to \mathbf{R}^3$ with $|u| = 1$, α_2 is the exchange constant and $\alpha_1 > 0$ the Gilbert damping constant. The continuous Heisenberg spin chain has aroused considerable interest among physicists. The above equation (1.1) of motion was first derived on phenomenogical grounds by Landau-Lifshitz[1]. It bears on a fundamental role in the understanding of nonequilibrium magnetism. A lot of work on the study of the soliton, existence and regularity of solutions for one or two dimensional problems has been made by physicists and mathematicians (see [2 − 7] for example). For the higher dimensional problem, the only known result is the existence of weak solutions[8, 9].

It is easy to see from [6] that u is a solution of (1.1) if and only if u is a solution in the classical sense of the following equation

$$\partial_t u = \alpha_1 \Delta_M u + \alpha_2 (u \times \Delta_M u) + \alpha_1 |du|^2 u, \tag{1.2}$$

or

$$\partial_t u = \alpha u \times \partial_t u + \beta \Delta_M u + \beta |du|^2 u, \tag{1.3}$$

for some constant α, β with $\beta > 0$.

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Acta Mathematica Sinica (New Series), 1998, 14(3): 423 − 432.

In this note, we shall discuss the partial regularity of any weak solution of above equation having finite energy.

Consider the following Cauchy problem of the Landau-Lifshitz equation on $M\times\mathbf{R}^+$:

$$\partial_t u = \alpha u\times\partial_t u+\beta\Delta_M u+\beta|\mathrm{d}u|^2 u,\quad x\in M,\quad t>0, \tag{1.4}$$

$$u(x,0)=u_0,\quad x\in M, \tag{1.5}$$

where $u_0\in H^1(M,S^2)$, $\dim(M)=2$. The weak solution of this problem is defined by the following.

Definition A map $u: M\times[0,T]\to S^2$ is said to be a weak solution of $(1.4)\sim(1.5)$, if $u\in W_T$, where

$$W_T=:\{\partial_t u\in L^2(0,T;L^2(M)),\mathrm{d}u\in L^\infty((0,T),L^2(M)),$$
$$|u|=1,\ \text{a.e. on } M\times[0,T]\}$$

and if u satisfies $(1.4)\sim(1.5)$ in the sense of distribution.

The following existence result was proved in [6]:

Theorem 1.1[6] Let M be a compact two dimensional Riemannian manifold without boundary. For any initial value $u_0\in H^1(M,S^2)$, there exists a unique solution u with finite energy of $(1.4)\sim(1.5)$ on $M\times\mathbf{R}_+$, which is regular on $M\times\mathbf{R}_+$ with the exception of at most finitely many points.

From now on, we call this solution as the "almost smooth" solution.

Recently, Chen and Guo[5] found that any weak solution of $(1.4)\sim(1.5)$ from a Riemannian surface satisfying the energy inequality

$$\int_0^t\int_M|\partial_t u|^2+\int_M|\mathrm{d}u|^2(\cdot,t)\leqslant\int_M|\mathrm{d}u_0|^2,\quad\text{for }\forall t>0 \tag{1.6}$$

is "almost smooth". In this note we shall use a different approach which is simpler than that used in [5], to prove that any weak solution of $(1.4)\sim(1.5)$ with finite energy is "almost smooth". Therefore, the assumption (1.6) is weakened in this note. More precisely our main result is the following.

Theorem 1.2 Let M be a compact two dimensional Riemannian manifold without boundary. For any $T>0$, if $u\in W_T$ is a weak solution of $(1.4)\sim(1.5)$ with finite energy $E(u(t))=\int_M|\mathrm{d}u|^2(\cdot,t)\leqslant E_0$ for some constant E_0 and for $t\in[0,T]$, then u is unique and is smooth on $M\times[0,T]$ with the exception of at most finitely many points.

This result is similar to the result of [10] on the uniqueness of heat flow of harmonic maps. However, the extra nonlinear term $u\times u_t$ brings more difficulty for the proof of our result. Our proof is completed by first showing that any weak solution of $(1.4)\sim(1.5)$ with finite energy is in the space $L^2(0,T;W^{1,4}(M))$, and secondly by proving such a solution in fact is in the space $L^4(0,T;W^{1,4}(M))$. Then, we can prove the uniqueness for the weak solutions of $(1.4)\sim(1.5)$. At last, the regularity result is the consequence of the uniqueness result and the existence result (Theorem 1.1).

Notation The Sobolev space $W^{k,p}(M,S^2)$ is defined by

$$W^{k,p}(M,S^2) = \{u \in W^{k,p}(M,\mathbf{R}^3) \mid u \in S^2 \text{ a.e. } x \in M\}.$$

The norm $\| u \|_{W^{k,p}(M,S^2)}$ is simply denoted by $\| u \|_{W^{k,p}}$, here the domain M and the target S^2 are usually omitted. We use $W^{k,p}(M, \Lambda^l)$ to denote the space of differential l-forms on M with coefficients in the Sobolev space $W^{k,p}(M)$. The exterior derivative operator is denoted by d and the conjugate operator of the exterior differential operator d in the metric of M is denoted by d^*. It is well known that $\mathrm{dd}=0$, $\mathrm{d}^*\mathrm{d}^* = 0$ and the Laplace operator for differential forms is given by $\Delta_M = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$.

2 Preliminaries

In this section we shall give some lemmas, that are used in the proof of the main theorem.

Lemma 2.1 (Hodge decomposition theorem)[11] Let M be an m dimensional compact Riemannian manifold without boundary and W be an l-form on M in $L^p(M, \Lambda^l)$. Then, there is a $(l-1)$-form A and $(l+1)$-form B such that $W = \mathrm{d}A + \mathrm{d}^*B$, and $A \in W^{1,p}(M, \Lambda^{l-1})$, $B \in W^{1,p}(M, \Lambda^{l+1})$ satisfy

$$\| A \|_{W^{1,p}} + \| B \|_{W^{1,p}} \leqslant C \| W \|_{L^p}, \tag{2.1}$$

where $C>0$ is a constant depending on M and p. Moreover A and B are unique and satisfy

$$\mathrm{d}^*A = \mathrm{d}B = 0.$$

This result is due to Iwaniec and Martin[11] and the original proof is for the case that $M = \mathbf{R}^m$, but it is not difficult to see that the lemma is also true in this case.

Lemma 2.2 (Wente's theorem)[12] Let M be a two dimensional compact Riemannian manifold without boundary. If $g \in H^1(M, \Lambda^2)$ and $h \in H^1(M, \Lambda^0)$, then $\langle \mathrm{d}^*g, \mathrm{d}h \rangle \in H^{-1}(M, \Lambda^0)$ and

$$\| \langle \mathrm{d}^*g, \mathrm{d}h \rangle \|_{H^{-1}} \leqslant C \| \mathrm{d}^*g \|_{L^2} \| \mathrm{d}h \|_{L^2}, \tag{2.2}$$

where $C>0$ is a constant depending only on M. This result is due to Wente, the original proof is for the case that $M = \mathbf{R}^2$, but the argument also can be applied to this case.

The next lemma is on the uniqueness and regularity for weak solutions to the Cauchy problem of the following system:

$$\partial_t u^i - \beta \Delta_M u^i = \beta \sum_{j=1}^{3} \mathrm{d}^* g^{ij} \cdot \mathrm{d}u^i + f^i, \quad x \in M, \quad t > 0, \tag{2.3}$$

$$u^i(x,0) = u_0^i(x), \quad x \in M, \quad i = 1,2,3, \tag{2.4}$$

where $\beta>0$ is a absolute constant, g^{ij} $(i,j=1,2,3)$ are differential 2-forms on M for each $t \in (0,T)$ and f^i $(i=1,2,3)$ are functions of x and t.

Let

$$X_T = \{u : M \to \mathbf{R}^3 \mid u \in L^2([0,T], H^{1,2}(M)), \partial_t u \in L^2([0,T], H^{-1}(M))\},$$

and for $u \in X_T$, define the norm of u in X_T by

$$\| u \|_{X_T} = \| \partial_t u \|_{L^2([0,T],H^{-1})} + \| u \|_{L^2([0,T],H^{1,2})}.$$

Let also that

$$Y_T = \{u:M\to\mathbf{R}^3 \mid u\in L^2([0,T],W^{1,4}(M)),\partial_t u\in L^2([0,T],L^{\frac{4}{3}}(M))\}, \tag{2.5}$$

and for $u\in Y_T$, define the norm of u in Y_T by

$$\|u\|_{Y_T} = \|\partial_t u\|_{L^2([0,T],L^{\frac{4}{3}})} + \|u\|_{L^2([0,T],W^{1,4})}.$$

We have the following lemma:

Lemma 2.3 Let M be a two dimensional compact Riemannian manifold without boundary. Let also that in (2.3)~(2.4) $f^i(i=1,2,3)\in L^2([0,T],L^{\frac{4}{3}}(M,\Lambda^0)),g^{ij}(1\leqslant i,j\leqslant 3)\in L^\infty([0,T],H^1(M,\Lambda^2))$ and $u_0\in H^1(M,S^2)$. Then, there exists an $\varepsilon_0>0$, depending only on M, T and β, such that if

$$\|g\|_{L^\infty([0,T],H^1)} = \max_{1\leqslant i,j\leqslant 3}\|g^{ij}\|_{L^\infty([0,T],H^1)} < \varepsilon_0, \tag{2.6}$$

then any solution u of (2.3)~(2.4) in the space X_T must belong to the space Y_T.

Proof It is sufficient to prove

Claim 1 The problem (2.3)~(2.4) has a unique solution u in X_T.

Claim 2 The problem (2.3)~(2.4) has a unique solution w in Y_T.

In fact, from the second claim, the problem (2.3)~(2.4) admits a unique solution w in Y_T. Since $Y_T\subset X_T$, so $w\in X_T$. Then, from the first claim, if u is a solution of (2.3)~(2.4) in X_T, u must coincide with w. Therefore, $u\in Y_T$.

Now let us prove these claims by the contraction mapping principle. Endowed with the metric $\rho_1(u_1,u_2)=\|u_1-u_2\|_{X_T}$ on X_T and with the metric $\rho_2(u_1,u_2)=\|u_1-u_2\|_{Y_T}$ on Y_T, X_T and Y_T are nonempty complete metric spaces. For any $v\in X_T$, by solving the Cauchy problem of the following parabolic system:

$$\partial_t u^i - \beta\Delta_M u^i = \beta\sum_{j=1}^{3} \mathrm{d}^* g^{ij}\cdot \mathrm{d}v^j + f^i, \quad \text{in } M\times(0,T], \tag{2.7}$$

$$u^i(x,0) = u_0^i(x), \quad \text{on } M,\ i=1,2,3, \tag{2.8}$$

we define a map $L: v\to u = Lv$.

By the theory for linear parabolic equations[13], the problem (2.7)~(2.8) admits a unique solution $u\in X_T$ and

$$\|u\|_{X_T} = \|\partial_t u\|_{L^2([0,T],H^{-1})} + \|\mathrm{d}u\|_{L^2([0,T],L^2)}.$$

$$\leqslant C\Big(\sum_{j=1}^{3}\|\mathrm{d}^* g^{ij}\cdot\mathrm{d}v^j\|_{L^2([0,T],H^{-1})} + \|f\|_{L^2([0,T],H^{-1})} + \|u_0\|_{L^2}\Big).$$

Applying Lemma 2.2 and the Sobolev embedding theorem to the right side of this inequality, we obtain that

$$\|u\|_{X_T} \leqslant C\big(\|\mathrm{d}^* g\|_{L^\infty([0,T],L^2)}\|\mathrm{d}v\|_{L^2([0,T],L^2)} + \|f\|_{L^2([0,T],L^{\frac{4}{3}})} + \|u_0\|_{H^1}\big), \tag{2.9}$$

where $C>0$ is a constant depending only on M and β. Therefore, the mapping L maps X_T into itself. Moreover, we would like to show that L is a contraction mapping with respect to the metric of X_T. For any $\bar{v},\bar{\bar{v}}\in X_T$, let $\bar{u}=L\bar{v}$, $\bar{\bar{u}}=L\bar{\bar{v}}$ and $u=\bar{u}-\bar{\bar{u}}$, $v=\bar{v}-\bar{\bar{v}}$. Since L

maps X_T into itself, $\bar{u}$, $\bar{u} \in X_T$ and u satisfies

$$\partial_t u^i - \beta \Delta_M u^i = \beta \sum_{j=1}^{3} \mathrm{d}^* g^{ij} \cdot \mathrm{d} v^j, \quad \text{in } M \times (0, T], \tag{2.10}$$

$$u^i(x, 0) = 0, \quad \text{on } M, \quad i = 1,2,3. \tag{2.11}$$

Applying the estimate (2.9) to the problem (2.10)~(2.11) and noticing that $\| \mathrm{d} v \|_{L^2([0,T],L^2)} \leqslant T \| v \|_{X_T}$, we find that under the assumption (2.6),

$$\| u \|_{X_T} \leqslant C\varepsilon_0 \| v \|_{X_T},$$

where $C > 0$ is a constant depending only on M, T and β. Therefore, if ε_0 is chosen to be equal to $\frac{1}{2C}$, L is a contraction mapping on X_T. By the contraction mapping principle, there exists an unique solution $u \in X_T$ to the problem (2.3) ~ (2.4). The proof of Claim 1 is completed.

Claim 2 can be proved by the same method. In fact, by the regularity theory for linear parabolic equations[14], the problem (2.7)~(2.8) admits a unique solution $u \in Y_T$ and

$$\| u \|_{Y_T} \leqslant \| \partial_t u \|_{L^2([0,T],L^{4/3})} + \| \mathrm{d}^2 u \|_{L^2([0,T],L^{4/3})}$$

$$\leqslant C\left(\| \mathrm{d}^* g \|_{L^\infty([0,T],L^2)} \| \mathrm{d} v \|_{L^2([0,T],L^4)} + \| f \|_{L^2([0,T],L^{4/3})} + \| u_0 \|_{H^1} \right). \tag{2.12}$$

By the estimate (2.12), it easy to see that if ε_0 in (2.6) is sufficiently small, then the map L is a contraction mapping on Y_T. Therefore, there exists a unique solution $u \in Y_T$ to the problem (2.3)~(2.4).

The last lemma in this section is regarding to the following Cauchy problem:

$$\partial_t u^i - G_1(x, t) \Delta u = G_2(x, t) \Delta u + g(x, t), \quad x \in M, \quad t > 0, \tag{2.13}$$

$$u(x, 0) = u_0(x), \quad x \in M, \tag{2.14}$$

where $G_i(x, t)(i = 1, 2)$ are matrices and $g(x, t)$ is a vector.

Lemma 2.4 Let M be a two dimensional compact Riemannian manifold without boundary. Suppose that in (2.13)~(2.14),

(1) $G_1(x, t) \Delta u$ is strongly elliptic;

(2) $G_1 \in C^\infty(M \times (0, T))$, $G_2 \in L^\infty(M \times (0, T))$, $g \in L^4(0, T; L^{4/3}(M))$ and $u_0 \in H^1(M)$. Then, there exists a constant $\varepsilon_1 > 0$ depending only on M and T, such that if

$$\| G_2 \|_{L^\infty(M \times (0, T))} \leqslant \varepsilon_1, \tag{2.15}$$

then the problem (2.13) ~ (2.14) has unique solution in $L^s(0, T; W^{2,4/3}(M))$, for any $s \in [2, 4]$.

Proof Again we will use the contraction mapping principle to prove this lemma.

Fix an $s \in [2, 4]$. For any $v \in L^s(0, T; W^{2,4/3}(M))$ by solving the following linear strongly parabolic system

$$\partial_t u - G_1(x, t) \Delta u = G_2(x, t) \Delta v + g(x, t), \quad x \in M, \quad t > 0, \tag{2.16}$$

with the initial condition (2.14), we define a map $L: v \to u = Lv$.

By the theory for linear parabolic equations (see [14] Theorem 9.3, Remark 9.14 and Remark 9.15), the problem (2.16) and (2.14) admits a unique solution $u \in L^s(0, T; W^{2,4/3}$

$(M))$ and

$$\begin{aligned}&\| u\|_{L^s(0,\ T;W^{2,4/3}(M))}\\&=C\{\| G_2\|_{L^\infty(M\times[0,T])}\| v\|_{L^s(0,\ T;W^{2,4/3}(M))}+\| g\|_{L^4(0,\ T;L^{4/3}(M))}+\| u_0\|_{H^1(M)}\}\\&\leqslant C\{\varepsilon_1\| v\|_{L^s(0,\ T;W^{2,4/3}(M))}+\| g\|_{L^4(0,T;L^{4/3}(M))}+\| u_0\|_{H^1(M)}\},\end{aligned}\tag{2.17}$$

where $C>0$ depends only on M and T and in the last inequality we have used (2.15). Therefore, the mapping L maps $L^s(0,\ T;W^{2,4/3}(M))$ into itself. Next we show that L is a contraction mapping with respect to the metric of $L^s(0,\ T;W^{2,4/3}(M))$. For any $\bar{v},\bar{\bar{v}}\in L^s(0,\ T;W^{2,4/3}(M))$, let $\bar{u}=L\bar{v}$, $\bar{\bar{u}}=L\bar{\bar{v}}$ and $u=\bar{u}-\bar{\bar{u}}$, $v=\bar{v}-\bar{\bar{v}}$. Since L maps $L^s(0,\ T;W^{2,4/3}(M))$ into itself, $\bar{u},\bar{\bar{u}}\in L^s(0,\ T;W^{2,4/3}(M))$ and u satisfies (2.16) and (2.14) with $g=0$ and $u_0=0$. It is easy to see from (2.17) that

$$\| u\|_{L^s(0,\ T;W^{2,4/3}(M))}\leqslant\frac{1}{2}\| v\|_{L^s(0,\ T;W^{2,4/3}(M))},$$

if ε_1 is sufficiently small. Therefore, L is a contraction mapping. By the contraction mapping principle, there exists a unique solution $u\in L^s(0,\ T;W^{2,4/3}(M))$ to the problem (2.13)~(2.14), for any $s\in[2,4]$.

3 The Proof of the Main Result

In this section we shall prove Theorem 1.2.

Proof of Theorem 1.2

Step 1 In this step we will show that if $u\in W_T$ is a weak solution of (1.4)~(1.5) with finite energy, then $u\in Y_T\cap W_T$.

Let

$$W^{ij}=u^i\mathrm{d}u^j-u^j\mathrm{d}u^i,\quad i,j=1,2,3.\tag{3.1}$$

Since $E(u(t))\leqslant E_0$ for a.e. $t\in[0,T]$, we have that

$$W^{ij}\in L^\infty([0,T],L^2(M,\ \wedge^1))$$

and

$$\| W^{ij}\|_{L^\infty([0,T],L^2)}\leqslant 2\sqrt{E_0}.$$

By using Hélein's trick[15] and the fact that $|u|=1$, the equation (1.4) can be written as

$$\partial_t u^i-\beta\Delta_M u^i=\alpha(u\times\partial_t u)^i+\beta\sum_{j=1}^{3}W^{ij}\cdot\mathrm{d}u^j.\tag{3.2}$$

Applying the Hodge decomposition theorem (Lemma 2.1) to W^{ij} at each time slice $t\in(0,T)$, one may find that $A^{ij}\in L^\infty([0,T],H^1(M,\ \wedge^0))$ and $B^{ij}\in L^\infty([0,T],H^1(M,\ \wedge^2))$ such that

$$W^{ij}=\mathrm{d}A^{ij}+\mathrm{d}^*B^{ij},\quad\text{for a.e. } t\in[0,T],\tag{3.3}$$

and

$$\| A^{ij}\|_{L^\infty([0,T],H^1)}+\| B^{ij}\|_{L^\infty([0,T],H^1)}\leqslant C\| W^{ij}\|_{L^\infty([0,T],L^2)}\leqslant C\sqrt{E_0}.\tag{3.4}$$

From (3.3), (3.1) and (1.4), we get

$$\Delta A^{ij} = \mathrm{d}^* W^{ij} = u^i \Delta_M u^j - u^j \Delta_M u^i + \beta^{-1}(u^i \partial_t u^j - u^j \partial_t u^i) -$$
$$\frac{\alpha}{\beta}(u^i(u \times \partial_t u)^j - u^j(u \times \partial_t u)^i) \in L^2([0,T], L^2(M, \Lambda^0)).$$

By the Calderón-Zygmund inequality, $\mathrm{d}A^{ij} \in L^2([0,T], H^1(M, \Lambda^0))$ and

$$|\mathrm{d}A^{ij}|_{L^2([0,T],H^1)} \leqslant C |\partial_t u|_{L^2(M\times[0,T])}, \tag{3.5}$$

where $C>0$ is a constant depending only on α, β and M.

On the other hand, since $B^{ij} \in L^\infty([0,T], H^1(M, \Lambda^2))$, for any $\varepsilon > 0$, we always can find $B_1^{ij} \in L^\infty([0,T], C^\infty(M, \Lambda^2))$ and $B_2^{ij} \in L^\infty([0,T], H^1(M, \Lambda^2))$, such that

$$B^{ij} = B_1^{ij} + B_2^{ij}, \tag{3.6}$$

and

$$\| B_2^{ij} \|_{L^\infty([0,T],H^1)} < \varepsilon. \tag{3.7}$$

Now, by using (3.3) and (3.6), we may rewrite the equation (3.2) to the following form:

$$\partial_t u^i - \beta \Delta_M u^i = \beta \sum_{j=1}^{3} \mathrm{d}^* B_2^{ij} \cdot \mathrm{d}u^j + f^i, \tag{3.8}$$

where

$$f^i = \alpha(u \times \partial_t u)^i + \beta \sum_{j=1}^{3} \mathrm{d}A^{ij} \cdot \mathrm{d}u^j + \beta \sum_{j=1}^{3} \mathrm{d}^* B_1^{ij} \cdot \mathrm{d}u^j. \tag{3.9}$$

By the Sobolev embedding theorem and (3.5), we have

$$\Big\| \sum_{j=1}^{3} \mathrm{d}A^{ij} \cdot \mathrm{d}u^j \Big\|_{L^2([0,T],L^{\frac{4}{3}})} \leqslant \sum_{j=1}^{3} \| \mathrm{d}A^{ij} \|_{L^2([0,T],L^4)} \| \mathrm{d}u^j \|_{L^\infty([0,T],L^2)}$$
$$\leqslant \sum_{j=1}^{3} \| \mathrm{d}A^{ij} \|_{L^2([0,T],H^1)} \| \mathrm{d}u^j \|_{L^\infty([0,T],L^2)} \leqslant C_1 \sqrt{E_0}, \tag{3.10}$$

where $C_1 > 0$ is a constant depending only on M and $\| \partial_t u \|_{L^2(M\times[0,T])}$.

On the other hand, noticing that $B_1^{ij} \in L^\infty([0,T], C^\infty(M, \Lambda^2))$, we have

$$\| \mathrm{d}^* B_1^{ij} \|_{L^2([0,T],L^4)} \leqslant C(T,M) \| \mathrm{d}^* B_1^{ij} \|_{L^\infty(M\times[0,T])} \leqslant C(T,M).$$

This leads to the estimate

$$\Big\| \sum_{j=1}^{3} \mathrm{d}^* B_1^{ij} \cdot \mathrm{d}u^j \Big\|_{L^2([0,T],L^{\frac{4}{3}})} \leqslant \sum_{j=1}^{3} \| \mathrm{d}^* B_1^{ij} \|_{L^2([0,T],L^4)} \| \mathrm{d}u^j \|_{L^\infty([0,T],L^2)} \leqslant C_2 \sqrt{E_0}, \tag{3.11}$$

where $C_2 > 0$ is a constant depending only on M, T.

It is obvious that

$$u \times \partial_t u \in L^2([0,T], L^2) \subset L^2([0,T], L^{\frac{4}{3}}). \tag{3.12}$$

From (3.9)~(3.12), we get that

$$f^i \in L^2([0,T], L^{\frac{4}{3}}), \quad i = 1,2,3, \tag{3.13}$$

and $\| f^i \|_{L^2([0,T],L^{\frac{4}{3}})}$ is bounded by a constant depending only on M, T, ε, E_0 and $\| \partial_t u \|_{L^2(M\times[0,T])}$.

Applying Lemma 2.3 to the problem (3.8) and (1.5), and from (3.7) and (3.13), we can conclude that if $\varepsilon \leqslant \varepsilon_0$, where ε_0 is determined in Lemma 2.3, then any solution of (3.8) and (1.5) in X_T is in Y_T. On the other hand, if u is a weak solution of (1.4)～(1.5), then u is also a solution of (3.8) and (1.5) in X_T. Therefore, $u \in Y_T$.

Step 2 In this step we will show that if $u \in Y_T \cap W_T$ is a weak solution of (1.4)～(1.5), then $u \in L^4(0,T;W^{2,4/3}) \cap W_T$.

For $u \in Y_T \cap W_T$, u is defined a.e. on $M \times [0,T]$. From (1.4), one can have

$$u \times \partial_t u = \beta u \times \Delta u - \alpha \partial_t u.$$

Inserting this to (1.4), (1.4) reduces to the form:

$$(1+\alpha^2)\partial_t u = \beta \Delta u + \beta |\mathrm{d}u|^2 u + \alpha\beta u \times \Delta u,$$

i.e.,

$$\partial_t u - G(u)\Delta u = \tilde{\beta} |\mathrm{d}u|^2 u, \tag{3.14}$$

where $\tilde{\beta} = \beta/(1+\alpha^2)$ and

$$G(u) = \frac{1}{1+\alpha^2}\begin{pmatrix} \beta & -\alpha\beta u^3 & \alpha\beta u^2 \\ \alpha\beta u^3 & \beta & -\alpha\beta u^1 \\ -\alpha\beta u^2 & \alpha\beta u^1 & \beta \end{pmatrix}. \tag{3.14′}$$

Since $|u| = 1$, for any $\varepsilon > 0$ we can decompose u into the form:

$$u = m + n, \tag{3.15}$$

where $m \in C^\infty(M \times (0,T))$, $n \in L^\infty(M \times (0,T))$ with

$$\| m \|_{L^\infty(M\times(0,T))} \leqslant C \| u \|_{L^\infty(M\times(0,T))}, \tag{3.16}$$

$$\| n \|_{L^\infty(M\times(0,T))} \leqslant \varepsilon. \tag{3.17}$$

Inserting (3.15) into (3.14), it yields that

$$\partial_t u - G(m)\Delta u = G(n)\Delta u + \tilde{\beta} |\mathrm{d}u|^2 u. \tag{3.18}$$

Letting $G_1(x,t) = G(m(x,t))$, $G_2(x,t) = G(n(x,t))$, and $g(x,t) = \tilde{\beta}|\mathrm{d}u|^2 u$, then, (3.18) takes the same form as (2.13). Moreover,

$$\| G_2 \|_{L^\infty(M\times(0,T))} \leqslant C \| n \|_{L^\infty(M\times(0,T))},$$

$$\| g \|_{L^4(0,T;L^{4/3})} \leqslant C \| \mathrm{d}u \|_{L^\infty([0,T],L^2)} \| \mathrm{d}u \|_{L^2([0,T],L^4)}.$$

We can see from (3.14′), (3.16) and (3.17), that if ε is sufficiently small, then the assumptions in Lemma 2.4 and (2.15) are satisfied. Therefore, by Lemma 2.4, the problem (3.18) with the initial condition (1.5) admits a unique solution $v \in L^4(0,T;W^{2,4/3}) \subset L^2(0,T;W^{2,4/3})$ and the solution in the space $L^2(0,T;W^{2,4/3})$ is also unique. On the other hand, it is obvious that if $u \in Y_T \cap W_T$ is a weak solution of (1.4)～(1.5), then u is a solution of (3.18) and (1.5) in $L^2(0,T;W^{2,4/3})$. Therefore, $u = v \in L^2(0,T;W^{2,4/3})$.

Step 3 In this step we will complete our proof for Theorem 2.1 by showing the following.

Lemma 3.1 Let $u \in L^4(0,T;W^{2,4/3}) \cap W_T$ and $v \in L^4(0,T;W^{2,4/3}) \cap W_T$ be two solutions of (1.4)～(1.5). Then, $u = v$ a.e. on $M \times [0,T]$.

Proof For simplicity we assume $\alpha = 1$, $\beta = 1$. Now since $w = u - v$ solves the following equation