Textbook for Bilingual Course of Differential Equations in College

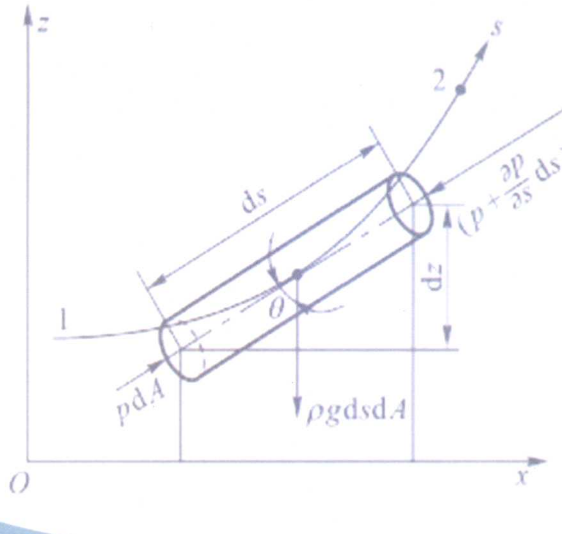
高等学校"微分方程"课程双语教学教材

DIFFERENTIAL EQUATIONS

微分方程

Song Yingqing Cao Fuhua Huang Xin

宋迎清 曹付华 黄 新 编著



Wuhan University of Technology Press

武汉理工大学出版社

Textbook for Bilingual Course of Differential Equations in College
高等学校“微分方程”课程双语教学教材

DIFFERENTIAL EQUATIONS
微分方程

Song Yingqing **Cao Fuhua** **Huang Xin**
宋迎清 曹付华 黄 新

Wuhan University of Technology Press
武汉理工大学出版社

内 容 简 介

本书是高等学校本科生“微分方程”课程双语教学的教材，主要介绍各类微分方程的解法，全书共分6章，主要包括：微分方程模型与基本概念；一阶常微分方程（包括一阶显式常微分方程和一阶隐式常微分方程）的解法；常系数高阶线性微分方程的解法、变系数微分方程的解法以及边值问题和可降阶的高阶微分方程的解法；线性方程组的基本原理、常系数齐次线性方程组的解法、常系数非齐次线性方程组的解法；首次积分；解的定性分析方法和稳定性原理；一阶和二阶偏微分方程的解法。

全书各章均编写了习题（答案附在全书的最后）。

本书除了适合作为高等学校本科生“微分方程”双语课程教学使用外，也可作为自学读本和研究生参考书。

图书在版编目（CIP）数据

微分方程/宋迎清，曹付华，黄新. —武汉：武汉理工大学出版社，2009.8
ISBN 978-7-5629-2916-1

Ⅰ.微…
Ⅱ.①宋… ②曹… ③黄…
Ⅲ.微分方程-双语教学-高等学校-教材
Ⅳ.O175

中国版本图书馆 CIP 数据核字（2009）第 083562 号

出版发行：武汉理工大学出版社
武汉市武昌珞狮路 122 号　邮编：430070
http://www.techbook.com.cn　理工图书网
经 销 者：各地新华书店
印 刷 者：京山德兴印刷有限公司
开　　本：787×1092　1/16
印　　张：14.25
字　　数：365 千字
版　　次：2009 年 8 月第 1 版
印　　次：2009 年 8 月第 1 次印刷
定　　价：28.00 元
凡购本书，如有缺页、倒页、脱页等印装质量问题，请向出版社发行部调换。
本社购书热线电话：（027）87394412　87397097

Preface

This book is intended as textbook for the bilingual-course on theory and applications of differential equations. It is an outgrowth of course given by us in the last several years to students majoring in mathematics at Hunan City University.

We believe that the knowledge and appreciation of the basic theory of differential equations are important for scientists and engineers as well as mathematicians. Accordingly we have tried to present this theory in a careful and straightforward manner. The meaning of a solution of a differential equation and methods to find information about the solutions are discussed more thoroughly than is customary in an elementary text. Considerable emphasis is given to techniques of solution. Many important points of view are well motivated and explained. We have included much more material than is usual for a book on this level. This has been done to allow greater flexibility in the use of the book for students of varying backgrounds and interests. Furthermore, the value of this book as a reference for students in later work and for practicing scientists and engineers is enhanced.

The book is divided into six chapters.

Chapter 1 covers models, definitions, classifications, and simple illustrations of solutions of differential equations.

Chapter 2 is connected with first-order differential equations. We give some special methods of solving first-order explicit form then deal with existence and uniqueness of solutions and methods of approximating solutions. Some viewpoints treat with the general solution and the singular integral of first-order implicit form are introduced at the final of the chapter.

In Chapter 3, we study high-order equations. Some important methods to solve linear equations are developed. These methods include undetermined coefficients, variation of parameters, Laplace transform, power series, and so on. The last section is devoted to linear boundary-value problem and reducible high-order equations.

Many methods of finding solutions of first-order ordinary differential system are summarized in Chapter 4. Matrix expression is an excellent way of finding solutions of first-order linear system with constant coefficients. Finding the first integrals can solve some particular systems.

Chapter 5 emphasizes to predict behavior of solutions without attempting to find them. Some important methods of obtaining information about the solutions are introduced. The last two sections are devoted to study two-dimensional autonomous systems.

Chapter 6 introduces partial differential equations. We focus on solving first- and second-order linear forms.

In addition, some exercises are provided at the end of each chapter and answers or hints to most of these problems are collected at the end of the book.

The book is designed to serve as a textbook for 64 hours.

Some chapters or sections marked " * " should be selected when teaching time is more than 64 hours.

It is pleasure to express our gratitude to many friends and colleagues and to generations of students at Hunan City University for their valuable criticism of a preliminary version of this book.

By Song Yingqing, Cao Fuhua and Huang Xin

Contents

CHAPTER 1

INTRODUCTION TO DIFFERENTIAL EQUATIONS

In mathematical analysis, we have studied many functions which can reflect the relations between quantities and quantities on moves of materials. However, a few facts show that many relations can not directly be expressed in such a function form. They are usually expressed as the equations which **relate independent variable(s), unknown function(s) and the derivatives. Such an equation is called differential equation (DE).**

Essentially, a differential equation implies a relation between variables, which is not clear as that of a function. Therefore, it is necessary to study further the relation after building up a mathematical model of differential equation. However, this is not a simple work. Sometimes, a clear relation, say a solution of the equation, can be obtained; sometimes we can only do some qualitative analysis.

In our textbook, we will mainly study the basic theory and methods of solution of differential equations.

1.1 MODELS ON DIFFERENTIAL EQUATIONS

The importance of differential equations is attested to by the frequency with which they occur in scientific phenomena; in fact, many of the fundamental laws of science are formulated in terms of differential equations.

Example 1.1 Newton's law of cooling

This law states that the time rate of change of the temperature of a cooling body is proportional to the temperature difference between the body and its surroundings. Therefore the differential equation for the temperature of the cooling body is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_s)$$

where T_s is the temperature of surroundings and k is a positive constant.

Example 1.2 Dilution problems

A tank initially contains V gal of fresh water. At $t=0$ a brine solution, containing c lb of salt per gallon, is poured into the tank at a rate of a gal/min. The contents of the tank are stirred to maintain homogeneity, and the dilute solution is pumped out at a rate of a gal/min. The problem is to find the amount of salt in the tank at any time.

Let x (in pounds) be the amount of salt in the tank at any time. The volume of the solution in the tank is always V; therefore the concentration of salt is x/V (in pounds per gallon). Salt leaves the tank at the rate of ax/V lb/min and salt enters the tank at ca lb/min. The rate of change of the amount of salt in the tank is therefore

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ca - \frac{ax}{V}$$

Example 1.3 Electrical circuits

The circuit shown in Fig1.1 contains a resistance R, an inductance L, a capacitance C and a source of voltage $E(t)$. According to Kirchhoff's law: **the sum of the voltage drops around a simple closed series circuit equals zero**, the charge $Q(t)$ satisfies the differential equation:

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = E(t)$$

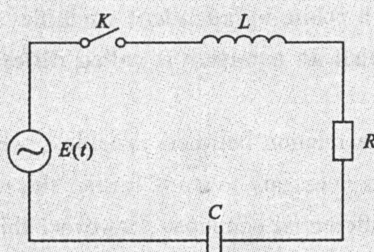


Fig 1.1 A Simple Electrical Circuit

Example 1.4* Population growth

Some interesting mathematical problems arise in biology. One example is the problem of predicting the growth of population. This is, of course, an extremely difficult problem which depends on many complicated parameters.

Let $N(t)$ be the number of people in an isolated population at time t. The function $N(t)$ is clearly an integer-valued function; it is constant in intervals of time when the population does not change, and it is discontinuous when a birth or death changes the population. Such discrete-valued functions are difficult to analyze. Our first simplifying approximation is to replace $N(t)$ by a continuous and differentiable function. This allows us to use the powerful tools of calculus. The function $N(t)$ can be thought of as a smoothed-out version of the actual population.

Our next simplifying assumption is that the time rate of change of $N(t)$ is a function only of N, that is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = F(N)$$

The function $F(N)$ is called the growth function.

We shall further assume that

$$F(0) = 0$$

$$F(C) = 0$$

where $C > 0$. Assumption $F(0) = 0$ implies that if no people are initially present, then no people will ever be present. Assumption $F(C) = 0$ can be thought of as an upper limit to the population which could occur, for example, because of a limited food supply.

The simplest differential equation for population growth is

$$\frac{dN}{dt} = kN(C - N)$$

where $k > 0$ stands for a growing population.

This can be rewritten as

$$\frac{dN}{dt} = kCN - KN^2$$

The first term on the right-hand side can be considered the **birth rate** and the second term **death rate**.

Example 1.5* Multiple-species growth

In 1924, an Italian biologist D. Ancona (Volterra's future son-in-law) introduced Volterra to problems in ecology that in the years after the First World War the proportion of predatory fishes caught in the Upper Adriatic was found to be considerably higher than in the years before the war, whereas the proportion of prey fishes was down. Of course, the war between Austria and Italy made fishery in the Adriatic impossible, but why did this give much benefit to predatory fishes than to their prey?

Volterra denoted by x the density of the prey fishes, also by y that of predators and proposed differential equations for growth of predators and prey. He assumed that the growth rate of prey is a positive constant a if there exists no predator. Further, it is assumed that the rate decreases linearly as a function of the predator density. This leads to the prey equation:

$$\frac{1}{x}\frac{dx}{dt} = a - by, \quad a, b > 0$$

For the predatory fishes, it is assumed that the predators will decay to zero exponentially in the absence of the prey and the growth rate is enhanced with the density x. This leads to the predator equation:

$$\frac{1}{y}\frac{dy}{dt} = -c + dx, \quad c, d > 0$$

Rewritten these two equations, we have a system of differential equations:

$$\begin{cases} x'(t) = x(a - by) & a, b > 0 \\ y'(t) = y(-c + dx) & c, d > 0 \end{cases}$$

Example 1.6* A two-degree-of-freedom vibration

Consider a mechanical system consisting of two masses connected by three springs as shown in

Fig 1.2.

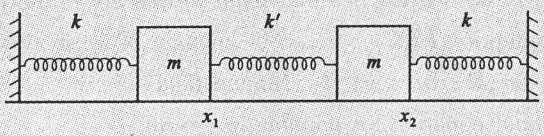


Fig 1.2 A two-degree-of-freedom vibration

The springs have spring constants of k, k', and the masses are equal. We assume that the motion of the masses takes place in a straight line on a smooth frictionless plane. The only force on the masses is the spring forces.

Let x_1 and x_2 denote the displacement of the masses from their equilibrium positions. From Newton's second law we obtain

$$\begin{cases} m\dfrac{d^2x_1}{dt^2} = -kx_1 + k'(x_2 - x_1) \\ m\dfrac{d^2x_2}{dt^2} = -k'(x_2 - x_1) - kx_2 \end{cases}$$

It follows that

$$\begin{cases} m\dfrac{d^2x_1}{dt^2} + (k + k')x_1 - k'x_2 = 0 \\ m\dfrac{d^2x_2}{dt^2} - k'x_1 + (k + k')x_2 = 0 \end{cases}$$

Example 1.7* The vibrating string

Suppose that a thin flexible string of mass m and length L is tightly stretched horizontally between two supports (See Fig 1.3).

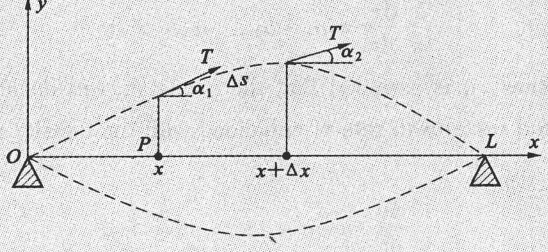


Fig 1.3 The Vibrating String

If the string is given a small vertical displacement and released, it will vibrate up and down. A point P on the string, at a distance x from the left end of the string, will vibrate vertically. That is, its vertical displacement y will vary with the time t. Thus, in general, the displacement y will depend on both the position x and the time t. That is, y will be a function of x and t which we shall designate by $y(x,t)$. We shall usually assume that $y(x,t)$ has continuous second partial derivatives.

In order to derive the differential equation for $y(x,t)$, we shall make a few simplifying assumptions. We assume that the only force that is exerted on the string is the constant tension T lb. That is, we neglect the effects of gravity, friction, etc. We assume that the mass per unit length δ is constant along the string ($\delta = m/L$). Finally, we assume that the displacement $y(x,t)$ is small in such a way that the angle of inclination α of the displacement curve is always small.

Consider a typical element of length Δs, of the string between x and $x+\Delta x$, the vertical component of the force on the element is

$$F = T\sin\alpha_2 - T\sin\alpha_1$$

Since we have assumed α to be small, $\sin\alpha \approx \tan\alpha$ we have approximately

$$F = T(\tan\alpha_2 - \tan\alpha_1)$$

or, by the definition of the partial derivative y_x'

$$F = T[y_x'(x+\Delta x,t) - y_x'(x,t)]$$

The mass of the element is $\delta \cdot \Delta s$, where Δs is the length. Our assumptions ensure that $\Delta s \approx \Delta x$, and hence the mass is approximately $\delta \cdot \Delta x$. Applying Newton's law, we have

$$T[y_x'(x+\Delta x,t) - y_x'(x,t)] = (\delta \cdot \Delta x)y_{tt}''(x,t)$$

where we assume that the element is small enough so that the vertical acceleration at all points of the element is approximately $y_{tt}''(x,t)$.

Dividing the equation by $\delta \cdot \Delta x$, we have

$$\frac{T}{\delta}\left[\frac{y_x'(x+\Delta x,t) - y_x'(x,t)}{\Delta x}\right] = y_{tt}''(x,t)$$

If we now take the limit as $\Delta x \to 0$, the expression in brackets becomes y_{xx}''.

Writing down $a^2 = T/\delta$, we can derive the differential equation for the displacement of a vibrating string:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

Example 1.8* Heat conduction

Whenever there is a difference in temperature between two parts of a solid object, there will be a flow of heat from the hotter part to the cooler. This fact is essentially the content of the second law of thermodynamics. Implied in the statement is the fact that the temperature u (degrees centigrade) is a function of both position and time. Hence the variations of u will be governed by a partial differential equation.

We shall examine first the simple special case in which the heat conductor is a cylinder parallel to the x axis. If the lateral surface of the cylinder is insulated, and if the ends are perpendicular to the x axis and are kept at constant temperature, then the temperature is a function of x and t only, i.e. $u = u(x,t)$. We also assume that there is no source of heat inside the cylinder. The differential equation governing the changes in u can be derived from the following two laws of physics, which we assume to be known.

① The heat content Q of a solid of mass m and specific heat c is

$$Q = cmu$$

if the entire body is at temperature u.

② The rate at which heat flows out of a body through a plane surface of area A is given by

$$-KA\frac{\partial u}{\partial n}$$

where K is the thermal conductivity (Assumed to be a positive constant) and $\partial u/\partial n$ is the outward normal derivative.

These two laws will be applied to the thin slice of the cylinder between x and $x+\Delta x$ (See Fig 1.4)

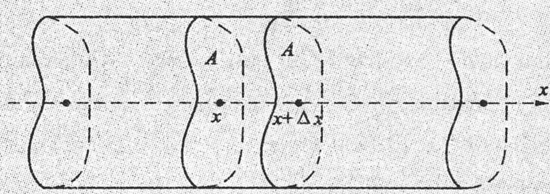


Fig 1.4 Heat conduction between x and $x+\Delta x$

If the mass density of the solid is ρ, then the mass Δm of the slice is $\Delta m = \rho A\Delta x$, where A is the (constant) cross-sectional area.

Equation $Q = cmu$ implies that the heat content of the slice is

$$\Delta Q = c\rho A(\Delta x)u(x,t)$$

where we assume that the temperature at all points in the slice is approximately $u(x,t)$.

At the right-hand boundary of the slice, i. e., at $x+\Delta x$, the outer normal derivative is simply the derivative with respect to x. Hence the rate of heat flow out through the face at $x+\Delta x$ is given by $-KAu_x'(x+\Delta x,t)$.

Similarly, at the left face, at x the outward normal derivative of u is $-\partial u/\partial x$ and hence the rate of heat loss equals $KAu_x'(x,t)$.

Expressions $-KAu_x'(x+\Delta x,t)$ and $KAu_x'(x,t)$ represent the rate at which heat flows out of the element. Hence, if we change the signs of both expressions and add, we obtain the rate of increase of heat in the slice.

$$KA[u_x'(x+\Delta x,t) - u_x'(x,t)]$$

From $\Delta Q = c\rho A(\Delta x)u(x,t)$ we can obtain another expression for the time rate Q_t' of change of heat content in the slice, i. e.

$$Q_t' = c\rho Au_t'(x,t)\Delta x$$

Combining the results of $KA[u_x'(x+\Delta x,t) - u_x'(x,t)]$ and $Q_t' = c\rho Au_t'(x,t)\Delta x$, we have

$$c\rho Au_t'(x,t)\Delta x = KA[u_x'(x+\Delta x,t) - u_x'(x,t)]$$

Dividing by $c\rho A\Delta x$ and taking the limit as $\Delta x\to 0$ yields

$$u_t'(x,t) = \frac{K}{c\rho}u_{xx}''(x,t)$$

The positive constant $K/c\rho = k$ is called **the thermal diffusivity**, and the above equation can be written in the standard form of

$$u_{xx}'(x,t) = \frac{1}{k}u_t'(x,t)$$

1.2 BASIC CONCEPTS OF DIFFERENTIAL EQUATIONS

1.2.1 Classifications of Differential Equations

Definition 1 An equation which relates derivative (s) of unknown function (s) is called a differential equation.

The differential equations can be classified in terms of number of independent variables, number of dependent variable (s), the order of the highest derivative, and the linearity of equations.

Classification 1 Classifying according to number of independent variables

According to number of independent variables, the differential equations can be classified to **ordinary** ones and **partial** ones.

Definition 2 If a differential equation depends on only one independent variable, then the derivatives of unknown function (s) are ordinary derivatives, and the equation is called an **ordinary differential equation** (ODE).

If the equation depends on two or more independent variables, then the derivatives of unknown function (s) are partial derivatives, and the equation is called a **partial differential equation** (PDE).

For example, the equations in example 1.1, example 1.2, example 1.3, example 1.4, example 1.5, and example 1.6 are ordinary differential equations, and the equations in example 1.7 and example 1.8 are partial differential equations(See section 1.1).

Classification 2 Classifying according to number of dependent variables

According to number of dependent variables (unknown functions), the equations can be classified to **differential equations** and **differential systems**.

Definition 3 Number of dependent variables (unknown functions) appearing in a differential equation is called **dimension** of the equation.

For example, the equations in example 1.1, example 1.2, example 1.3, example 1.4, example 1.7 and example 1.8 are one-dimensional equations, and the equations in example 1.5 and example 1.6 are two-dimensional equations (See section 1.1).

Generally, in a problem, if there are n unknown functions, then we shall obtain n differential equations. These equations make up of a **differential system.**

Classification 3 Classifying according to the Order

The differential equations are also classified according to their **order**.

Definition 4 The order of a differential equation means the order of the highest derivative appearing in the equation.

According to the order of a differential equation, the equations have two types: **first-order** ones and **high-order** ones.

For example, the equations in example 1.1 and example 1.2 are first-order ones; the equations in example 1.3 and example 1.4 are second-order ones. The equation in example 1.5 is a first-order differential system. The equation in example 1.6 is a second-order differential system. The equations in example 1.7 and example 1.8 are second order partial differential equations (See section 1.1).

An ordinary differential equation of the nth-order can be described as an equation in the form:

$$f(x, y, y', \cdots, y^{(n)}) = 0$$

where f is a well-defined function of its various arguments.

Similar expressions could be written for an n-dimensional system and a partial differential equation of the nth-order.

Classification 4 Classifying according to the linearity

According to the linearity, the differential equations can be classified to **linear** ones and **nonlinear** ones.

Definition 5 If unknown function and its derivatives appear linearly, the equation is called a **linear** differential equation. Otherwise it is **nonlinear.**

For example, the equations in example 1.1, example 1.2, example 1.3, example 1.6, example 1.7, and example 1.8 are **linear** ones; the equations in example 1.4 and example 1.5 are **nonlinear** ones (See section 1.1).

An nth-order linear ordinary differential equation can be described as an equation in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x)$$

In particular, if $f(x) = 0$, the equation is called a **homogeneous** linear ones;