量子力学题解二百例

（下 册）

熊钰庆 周笣声编解

华南师院物理系

一九八一年八月

§4 表象理论

107. (1) 矩阵形式

根据 $F_{\lambda'\lambda}=\int u^*_{\lambda'}(x)\hat{F}(x,\frac{\hbar}{i}\frac{\partial}{\partial x})u_\lambda(x)dx$

有
$$
\begin{aligned}
x_{p'p}&=\int\psi^*_{p'}x\psi_p dx\\
&=\frac{1}{2\pi\hbar}\int e^{-\frac{i}{\hbar}p'x}\cdot x\cdot e^{\frac{i}{\hbar}px}dx\\
&=\frac{1}{2\pi\hbar}\int xe^{\frac{i}{\hbar}(p-p')x}dx\\
&=\frac{-i\hbar}{2\pi\hbar}\frac{\partial}{\partial p}\int e^{\frac{i}{\hbar}(p-p')x}dx\\
&=-i\hbar\frac{\partial}{\partial p}\frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}(p-p')x}dx\\
&=-i\hbar\frac{\partial}{\partial p}\delta(p-p').
\end{aligned}
$$

(2) 微分算符形式:

根据 $b(p')=\int F_{p'p}a(p)dp$

有
$$
\begin{aligned}
b(p')&=\int x_{p'p}a(p)dp\\
&=-i\hbar\int[\frac{\partial}{\partial p}\delta(p-p')]a(p)dp\\
&=-i\hbar[\delta(p-p')a(p)]^{\infty}_{-\infty}\\
&\quad-(-i\hbar)\int\delta(p-p')\frac{\partial}{\partial p}a(p)dp\\
&=i\hbar\frac{\partial}{\partial p'}a(p')
\end{aligned}
$$

故 $\hat{x}$ 在 "P" 表象中的微分算符形式为

$$\hat{x}=i\hbar\frac{\partial}{\partial p}$$

另法 $\varphi(\vec{r})=\int c(\vec{p})\psi_{\vec{p}}(\vec{r})d^3p$

$$=\frac{1}{(2\pi\hbar)^{3/2}}\int c(\vec{p})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}d^3p, \quad (1)$$

而 $C(\vec{p})=\int\psi(\vec{r})\psi_{\vec{p}}^{*}(\vec{r})\,dv$

$$=\frac{1}{(2\pi\hbar)^{3/2}}\int\psi(\vec{r})e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}}dv, \qquad (2)$$

$C(\vec{p})$ 可看作是"动量表象"中的波函数。在动量表象中，粒子坐标算符 $\hat{\vec{r}}$ 由下式定义：

$$\overline{\vec{r}}=\int C^{*}(\vec{p})\hat{\vec{r}}C(\vec{p})d^{3}p \qquad (3)$$

另一方面，在"坐标表象"中：

$$\overline{\vec{r}}=\int\psi^{*}(\vec{r})\vec{r}\psi(\vec{r})dv \qquad (4)$$

将(1)式代入(4)式，得

$$\overline{\vec{r}}=\frac{1}{(2\pi\hbar)^{3}}\iiint C^{*}(\vec{p}\,')e^{-\frac{i}{\hbar}\vec{p}\,'\cdot\vec{r}}\vec{r}C(\vec{p})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}d^{3}p'd^{3}p\,dv$$

-----(5)

上式中

$$\int\vec{r}C(\vec{p})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}d^{3}p=\int\frac{\hbar}{i}C(\vec{p})\frac{\partial}{\partial\vec{p}}e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}d^{3}p$$

$$=\frac{\hbar}{i}C(\vec{p})e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\Big|_{-\infty}^{\infty}-\frac{\hbar}{i}\int e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\frac{\partial C(\vec{p})}{\partial\vec{p}}d^{3}p$$

$$=i\hbar\int e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\frac{\partial C(\vec{p})}{\partial\vec{p}}d^{3}p,$$

代入(5)式得

$$\overline{\vec{r}}=\frac{1}{(2\pi\hbar)^{3}}\iiint C^{*}(\vec{p}\,')e^{-\frac{i}{\hbar}\vec{p}\,'\cdot\vec{r}}e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}i\hbar\frac{\partial}{\partial\vec{p}}C(\vec{p})d^{3}p'd^{3}p\,dv$$

$$=\iint C^{*}(\vec{p}\,')\left[\frac{1}{(2\pi\hbar)^{3}}\int e^{\frac{i}{\hbar}(\vec{p}-\vec{p}\,')\cdot\vec{r}}dv\right]i\hbar\frac{\partial}{\partial\vec{p}}C(\vec{p})d^{3}p'd^{3}p$$

$$=\iint C^{*}(\vec{p}\,')\delta(\vec{p}-\vec{p}\,')i\hbar\frac{\partial}{\partial\vec{p}}C(\vec{p})d^{3}p'd^{3}p$$

$$=\int C^{*}(\vec{p})i\hbar\frac{\partial}{\partial\vec{p}}C(\vec{p})d^{3}p. \qquad (6)$$

将(3)式与(6)式比较可得

$$\hat{\vec{r}}=i\hbar\frac{\partial}{\partial\vec{p}},$$

这就是动量表象中，粒子坐标算符的微分形式。

108．算符 $\hat{F}$ 在 Q 表象中的矩阵元是

$$F_{mn}=\int u_m^*(x)\hat{F}u_n(x)dx.$$

其中 $u_n(x)$ 是 $\hat{Q}$ 的本征函数．

若 $\hat{Q}$ 的本征值组成连续谱，则为

$$F_{\lambda'\lambda}=\int u_{\lambda'}^*(x)\hat{F}u_\lambda(x)dx. \qquad (2)$$

现在要求 $\hat{L}_x$ 在动量表中的矩阵元，可设

$$\hat{F}=\hat{L}_x,\quad \hat{Q}=\hat{\vec{p}}.$$

因为动量算符 $\hat{\vec{p}}$ 的本征值组成连续谱，故采用（2）式，且 $u_\lambda(x)$ 就是平面波

$$\psi_{\vec{p}}=\frac{1}{(2\pi\hbar)^{3/2}}e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \qquad (3)$$

利用（2）式，我们有

$$(L_x)_{p'p}=\int\psi_{\vec{p}'}^*\hat{L}_x\psi_{\vec{p}}d\tau. \qquad (4)$$

将（3）式及

$$\hat{L}_x=y\hat{p}_z-z\hat{p}_y$$

代入（4）式得

$$(L_x)_{p'p}=\frac{1}{(2\pi\hbar)^3}\int e^{-\frac{i}{\hbar}\vec{p}'\cdot\vec{r}}(y\hat{p}_z-z\hat{p}_y)e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}d\tau \quad \text{-------}(5)$$

注意到

$$\hat{p}_z=\frac{\hbar}{i}\frac{\partial}{\partial z},\quad \hat{p}_y=\frac{\hbar}{i}\frac{\partial}{\partial y}$$

及

$$\hat{p}_z e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}=\frac{\hbar}{i}\frac{\partial}{\partial z}e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}=\frac{\hbar}{i}\cdot\frac{i}{\hbar}p_z e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}=p_z e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$

$$\hat{p}_y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}=p_y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$

则（5）式变成

$$(L_x)_{p'p}=\frac{1}{(2\pi\hbar)^3}\int e^{-\frac{i}{\hbar}\vec{p}'\cdot\vec{r}}(yp_z-zp_y)e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}d\tau$$

但

$$y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = \frac{\hbar}{i}\frac{\partial}{\partial p_y} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$

$$z e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = \frac{\hbar}{i}\frac{\partial}{\partial p_z} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$

$$\therefore\ (L_x)_{p'p} = \frac{p_z}{(2\pi\hbar)^3}\int e^{-\frac{i}{\hbar}\vec{p}'\cdot\vec{r}}\frac{\hbar}{i}\frac{\partial}{\partial p_y} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} d\tau$$

$$-\frac{p_y}{(2\pi\hbar)^3}\int e^{-\frac{i}{\hbar}\vec{p}'\cdot\vec{r}}\frac{\hbar}{i}\frac{\partial}{\partial p_z} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} d\tau$$

$$= \frac{\hbar}{i}\left(p_z\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_z}\right)\cdot\frac{1}{(2\pi\hbar)^3}\int e^{\frac{i}{\hbar}(\vec{p}-\vec{p}')\cdot\vec{r}} d\tau$$

$$= \frac{\hbar}{i}\left(p_z\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_z}\right)\delta(\vec{p}-\vec{p}')$$

$$= \frac{\hbar}{i}\left(p_z\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_z}\right)\delta(\vec{p}'-\vec{p})$$

下面求$\hat{L}^2_x$的矩阵元：

$$因\ \hat{L}^2_x = \hat{L}_x\hat{L}_x = (y\hat{p}_z - z\hat{p}_y)(y\hat{p}_z - z\hat{p}_y)$$

$$= y^2\hat{p}^2_z + z^2\hat{p}^2_y - z\hat{p}_y\, y\hat{p}_z - y\hat{p}_z\, z\hat{p}_y,$$

利用对易关系：

$$y\hat{p}_y - \hat{p}_y y = i\hbar,\quad z\hat{p}_z - \hat{p}_z z = i\hbar,$$

得

$$\hat{L}^2_x = y^2\hat{p}^2_z + z^2\hat{p}^2_y + i\hbar z\hat{p}_z - zy\hat{p}_y\hat{p}_z$$

$$+ i\hbar y\hat{p}_y - yz\hat{p}_z\hat{p}_y$$

$$= y^2\hat{p}^2_z + z^2\hat{p}^2_y + i\hbar z\hat{p}_z + i\hbar y\hat{p}_y - 2yz\hat{p}_z\hat{p}_y$$

于是

$$(L^2_x)_{p'p} = \int \psi^*_{\vec{p}'}\hat{L}^2_x\psi_{\vec{p}}\, d\tau$$

$$= \frac{1}{(2\pi\hbar)^3}\int e^{-\frac{i}{\hbar}\vec{p}'\cdot\vec{r}}\hat{L}^2_x e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} d\tau$$

$$= \frac{1}{(2\pi\hbar)^3}\int e^{-\frac{i}{\hbar}\vec{p}'\cdot\vec{r}}(y^2\hat{p}_z^2 + z^2\hat{p}_y^2 + i\hbar z\hat{p}_z$$

$$+ i\hbar y\hat{p}_y - 2yz\hat{p}_z\hat{p}_y)e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\,d\tau \qquad (6)$$

与上面类似，有

$$\begin{cases}
\hat{p}_z^2 e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = p_z^2 e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
\hat{p}_y^2 e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = p_y^2 e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
\hat{p}_z e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = p_z e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
\hat{p}_y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = p_y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
\hat{p}_z\hat{p}_y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = p_z p_y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}
\end{cases}$$

及

$$\begin{cases}
z e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = \frac{\hbar}{i}\frac{\partial}{\partial z} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
y e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = \frac{\hbar}{i}\frac{\partial}{\partial y} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
y^2 e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = -\hbar^2\frac{\partial^2}{\partial p_y^2} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
z^2 e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = -\hbar^2\frac{\partial^2}{\partial p_z^2} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \\
yz e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = -\hbar^2\frac{\partial^2}{\partial p_y\partial p_z} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}
\end{cases}$$

又注意到

$$\frac{1}{(2\pi\hbar)^3}\int e^{\frac{i}{\hbar}(\vec{p}-\vec{p}')\cdot\vec{r}}\,d\tau = \delta(\vec{p}-\vec{p}')$$

$$= \delta(\vec{p}'-\vec{p})$$

将以上诸式代入(6)式，最后得到

$$(L_x^2)_{p'p} = \Big\{-\hbar^2\Big(p_z^2\frac{\partial^2}{\partial p_y^2} + p_y^2\frac{\partial^2}{\partial p_z^2}\Big)$$

$$+\hbar^2\left(p_z\frac{\partial}{\partial p_z}+p_y\frac{\partial}{\partial p_y}\right)$$

$$+2\hbar^2 p_z p_y\frac{\partial^2}{\partial p_y\partial p_z}\Big\}\delta(\vec{p}'-\vec{p})\text{.}$$

109. 在坐标表象中,线性 谐振子哈密顿量是

$$\hat{H}=\frac{1}{2\mu}\hat{p}^2+\frac{1}{2}\mu\omega^2x^2$$

动量算符的本征函数是

$$\psi_p=\frac{1}{(2\pi\hbar)^{1/2}}e^{\frac{i}{\hbar}px}$$

在动量表象中哈密顿量的矩阵元是

$$H_{p'p}=\int_{-\infty}^{\infty}\psi_{p'}^*\hat{H}\psi_p dx$$

$$=\int_{-\infty}^{\infty}\psi_{p'}^*\frac{1}{2\mu}\hat{p}^2\psi_p dx+\int_{-\infty}^{\infty}\psi_{p'}^*\frac{1}{2}\mu\omega^2x^2\psi_p dx$$

注意到

$$\hat{p}^2\psi_p=\hat{p}^2\frac{1}{(2\pi\hbar)^{1/2}}e^{\frac{i}{\hbar}px}=p^2\frac{1}{(2\pi\hbar)^{1/2}}e^{\frac{i}{\hbar}px}=p^2\psi_p$$

$$x^2\psi_p=x^2\frac{1}{(2\pi\hbar)^{1/2}}e^{\frac{i}{\hbar}px}=-\hbar^2\frac{\partial^2}{\partial p^2}\frac{1}{(2\pi\hbar)^{1/2}}e^{\frac{i}{\hbar}px}$$

$$=-\hbar^2\frac{\partial^2}{\partial p^2}\psi_p$$

$$H_{p'p}=\frac{p^2}{2\mu}\int_{-\infty}^{\infty}\psi_{p'}^*\psi_p dx-\frac{1}{2}\mu\omega^2\hbar^2\frac{\partial^2}{\partial p^2}\int_{-\infty}^{\infty}\psi_{p'}^*\psi_p dx$$

$$=\frac{p^2}{2\mu}\delta(p'-p)-\frac{1}{2}\mu\omega^2\hbar^2\frac{\partial^2}{\partial p^2}\delta(p'-p)\text{.}$$

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讨论

上面求出了动量表象中线性谐振子的哈密顿量的矩阵元，利用它可以写出动量表象中振子满足的薛定格方程，设波函数为 $C(p,t)$.

在分立谱情况下，薛定格方程在 Q表象中的形式为

$$i\hbar\frac{\partial a_m(t)}{\partial t}=\sum_m H_{mn}a_n(t),\qquad (m=1,2,\cdots\cdots)$$

对连续谱，总和应被积分代替，对动量表象，应为

$$i\hbar\frac{\partial C(p,t)}{\partial t}=\int H_{p'p}C(p',t)dp'.$$

将已求得的 $H_{p'p}$ 代入上式得

$$i\hbar\frac{\partial C(p,t)}{\partial t}=\int\left[\frac{p^2}{2\mu}\delta(p'-p)-\frac{1}{2}\mu\omega^2\hbar^2\frac{\partial^2}{\partial p^2}\delta(p'-p)\right]\times$$

$$\times C(p',t)dp'$$

$$=\frac{p^2}{2\mu}C(p,t)-\frac{1}{2}\mu\omega^2\hbar^2\frac{\partial^2}{\partial p^2}C(p,t)$$

即 $$i\hbar\frac{\partial C(p,t)}{\partial t}=\left(\frac{p^2}{2\mu}-\frac{1}{2}\mu\omega^2\hbar^2\frac{\partial^2}{\partial p^2}\right)C(p,t)$$

令 $$\frac{1}{\mu}=\mu'\omega^2,\qquad \mu\omega^2=\frac{1}{\mu'}$$

则 $$i\hbar\frac{\partial C(p,t)}{\partial t}=\left(-\frac{\hbar^2}{2\mu'}\frac{\partial^2}{\partial p^2}+\frac{1}{2}\mu'\omega^2p^2\right)C(p,t).$$

此方程与坐标表象中振子的薛定格方程形式完全相同，只须作如下代换：

$$C(p,t)\longrightarrow\Psi(x,t)\qquad \mu'\longrightarrow\mu$$

$$\frac{\partial^2}{\partial p^2}\longrightarrow\frac{\partial^2}{\partial x^2}$$

$$p^2\longrightarrow x^2$$

因此，坐标表象中振子薛定格方程的解

$$H_n(\xi)=\sqrt{\frac{\alpha}{\pi^{1/2}n!2^n}}e^{-\frac{\xi^2}{2}}H_n(\xi)$$

作上述代换便是动量表象中振子薛定格方程的解。

注意： $\alpha \longrightarrow \alpha'$

因 $$\alpha=\sqrt{\frac{\mu w}{\hbar}}$$

故 $$\alpha'=\sqrt{\frac{\mu' w}{\hbar}}=\sqrt{\frac{1}{\mu w^2}\cdot\frac{w}{\hbar}} \qquad \left(\mu'=\frac{1}{\mu w^2}\right)$$

$$=\sqrt{\frac{1}{\mu w}\cdot\frac{1}{\hbar}}=\sqrt{\frac{\hbar}{\mu w}\cdot\frac{1}{\hbar^2}}=\frac{1}{\alpha\hbar}\equiv\beta$$

又 $\xi\longrightarrow\xi'$

因 $\xi=\alpha x$

故 $$\xi=\alpha' p=\frac{p}{\alpha\hbar}\equiv\eta$$

这样便可得
$$c(\eta)=\sqrt{\frac{\beta}{\pi^{1/2}n!\,2^n}}\,e^{-\frac{\eta^2}{2}}H_n(\eta)$$

与上题结果差一常数因子 $(-i)^n$，但两者表示同一状态

110　在坐标表象中谐振子的能量本征函数为

$$\psi_n=\sqrt{\frac{\alpha}{\pi^{1/2}n!\,2^n}}\,e^{-\frac{\xi^2}{2}}H_n(\xi)=N_n e^{-\frac{\xi^2}{2}}H_n(\xi)$$

其中 $\alpha=\sqrt{\frac{\mu w}{\hbar}}$，$\xi=\alpha x$

在坐标表象中，动量算符的本征函数是

$$\psi_p=\frac{1}{(2\pi\hbar)^{1/2}}e^{\frac{i}{\hbar}px}$$

设
$$\psi_n=\int c(p)\psi_p(x)\,dp$$

其中
$$c(p)=\int\psi_n\psi_p^*\,dx$$

就是动量表象中谐振子能量的本征函数

$$c(p)=\int_{-\infty}^{\infty}\psi_n(x)\psi_p^*(x)\,dx$$

$$=\frac{N_n}{(2\pi\hbar)^{1/2}}\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} H_n(\xi) e^{-\frac{i}{\hbar}px} dx$$

令

$$\eta=\frac{p}{\alpha\hbar}$$

则 $\frac{p}{\hbar}x=\frac{p}{\hbar}\cdot\frac{\xi}{\alpha}=\eta\xi$

$$d\xi=d(\alpha x)=\alpha dx$$

于是

$$C(\eta)=\frac{N_n}{\alpha\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} H_n(\xi) e^{-i\eta\xi} d\xi .$$

上式的积分计算如下：

利用厄米多项式的母函数

$$e^{-t^2+2t\xi}=\sum_{l}\frac{H_l(\xi)}{l!}t^l$$

两边乘以 $e^{-\frac{\xi^2}{2}}e^{-i\eta\xi}$ 并对 ξ 积分得

$$\sum_l\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} H_l(\xi)\frac{t^l}{l!} e^{-i\eta\xi} d\xi$$

$$=\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} e^{-t^2+2t\xi} e^{-i\eta\xi} d\xi$$

$$=e^{-t^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}[\xi-(2t-i\eta)]^2} d\xi\cdot e^{\frac{1}{2}(2t-i\eta)^2}$$

$$=e^{-t^2+\frac{1}{2}(2t-i\eta)^2}\sqrt{\frac{\pi}{(\frac{1}{2})}}$$

$$=\sqrt{2\pi}\, e^{t^2-2it\eta-\frac{\eta^2}{2}}$$

$$=\sqrt{2\pi}\, e^{-\frac{\eta^2}{2}} e^{-(it)^2+2(it)(-\eta)}$$

$$=\sqrt{2\pi}\, e^{-\frac{\eta^2}{2}}\sum_l\frac{H_l(-\eta)}{l!}(it)^l$$

比较两边 t^l 项的系数得

$$\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} H_n(\xi) e^{-i\eta\xi} d\xi=\sqrt{2\pi}\, e^{-\frac{\eta^2}{2}} H_n(-\eta)\, i^n$$

$$\therefore\quad C(\eta)=\frac{\sqrt{\beta}}{\sqrt{\pi^{1/2}n!2^n}} e^{-\frac{\eta^2}{2}} H_n(-\eta)\, i^n$$

式中

$$\beta=\frac{1}{\alpha\hbar}$$

但

$$Hn(\xi)=(-1)^{n}Hn(-\xi)$$

$$\therefore\quad C(\eta)=\frac{\sqrt{\beta}}{\sqrt{\pi^{1/2}n!\,2^{n}}}(-1)^{n}i^{n}e^{-\frac{\eta^{2}}{2}}Hn(\eta)$$

$$=\frac{\sqrt{\beta}}{\sqrt{\pi^{1/2}n!\,2^{n}}}(-i)^{n}e^{-\frac{\eta^{2}}{2}}Hn(\eta).$$

111. 在E表象中，算符$\hat{L}$可表示为一矩阵，矩阵元为

$$L_{mn}=\int\psi_{m}^{*}\hat{L}\psi_{n}dx$$

其中

$$\psi_{n}=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

(1) 坐标的矩阵元：

令$\hat{L}=\hat{x}=x$

则 当$m\neq n$时，

$$x_{mn}=\frac{2}{a}\int_{0}^{a}x\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}dx$$

$$=\frac{2}{a}\int_{0}^{a}\frac{x}{2}\cos\left[\frac{(m-n)\pi x}{a}\right]dx$$

$$-\frac{2}{a}\int_{0}^{a}\frac{x}{2}\cos\left[\frac{(m+n)\pi x}{a}\right]dx$$

$$=\frac{1}{a}\cdot\frac{a}{(m-n)\pi}\int_{0}^{a}x\,d\sin\frac{(m-n)\pi}{a}x$$

$$-\frac{1}{a}\cdot\frac{a}{(m+n)\pi}\int_{0}^{a}x\,d\sin\frac{(m+n)\pi}{a}x$$

$$=\frac{1}{(m-n)\pi}\left[x\sin\frac{(m-n)\pi}{a}x\right]_{0}^{a}$$

$$-\frac{1}{(m-n)\pi}\int_{0}^{a}\sin\frac{(m-n)\pi}{a}x\,dx$$

$$-\frac{1}{(m+n)\pi}\left[x\sin\frac{(m+n)\pi}{a}x\right]_{0}^{a}$$

$$+\frac{1}{(m+n)\pi}\int_0^a \sin\frac{(m+n)\pi}{a}x\,dx$$

$$=\frac{a}{(m-n)^2\pi^2}\cos\frac{(m-n)\pi}{a}x\Big|_0^a$$

$$-\frac{a}{(m+n)^2\pi^2}\cos\frac{(m+n)\pi}{a}x\Big|_0^a$$

$$=(-1)^{m-n}\frac{a}{(m-n)^2\pi^2}-\frac{a}{(m-n)^2\pi^2}$$

$$-(-1)^{m+n}\frac{a}{(m+n)^2\pi^2}+\frac{a}{(m+n)^2\pi^2}$$

$$=\frac{a}{\pi^2}\Big\{(-1)^{m+n}\frac{1}{(m-n)^2}-\frac{1}{(m-n)^2}$$

$$-(-1)^{m+n}\frac{1}{(m+n)^2}+\frac{1}{(m+n)^2}\Big\}\text{、}\qquad (-1)^{m+n}=(-1)^{m-n}$$

$$=\frac{a}{\pi^2}\Big\{(-1)^{m+n}-1\Big\}\Big\{\frac{1}{(m-n)^2}-\frac{1}{(m+n)^2}\Big\}$$

$$=\frac{a}{\pi^2}\cdot\frac{4mn}{(m^2-n^2)^2}\left[(-1)^{m+n}-1\right]$$

$$=\begin{cases}0 & \text{当 } m+n \text{ 为偶数时}\\ -\dfrac{a}{\pi^2}\cdot\dfrac{8mn}{(m^2-n^2)^2}, & \text{当 } m+n \text{ 为奇数时}\end{cases}$$

当 $m=n$ 时 $x_{nn}=\frac{2}{a}\int_0^a x\sin^2\frac{n\pi}{a}x\,dx=\frac{1}{a}\int_0^a x\,dx-\frac{1}{a}\int_0^a$

$$-\frac{1}{a}\int_0^a x\cos\frac{2n\pi}{a}x\,dx=\frac{a}{2}-0=\frac{a}{2}$$

例如 $x_{11}=x_{22}=\cdots=x_{nn}=\cdots=\frac{a}{2}$.

$$x_{12}=x_{21}=-\frac{16}{9\pi^2}a$$

$$x_{13}=x_{31}=0,\ \cdots\cdots$$

$$x_{14}=x_{41}=-\frac{32}{(15\pi)^2}a\cdots\cdots$$

可见所求矩阵为对称矩阵，又因矩阵元为实数，故又是厄米矩阵，矩阵形式为

$$\begin{pmatrix} \frac{a}{2} & -\frac{16}{9\pi^2}a & 0 & -\frac{32}{(15\pi)^2}a & \cdots \\ -\frac{16}{9\pi^2}a & \frac{a}{2} & \cdots & \cdots & \cdots \\ 0 & \cdots & \frac{a}{2} & \cdots & \cdots \\ -\frac{32}{(15\pi)^2}a & \cdots & \cdots & \frac{a}{2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \frac{a}{2} \end{pmatrix}$$

(2) 动量的矩阵元：

令 $\hat{L} = \hat{P}_x = -i\hbar\frac{\partial}{\partial x}$

则 当 $m \neq n$ 时

$$(P_x)_{mn} = \frac{2}{a}\int_0^a \sin\frac{m\pi x}{a}\left(-i\hbar\frac{\partial}{\partial x}\right)\sin\frac{n\pi x}{a}\,dx$$

$$= -\frac{2i\hbar}{a}\cdot\frac{n\pi}{a}\int_0^a \sin\frac{m\pi x}{a}\cos\frac{n\pi x}{a}\,dx$$

$$= -\frac{2i\hbar n\pi}{a^2}\int_0^a \frac{1}{2}\left[\sin\frac{(m+n)\pi}{a}x + \sin\frac{(m-n)\pi}{a}x\right]dx$$

$$= \frac{-i\hbar n\pi}{a^2}\cdot\frac{a}{(m+n)\pi}\int_0^a \sin\frac{(m+n)\pi}{a}x\,d\frac{(m+n)\pi}{a}x$$

$$-\frac{i\hbar n\pi}{a^2}\cdot\frac{a}{(m-n)\pi}\int_0^a \sin\frac{(m-n)\pi}{a}x\,d\frac{(m-n)\pi}{a}x$$

$$= \frac{-i\hbar n}{a(m+n)}\left[-\cos\frac{m+n}{a}\pi x\right]_0^a - \frac{i\hbar n}{a(m-n)}\left[-\cos\frac{m-n}{a}\pi x\right]_0^a$$

$$= \frac{-i\hbar n}{a(m+n)}\left[1-\cos(m+n)\pi\right] - \frac{i\hbar n}{a(m-n)}\left[1-\cos(m-n)\pi\right]$$

$$= \frac{-i\hbar n}{a(m+n)}\left[1-(-1)^{m+n}\right] - \frac{i\hbar n}{a(m-n)}\left[1-(-1)^{m-n}\right]$$

$$= \frac{-i\hbar n}{a}\left[1-(-1)^{m+n}\right]\times\left[\frac{1}{n+n}+\frac{1}{m-n}\right]$$

$$= \frac{2mn\,i\hbar}{a(m^2-n^2)}\left[(-1)^{m+n}-1\right]$$

$$=\begin{cases}0, & \text{当 } m+n \text{ 为偶数时} \\ -\dfrac{4mni\hbar}{a(m^2-n^2)}, & \text{当 } m+n \text{ 为奇数时}\end{cases}$$

当 $m=n$ 时 $(P_x)_{nn}=\frac{2}{a}\int_0^a \sin\frac{n\pi x}{a}\left(-i\hbar\frac{\partial}{\partial x}\right)\sin\frac{n\pi x}{a}\,dx$

$$=-\frac{2i\hbar}{a}\int_0^a \sin\frac{n\pi}{a}x\,d\sin\frac{n\pi x}{a}\cdots$$

$$=-\frac{i\hbar}{a}\sin^2\frac{n\pi x}{a}\Big|_0^a=0.$$

例如 $(P_x)_{11}=0$ $(P_x)_{22}=0$ $(P_x)_{33}=0$ ……

$(P_x)_{12}=\frac{8i\hbar}{3a}$ $(P_x)_{21}=-\frac{8i\hbar}{3a}=(P_x)_{12}^*$

$(P_x)_{13}=(P_x)_{31}=0$, ……

$(P_x)_{14}=\frac{16i\hbar}{15a}$ $(P_x)_{41}=-\frac{16i\hbar}{15a}=(P_x)_{14}^*$

可见所求矩阵为厄米矩阵，矩阵形式为

$$P_x=\begin{pmatrix} 0 & \frac{8i\hbar}{3a} & 0 & \frac{16i\hbar}{15a} & \cdots \\ -\frac{8i\hbar}{3a} & 0 & \cdots & & \\ 0 & 0 & 0 & \cdots & \\ -\frac{16i\hbar}{15a} & \cdots & 0 & \cdots & \\ \cdots & & & & \end{pmatrix}$$

112 $\because$ $\dot{x}=\frac{1}{i\hbar}(x\hat{H}-\hat{H}x)$

$$\therefore (\dot{x})_{nm}=\frac{1}{i\hbar}(x\hat{H}-\hat{H}x)_{nm}$$

$$=\frac{1}{i\hbar}\sum_k(x_{nk}H_{km}-H_{nk}x_{km}).$$

因为哈密顿量在能量表象中是对角矩阵，即

$H_{km}=E_k\delta_{km}$ $H_{nk}=E_k\delta_{nk}$

于是

$$(\dot{x})_{nm}=\frac{1}{i\hbar}\sum_k(x_{nk}E_k\delta_{km}-E_k\delta_{nk}x_{km})$$

$$=\frac{1}{i\hbar}(x_{nm}E_m-E_nx_{nm})$$

$$=\frac{1}{i\hbar}(E_m-E_n)x_{nm}.$$

或

$$x_{nm}=\frac{-i\hbar}{E_n-E_m}(\dot{x})_{nm}\qquad(1)$$

同理

$$(\dot{x}^*)_{nm}=\frac{1}{-i\hbar}(E_m-E_n)x^*_{nm}.$$

或

$$x^*_{nm}=\frac{i\hbar}{E_n-E_m}(\dot{x}^*)_{nm}\qquad(2)$$

现将$\sum_n(E_n-E_m)|x_{nm}|^2$写成I 即

$$I=\sum_n(E_n-E_m)|x_{nm}|^2$$

$$=\sum_n(E_n-E_m)x^*_{nm}x_{nm}\qquad(3)$$

将(1)式代入(3)式得

$$I=-i\hbar\sum_n x^*_{nm}(\dot{x})_{nm}$$

$$=-\frac{i\hbar}{\mu}\sum_n x^*_{nm}p_{nm}\qquad\left(\hat{\dot{x}}=\frac{\hat{p}}{\mu}\right)$$

$$=-\frac{i\hbar}{\mu}\sum_n x_{mn}p_{nm},\qquad(4)$$

其中利用了x的厄米性质

$$x^*_{nm}=x_{mn}.$$

将(2)式代入(3)式得

$$I=i\hbar\sum_n(\dot{x}^*)_{nm}x_{nm}$$

$$=\frac{i\hbar}{\mu}\sum_n p^*_{nm}x_{nm}$$

$$=\frac{i\hbar}{\mu}\sum_n p_{mn}x_{nm}.\quad(p^*_{nm}=p_{mn})\qquad(5)$$

(4) 式与（5）式相加，得

$$2I=\frac{-i\hbar}{\mu}\sum_n(x_{mn}p_{nm}-p_{mn}x_{nm})$$

$$=\frac{-i\hbar}{\mu}(x\hat{p}-\hat{p}x)_{mm}=\frac{-i\hbar}{\mu}\cdot i\hbar=\frac{\hbar^2}{\mu}$$

$$\therefore\quad I=\sum_n(E_n-E_m)|x_{nm}|^2=\frac{\hbar^2}{2\mu}.$$

113. (1) 在A表象中，算符$\hat{A}$的矩阵为对角矩阵，对角元素就是算符的本征值，现因

$$\hat{A}^2=1$$

故$\hat{A}$的本征值为±1，因而$\hat{A}$可用二阶矩阵表示，
由此得$\hat{A}$在A表象中的矩阵表示为

$$A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

设$\hat{B}$的矩阵表示为

$$B=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}$$

由于$\hat{A}\hat{B}+\hat{B}\hat{A}=0$，故

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}+\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=0.$$

即 $$\begin{pmatrix}b_{11}&b_{12}\\-b_{21}&-b_{22}\end{pmatrix}+\begin{pmatrix}b_{11}&-b_{12}\\b_{21}&-b_{22}\end{pmatrix}=\begin{pmatrix}2b_{11}&0\\0&-2b_{22}\end{pmatrix}=0$$

$$\therefore\quad b_{11}=b_{22}=0.$$

于是 $$B=\begin{pmatrix}0&b_{12}\\b_{21}&0\end{pmatrix}$$

再利用 $\hat{B}^2=1$，有

$$\begin{pmatrix}0&b_{12}\\b_{21}&0\end{pmatrix}\begin{pmatrix}0&b_{12}\\b_{21}&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

即

$$\begin{pmatrix}b_{12}b_{21}&0\\0&b_{21}b_{12}\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$\therefore\ \begin{cases} b_{12}\cdot b_{21}=1 \\ b_{21}\cdot b_{12}=1 \end{cases}$$

由此得
$$b_{12}=\frac{1}{b_{21}}.$$

于是
$$B=\begin{pmatrix} 0 & b_{12} \\ \frac{1}{b_{12}} & 0 \end{pmatrix}$$

因B是厄米矩阵，即 $B^{+}=B$，故

$$\begin{pmatrix} 0 & \frac{1}{b_{12}^{*}} \\ b_{12}^{*} & 0 \end{pmatrix}=\begin{pmatrix} 0 & b_{12} \\ \frac{1}{b_{12}} & 0 \end{pmatrix}$$

由此得
$$b_{12}=\frac{1}{b_{12}^{*}},$$

一般地选
$$b_{12}=e^{i\theta},$$

于是得算符 $\hat{B}$ 在A表象中的矩阵表示为

$$B=\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}.$$

(2) 同理，在B表象中，$\hat{B}$的矩阵表示为

$$B=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\hat{A}$的矩阵表示为

$$A=\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}.$$

(3) 设$\hat{B}$的本征值为β，本征函数为$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$

已知在A表象中$\hat{B}$的矩阵表示为

$$B=\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix},$$

故本征方程为：

$$\begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}=\beta\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$