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Distribution of Eigentones in a Rectangular Chamber at Low Frequency Range* †

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IT has been shown by various authors[1] that the number of eigentones in a rectangular chamber with frequencies less than a certain limiting value ν is given by

$$N=4\pi V\nu^3/3c^3 \tag{1}$$

where V is the volume of the chamber and c the wave velocity which, for sound waves, is 1125 feet per second at room temperature. This value given by (1) is correct only when the limiting wave-length is negligibly small compared to the dimensions of the chamber, as in case of light waves. When it is applied to problems in acoustics, as in the calculations of intensity distribution, growth and decay of sound energy in a room, etc., the results are far from satisfactory, because the wave-length now becomes comparable with the dimensions of the room. Mr. Bolt, by considering the density of representative points, arrived at a very satisfactory result, as shown in his diagrams.[2] From an alternative point of view, the following result

$$N=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}\right] \tag{2}$$

may be obtained, in which S is the surface area, L the sum of the three dimensions, V the volume of the chamber, and c the sound velocity. Although this result seems quite different from that obtained by Bolt:[2]

$$N=\frac{4\pi V\nu^3}{3c^3}\left[\frac{2\nu V+cR^{\frac{1}{2}}}{2\nu V+\frac{1}{2}cR^{\frac{1}{2}}}\right]^3 \tag{3}$$

(where R is the sum of the squares of three areas, $S_x^2+S_y^2+S_z^2$, and the other symbols have the same meaning as above), the two forms agree quite well. The values obtained from Eq. (2) are plotted as x's in the diagrams given by Bolt.[2]

* Part of this work was done while the author was in residence at the University of California at Los Angeles.

† Presented at the November meeting of the Acoustical Society of America. At the same meeting it was suggested that in the future, in acoustic terminology, the word "eigentone" should be replaced by the more specific terms "normal frequency" and "normal mode."

[1] See, e.g., Courant and Hilbert, *Methode der Mathematische Physik;* Morse, *Vibration and Sound.*

[2] See the preceding paper in this issue of the Journal: "Frequency Distribution of Eigentones in a Three-Dimensional Continuum," by R. H. Bolt.

DERIVATION

(A) The asymptotic formula (1)

Let the dimensions of the rectangular chamber be L_x, L_y, and L_z. If the surfaces of the chamber are all rigid, the sound wave in the chamber will be given by:

$$\nabla^2\phi-(1/c^2)\ddot{\phi}=0 \tag{4}$$

under the boundary conditions:

$$\partial\phi/\partial x=0 \text{ at } x=0 \text{ and } x=L_x,\ \partial\phi/\partial y=0 \text{ at } y=0 \text{ and } \cdots;$$

where ϕ represents the velocity potential, $\partial\phi/\partial x$ the particle velocity along x-direction, etc. The solution will be of the form:

$$\phi=\epsilon^{-i2\pi\nu t}\cos\frac{p_x\pi x}{L_x}\cos\frac{p_y\pi y}{L_y}\cos\frac{p_z\pi z}{L_z}$$

with $4\pi^2\nu^2/c^2=p_x^2\pi^2/L_x^2+p_y^2\pi^2/L_y^2+p_z^2\pi^2/L_z^2$,

where p_x, p_y, and p_z are integers or zero. For the eigentones with frequency less than ν, one has

$$4\nu^2/c^2>p_x^2/L_x^2+p_y^2/L_y^2+p_z^2/L_z^2. \tag{5}$$

Each set of positive integers (p_x, p_y, p_z) satisfying the relation (5) will give an eigentone of frequency less than ν. Therefore any point having positive integral coördinates in the (p_x, p_y, p_z) space will represent an eigentone and the number of eigentones having frequencies less than ν will be equal to the number of such representative points within the first octant of the ellipsoid

$$p_x^2/L_x^2+p_y^2/L_y^2+p_z^2/L_z^2=4\nu^2/c^2. \tag{6}$$

It is easy to see that there is, on the average, one point having integral coördinates in every unit volume in space, and the number of such points within a certain volume is equal to the volume itself. Therefore, the number of eigentones having frequency less than ν is equal to one-eighth of the volume enclosed by the ellipsoid (6), or

$$N=\frac{1}{8}\frac{4\pi}{3}\left(\frac{2L_x\nu}{c}\right)\left(\frac{2L_y\nu}{c}\right)\left(\frac{2L_z\nu}{c}\right)=\frac{4\pi}{3}\frac{V\nu^3}{c^3}, \tag{1}$$

where $V=L_xL_yL_z$ is the volume of the rectangular chamber.

(B) The correction

When the whole space was divided into eight octants, all the points on the coördinate planes were, effectively, cut into two halves and all the points on the coördinate axes were, effectively, cut into four quarters. So, in Eq. (1) only one-half of the points on the coördinate planes and only one-fourth of the points on the coördinate axes have been counted. The correction will be the addition of the missing points. Evidently there is, on the average, one point with integral coördinates in every unit area on the coördinate planes. The total number of representative points on the planes within the ellipsoid (6) is equal to the sum of the areas of the first quadrants of the ellipses

$$\frac{p_y^2}{(2L_y\nu/c)^2}+\frac{p_z^2}{(2L_z\nu/c)^2}=1,\ \cdots,$$

that is

$$\frac{1}{4}\left[\pi\frac{2L_y\nu}{c}\frac{2L_z\nu}{c}+\pi\frac{2L_z\nu}{c}\frac{2L_x\nu}{c}+\pi\frac{2L_x\nu}{c}\frac{2L_y\nu}{c}\right]=\frac{\pi\nu^2S}{2c},$$

where $S=2(L_yL_z+L_zL_x+L_xL_y)$ is the total surface area of the chamber. One-half of this number had been missed. So we have to add another half, i.e.,

$$\pi S\nu^2/4c^2 \tag{7}$$

to the value given by Eq. (1). As for the points on the axes, one-fourth were counted in Eq. (1) and another half in Eq. (7). One-fourth of the points are still uncounted. It is easily seen that the number of points on the axes is equal to the sum of lengths of the three semi-axes of the ellipsoid (6), i.e.,

$$2L_x\nu/c+2L_y\nu/c+2L_z\nu/c=2L\nu/c,$$

where $L=(L_x+L_y+L_z)$ is the sum of three dimensions of the chamber; and one-fourth of this number is

$$L\nu/2c. \tag{8}$$

Therefore, the total number of representative points, those within the positive octant of the ellipsoid, (6), or the number of eigentones of frequencies less than ν is given by

$$N=\frac{4\pi}{3}\frac{V\nu^3}{c^3}+\frac{\pi S\nu^2}{4c^2}+\frac{L\nu}{2c}$$
$$=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}\right]. \tag{2}$$

Or in terms of wave-length, the number of eigentones having wave-lengths greater than λ is

$$N=\frac{4\pi V}{3\lambda^3}\left[1+\frac{3S\lambda}{16V}+\frac{3L\lambda^2}{8\pi V}\right]. \tag{9}$$

Discussion

It will be seen that the correction terms depend only on the ratio of wave-length to the dimension of the chamber. In all practical cases, the second correction term is always very much smaller than one, except for the first few eigentones. For example, in a room $10'\times15'\times30'$ (discussed by Bolt)

$3L\lambda^2/8\pi V=0.19$ at 100 cycles per second

and 0.02 at 300 cycles per second.

So, in general, it is negligible. But the first correction term must be considered throughout most of the audiofrequency range. For example, in the same room, $10'\times15'\times30'$,

$3S\lambda/16V=1.13$ at 100 c.p.s.
0.38 at 300 c.p.s.

and 0.04 at 3000 c.p.s.

Since the mean free path of sound waves in the room is given by

$$\Lambda=4V/S,$$

the first correction term may also be put in the form $3\lambda/4\Lambda$, and Eq. (2) will become

$$N=(4\pi V/3\lambda^3)[1+3\lambda/4\Lambda+3L\lambda^2/8\pi V]. \tag{10}$$

From Eq. (2) many other useful relations can be derived; the following are a few examples:

(I) The number of eigentones having frequencies between ν and $\nu+\delta\nu$ is

$$\delta N=\frac{4\pi V}{3c^3}[(\nu+\delta\nu)^3-\nu^3]+\frac{\pi S}{4c^2}[(\nu+\delta\nu)^2-\nu^2]$$
$$+\frac{L}{2c}[(\nu+\delta\nu)-\nu],$$

or

$$\delta N=\frac{4\pi V\nu^2}{c^3}\left(1+\frac{Sc}{8V}\frac{1}{\nu}+\frac{Lc^2}{8\pi V}\frac{1}{\nu^2}\right)\delta\nu$$
$$+\frac{4\pi V\nu}{c^3}\left(1+\frac{Sc}{16V}\frac{1}{\nu}\right)\delta\nu^2+\frac{4\pi V}{3c^3}\delta\nu^3. \quad (11)$$

For very small values of $\delta\nu$, i.e., when $\delta\nu \ll \nu$, the higher order terms of $\delta\nu$ may be neglected, and we have

$$\delta N=\frac{4\pi V\nu^2}{c^3}\left(1+\frac{Sc}{8V}\frac{1}{\nu}+\frac{Lc^2}{8\pi V}\frac{1}{\nu^2}\right)\delta\nu. \quad (12)$$

The first correction term is just

$$\lambda/2\Lambda.$$

In the room $10'\times 15'\times 30'$, $\Lambda=10'$. Except when $\lambda \ll 20'$, or $\nu \gg 56$ c.p.s., the old form

$$\delta N=(4\pi V/c^3)\nu^2\delta\nu$$

will not be a good approximation. The second correction is still very small except when the frequency is very low.

(II) If the walls of the chamber are not totally reflective, there will be damping for any sound wave in the chamber and the above derivation will not be exact because the relation (5) is only an approximation. But when the absorption of the walls is small, no further correction will be required.

If, in such a room, a source of strength $Q_0\epsilon^{-i\omega t}$ is turned on, the final average energy density will be, according to Morse,[1]

$$W=\frac{4\rho c^2Q_0^2}{V^2}\sum_{(n)}\frac{\sigma_n\psi_n^2(S)}{(2\omega_nk_n/\omega)^2+(\omega-\omega_n^2/\omega)^2}, \quad (13)$$

where ρ is the density of air, k_n is the damping factor for the nth eigentone ω_n, S is the position of source,

$$\psi(x,y,z)=\cos\frac{p_x\pi x}{L_x}\cos\frac{p_y\pi y}{L_y}\cos\frac{p_z\pi z}{L_z}$$

and $\sigma_n=1, \frac{1}{2}, \frac{1}{4}$, according to whether none, one, or two of the p's are zero. If ν is high enough so that the eigentones around it are very close to one another, the average energy density will be

$$W=\frac{\pi\rho Q_0^2\nu^2}{4Vk}\left(1+\frac{Sc}{8V}\frac{2}{\nu}+\frac{Lc^2}{8\pi V}\frac{1}{\nu^2}\right) \quad (14)$$

by following Morse's method.

(III) The present result (2) is readily applicable to chambers of other shapes. It is suggested that when the formula is applied to a chamber of conventional type, V and S will still be taken as the total volume and surface area, respectively, and L will be the sum of the height and one-half of the perimeter of the chamber.

(IV) In the above derivation, no consideration has been given to the eigentone (0,0,0) which, in general, is not counted. Effectively, one-eighth of the eigentone (0,0,0) has been counted in the first term of N, three-eighths in the second and three-eighths in the third. If in the counting of the eigentones, the mode (0,0,0), which is not vibratory at all, is excluded, the number attributed by it, i.e.,

$$\tfrac{1}{8}+\tfrac{3}{8}+\tfrac{3}{8}=\tfrac{7}{8},$$

should be subtracted from the value of N obtained above and Eq. (2) will become

$$N=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}\right]-\frac{7}{8}$$

or

$$N=\frac{4\pi V\nu^3}{3c^3}\left[1+\frac{3Sc}{16V}\frac{1}{\nu}+\frac{3Lc^2}{8\pi V}\frac{1}{\nu^2}+\frac{21c^3}{32\pi V}\frac{1}{\nu^3}\right]. \quad (15)$$

The value of δN will be still given by Eq. (11) or (12). When the value of N is much larger than one, the last correction may be neglected. But if N is not so large, all the correction terms should be used.

(V) From the solution

$$\phi=\epsilon^{-i\omega t}\cos\frac{p_x\pi}{L_x}x\cos\frac{p_y\pi}{L_y}y\cos\frac{p_z\pi}{L_z}z$$

of the wave equation, it is evident that $(2L_x/p_x)$, $(2L_y/p_y)$, $(2L_z/p_z)$ are the "components" of the wave-length along the axes (here the "component" of a wave-length along a certain direction means the line segment along that direction cut by two successive wave fronts at a wave-length apart), hence the direction cosines of the wave normal of this particular eigentone are equal to, but may differ in signs from

$$\frac{\lambda}{2L_x/p_x},\quad \frac{\lambda}{2L_y/p_y},\quad \frac{\lambda}{2L_z/p_z}$$

or

$$\frac{p_x}{2L_x},\frac{p_y}{2L_y},\frac{p_z}{2L_z}\bigg/\left[\left(\frac{p_x}{2L_x}\right)^2+\left(\frac{p_y}{2L_y}\right)^2+\left(\frac{p_z}{2L_z}\right)^2\right]^{\frac{1}{2}},$$

the number, N_θ, of the eigentones with frequencies less than ν and with wave normals making angles less than θ with the x axis will be the number of representative points in the p space within the first octant of the ellipsoid (6) and also within the conical surface

$$\frac{p_x}{2L_x}\bigg/\left[\left(\frac{p_x}{2L_x}\right)^2+\left(\frac{p_y}{2L_y}\right)^2+\left(\frac{p_z}{2L_z}\right)^2\right]^{\frac{1}{2}}=\cos\theta. \tag{16}$$

Without the correction for the points on the coördinate planes and the axes, this number may be found by integration as

$$(4\pi V\nu^3/3c^3)(1-\cos\theta).$$

By the same method as before, the correction for the points on the coördinate planes is

$$\theta S'\nu^2/c^2$$

where $\qquad S'=L_xL_y+L_xL_z,$

for those on the p_x axis

$$2L_x\nu/c$$

and for the origin

$$\tfrac{1}{8}\cos\theta-\theta/2\pi-\tfrac{1}{4}.$$

Therefore

$$N_\theta=\frac{4\pi V\nu^3}{3c^3}(1-\cos\theta)+\frac{S'\nu^2}{c^2}\theta+\frac{2L\nu}{c}+\left(\tfrac{1}{8}\cos\theta-\frac{\theta}{2\pi}-\tfrac{1}{4}\right). \tag{17}$$

In general, the last term is neglected. For the eigentones in the directions between θ and $\theta+\delta\theta$ the number is

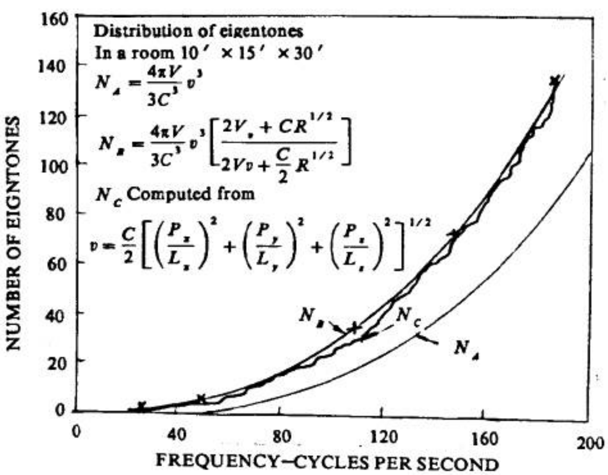
$$\delta N_\theta=\left[\frac{4\pi V\nu^3}{3c^3}\sin\theta+\frac{S'\nu^2}{c^2}-\tfrac{1}{8}\sin\theta-\frac{1}{2\pi}\right]\delta\theta. \tag{18}$$

If only the first term is taken, or what amounts to the same thing, if the eigentone with wave normal parallel to any of the walls be given a weight $\frac{1}{2}$ and the eigentone with wave normal perpendicular to any of them be given a weight $\frac{1}{4}$, Eq. (17) will give

$$N_\theta=(4\pi V\nu^3/3c^3)(1-\cos\theta). \tag{19}$$

This suggests a random distribution of the eigentones so far as the direction is concerned. Thus the usual way of assuming diffused waves in a room is justified provided the frequency is so high that the secondary terms are negligible or particular weights are given to the waves parallel to the walls or the edges, according to the above manner. In such a case the number of eigentones with frequencies between ν and $\nu+\delta\nu$ and in direction between θ and $\theta+\delta\theta$ will be given by

$$\delta^2N=(4\pi V\nu^2/c^3)\sin\theta\,\delta\nu\,\delta\theta. \tag{20}$$



Bolt's diagrams with x's computed from Eq (2) of the present paper

Analysis of Sound Decay in Rectangular Rooms*

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Sound decay in a rectangular room, each wall of which is uniform, is analyzed in terms of the damping of normal modes of aerial vibration, as influenced by the acoustical impedance of the boundary surfaces. From two to seven decay terms may be required in the complete decay equation under different circumstances, each term representing a group of the excited modes of vibration having common properties. The analysis is given in terms of r.m.s. pressure measured for specified locations of source and receiver and includes explicitly the relative dimensions of the room. The results of the mathematical analysis are presented in form to allow determination of the complex wall impedance from consideration of the initial and final slopes of the sound decay curve, and conversely. Experimental decay curves confirm both the detailed predictions of the theory and the sufficiency of the wall impedance as a unique characterization of the influence of the wall on the behavior of sound in the room.

INTRODUCTION

THIS paper presents a new analysis of reverberation which enables the complete decay curve to be precalculated from a knowledge of the room dimensions and a single invariant property of the wall surfaces. The theory is at present restricted, by lack of an applicable diffraction analysis, to the case in which each of the six bounding surfaces is uniform. The theory is also restricted to consider only rectangular rooms, although its extension to other simple shapes is straightforward. To a considerable degree the theory represents the implementation of the procedure described in a discussion of the general problem in a previous paper[1] and in the introductory remarks for this symposium. Briefly stated, the root-mean-square sound pressure at the measuring microphone is, at every instant during decay, considered as the summation of the contributions from each excited normal mode of vibration. In order to clarify the logical basis for the complete solution, we may recognize four subsidiary problems which can be discussed independently. These are:

I. Subdivision of the large number of excited modes of vibration into a manageable number of subgroups having common properties;

II. Computation of the number of modes of vibration in each subgroup;

III. Calculation of the weighting factor by which each mode of vibration enters into the final summation;

IV. Analysis of the decay rate for each mode of vibration in a subgroup in terms of some invariant property of the wall surfaces and the room geometry.

After consideration of these problems the complete decay equation can be assembled. It will contain one decay term for each subgroup (from I) and each term will contain the number of modes of vibration (from II), weighting factors (from III), and an exponential decay factor (from IV).

I. SUBDIVISION OF EXCITED MODES OF VIBRATION

Subdivision of the excited modes of vibration into groups is, basically, an analysis of the state of diffusion of sound energy in the room. Such an analysis can be made conveniently by utilizing the concept of frequency space,[2] an example of which is shown in Fig. 1. This diagram[3] represents the positive octant of a space lattice whose intersection points represent the normal frequencies of our reverberation chamber.[4] The length of a line joining the origin and any one of the lattice intersection points is equal to one of the normal or "proper" frequencies of the

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[1] F. V. Hunt, J. Acous. Soc. Am. **10**, 216 (1939).

[2] P. M. Morse, *Vibrations and Sound* (McGraw-Hill, New York, 1936), Chapter VIII.

[3] We are indebted to Mr. J. A. Pierce for preparing this drawing.

[4] This is the "constant temperature room" referred to in the *Collected Papers of W. C. Sabine*. The shape of the enclosure has recently been altered to that of a rectangular parallelepiped (20′×14′×8′) by closing off the vaulted ceiling with a 6″ concrete slab.

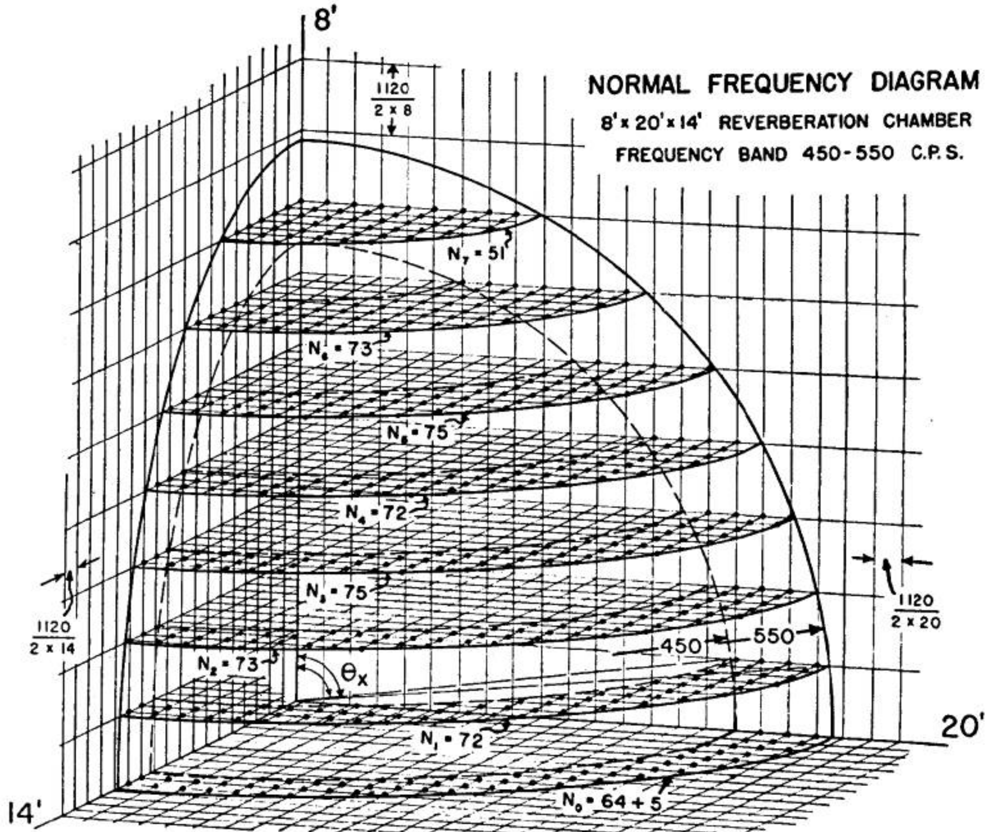

FIG. 1. Normal frequency diagram, drawn to scale for 20′×14′×8′ reverberation chamber. Most of the vertical lines are omitted to avoid confusion.

chamber; the angles which this line makes with the three principal axes are the angles between the three adjacent walls of the chamber and the direction of propagation of the plane waves constituting the standing wave, or normal mode of vibration, in question. Inasmuch as every condition of excitation of the chamber can be synthesized by combining the normal modes of vibration in proper amplitude and phase, it follows that the normal frequency diagram is sufficient for a complete study of sound diffusion in enclosed spaces. In passing, it may be remarked that the use of nonparallel or irregular boundaries does not impair the validity of the preceding statement, the principal effect of the irregularity being a distortion of the simple distribution of normal frequency points.

When a sound source having components at all frequencies within a certain band is placed in the chamber, it will excite all of those modes of vibration characterized by points in frequency space lying between two spherical shells having radii corresponding to the frequency band limits. This is illustrated in Fig. 1. It is immediately apparent that we do not have the uniform distribution-in-angle assumed in the classical theory. There is, instead, a step-wise distribution, some angles of incidence not being represented at all. For a given mid-band frequency the angular distribution becomes more uniform as the exciting band-width is increased, but there is always a discontinuity at zero and near ninety degrees.

The operation of dividing the large number of excited modes into a few groups implies the assumption that the members of the groups so formed will have common properties. The validity of this assumption will be established by subsequent computation of the decay constants for individual modes of vibration. We anticipate these results in proposing the following typical cases of appropriate subdivision. Two general distributions of absorbing material are considered, the first of which is designated as:

Case A. High absorption at one wall, low absorption at remaining walls

For the simplest treatment of this case the excited modes can be divided into two groups. One group contains the modes of vibration comprising waves at grazing incidence on the absorbing wall, while the other group includes all the modes of vibration at nongrazing incidence.[5] Fig. 1 is directly applicable if the absorbing material is placed on the wall perpendicular to the vertical axis of the drawing, corresponding to floor or ceiling coverage in our chamber. The first group then comprises the normal frequency points lying on the lowest (zero order) annular layer, and the other group contains all the rest. If one wall is very highly absorbing it may be desirable to use three groups, the first comprising the modes of vibration at grazing incidence as before, the second comprising the modes of vibration represented by points lying on the second annular layer (first-order modes having one nodal surface parallel to the highly absorbing wall) and the third including all the remaining modes of vibration. A slight further improvement in accuracy may be made by separating out from the zero layer the two small groups representing wave directions grazing at both the highly absorbing wall and one of the others. This makes a total of five groups which are, in order of increasing importance, the two representing grazing incidence on two wall pairs, one representing nearly grazing incidence on the sample, one representing grazing incidence on the sample, and the dominant group representing nongrazing incidence on the sample. Further subdivision could be made if necessary but the added complexity is seldom justified, and for most practical purposes the last two groups named are sufficient.

Case B. Similar absorption on all walls

The most common example of the acoustical situation in which all walls have comparable absorption is the untreated room or bare reverberation chamber, although some broadcasting studios and "dead" rooms are also typical. If the absorption is moderately low only four subgroups of the excited modes will need to be formed. Referring again to Fig. 1, three groups will contain the normal frequency points lying on the coordinate planes and representing grazing incidence on the three walls, respectively. The fourth group will contain the remaining points representing nongrazing incidence on all walls. An improvement in accuracy can be made by forming three additional groups containing the points on the coordinate axes representing the waves which are at grazing incidence on two walls. If the absorption is very high, or if still further improvement in accuracy is required, three more groups may be formed to include the first-order points lying on planes parallel to the coordinate planes and representing nearly grazing incidence on the three walls. Divisions into four or ten groups for this case correspond in approximation to the use of two or five groups, respectively, for Case A, the added complexity arising from the more complicated distribution of absorbing material.

II. Computation of the Number of Modes of Vibration in Each Group

In computing the number of modes of vibration in each group we follow the same procedure outlined in a previous paper.[6] Thus for Case A considered above we may write, for the number

[5] Throughout this paper the term "grazing incidence" will be used to refer to the modes of vibration characterized by normal frequency points lying in the lowest layer of the normal frequency diagram, even though the diagram may be so distorted by boundary absorption (see below) that the "grazing angle" is significantly greater than zero degrees.

[6] D. Y. Maa, J. Acous. Soc. Am. 10, 235 (1939).

of normal frequency points on the zero order annular layer,

$$N_{0_x} = 2\pi f \Delta f l_y l_z / c^2 - (l_y + l_z)\Delta f / c + 2l_y \Delta f / c + 2l_z \Delta f / c, \quad (1)$$

in which c is the velocity of sound, l_x, l_y, and l_z are the three dimensions of the chamber corresponding to the 8′, 14′, and 20′ axes, respectively, in Fig. 1, f is the mid-band frequency, and Δf is the band width of the exciting sound. The second, third, and fourth terms of (1) may be combined but, as written, the second term corrects the first term to make the sum of the first two yield the correct count of the number of normal frequency points lying on the ground plane exclusive of those on the coordinate axes. The last two terms add the points lying on the axes, or, for the five-group division, they count the two minor groups representing the modes whose wave directions are grazing at both the absorbing wall and one other wall.

The normal frequency count for the first annular layer above the y, z plane is given by

$$N_{1_x} = 2\pi f \Delta f l_y l_z / c^2 + \frac{l_y + l_z}{c} f \Delta f \left[f^2 - \left(\frac{c}{2l_x} \right)^2 \right]^{-\frac{1}{2}}. \quad (2)$$

More generally, we may include (1) and (2) in the following expression:

$$N_{n_x} = 2\pi f \Delta f l_y l_z / c^2 + \frac{l_y + l_z}{c} f \Delta f \left[f^2 - \left(\frac{n_x c}{2l_x} \right)^2 \right]^{-\frac{1}{2}}, \quad (3)$$

where n_x is the ordinal number designating the annular layer for which N_{n_x} is computed. n_x may also be identified as one of the three numbers (n_x, n_y, n_z) which designate a particular mode of vibration and which count, for the stationary wave system, the number of nodal surfaces perpendicular to the x, y, and z axes, respectively. The population of the group of modes whose wave directions are neither grazing nor "nearly grazing" at the absorbing wall may be computed by summing Eq. (3) from $n_x = 2$ to its maximum value, given by the first integer less than $(2f + \Delta f) l_x / c$. If the band width Δf is equal to or greater than the frequency space unit $c/2l_x$ the number $N_{n_{\max}}$ for the top layer will not be given correctly by (3), but must be determined by a separate computation (see, for example, N_7 of Fig. 1).

In considering Case B the preceding equations can be used with slight modifications. Thus the population of the three grazing incidence groups for the four-group division may be counted by modifying Eq. (1) to

$$N_0 = 2\pi f \Delta f l_y l_z / c^2 - (l_y - l_z)\Delta f / c \quad (4)$$

and advancing the subscripts x, y, z cyclically to generate the three equations. In this expression the normal frequency points lying on the z axis are counted as on the z, y plane, those on the x axis as on the x, z plane, and those on the y axis as on the y, x plane. In the more accurate seven-group division the points lying on the axes are counted in three separate expressions formed by cyclic advance of the subscripts in

$$N_{0_{xy}} = 2l_z \Delta f / c. \quad (5)$$

When this is used the sign before l_z in the second term of (4) should be reversed, in order to exclude the points counted by (5), and the modified equation designated as (4a) for reference. The population of the remaining nongrazing group of either the four- or seven-group division is determined, as before, by the summation of Eq. (3), except that the lower limit of n_x is to be taken as 1 and the algebraic sign of the second term is to be reversed to exclude the points already counted in (4) and (5). Alternatively, the total count for the nongrazing group may be computed directly from the expression

$$N_n = \frac{4\pi V f^2 \Delta f}{c^3} \left[1 - \frac{Sc}{8Vf} + \frac{Lc^2}{8\pi V f^2} \right], \quad (6)$$

where V is the volume, S is the total area, and L is the sum of the lengths of the three sides of the chamber.

In general, the ratio of the number of modes of vibration at grazing incidence to the number of modes of vibration not at grazing incidence increases almost linearly with the ratio of the wavelength to the smallest dimension of the room. It follows that, in ordinary rooms, the relative number of modes which involve grazing incidence will be small in the middle frequency range. However, the importance of these is enhanced by the fact that they usually decay in amplitude slowly and ultimately dominate in the residual sound. For example, in a room or office having

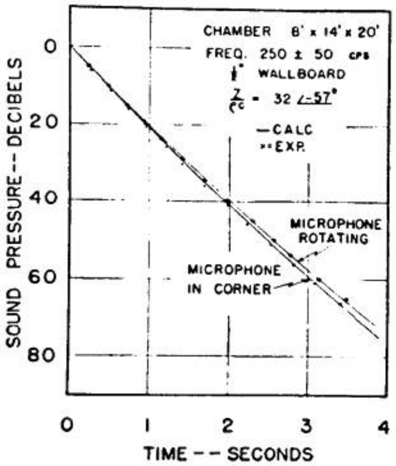

FIG. 2. Experimental confirmation of the predicted differences between two sound decay curves measured under different conditions of microphone placement.

a 9′ ceiling, *all* stationary wave systems for frequencies below 125 cycles represent grazing incidence on the ceiling, and the relative number of such grazing modes of vibration is significant throughout the middle audiofrequency range.

III. Weighting Factors

The weighting factor applying to each mode of vibration is determined by consideration of the type of electrical summation provided by the measuring apparatus, the location of the measuring microphone in the enclosure, and the steady-state pressure amplitude to which each mode of vibration is excited. We assume that our measuring apparatus is responsive to the sum of the squares of the amplitudes of all pressure components observable at the microphone. This assumption makes it possible to ignore the distribution of time phase among the various components and can be satisfactorily approximated in practice by using a diode rectifier at "small-signal" levels. It has been shown[1,2] that the steady-state pressure amplitude at a point x, y, z arising from excitation by a point source of angular frequency ω located at x_0, y_0, z_0 is given by

$$P = \text{constant} \frac{\omega\sigma_n\psi_n(x, y, z)\psi_n(x_0, y_0, z_0)}{2\omega_n k_n + j(\omega^2 - \omega_n^2 + k_n^2)}, \quad (7)$$

in which ω_n is 2π times the normal frequency of the nth mode of vibration. If we assume that all frequencies within a band are present in the exciting source the square of the pressure amplitude of each excited mode will take on the approximate resonance value given by

$$p_0^2 = \text{constant } \sigma_n^2\psi_n^2(x, y, z)\psi_n^2(x_0, y_0, z_0)/k_n^2, \quad (8)$$

where k_n is identified as the pressure decay constant (nepers per second), σ_n as a normalization constant discussed below, and ψ_n as the characteristic function describing the space distribution of pressure amplitude for the nth mode of vibration. We may now introduce the experimental circumstances involving placement of source and microphone. If either is located in a corner of the chamber the corresponding ψ_n function has the approximate value unity. Alternatively if either source or microphone is moved about in the chamber (or if several microphone outputs are commutated) to give a *space* average we must also average the corresponding ψ_n function throughout the volume of the room.

However, the normalization constant σ_n is defined in terms of the space average of ψ_n^2 so that Eq. (8) may readily be simplified to yield the following weighting factors: For both source and microphone fixed in corners of the chamber, use

$$p_n^2 \propto \sigma_n^2/k_n^2; \quad (9a)$$

for either source or microphone fixed in a corner and the other moved about to occupy an "average position," use

$$p_n^2 \propto \sigma_n/k_n^2; \quad (9b)$$

for both source and microphone moved about to occupy "average positions," use

$$p_n^2 \propto 1/k_n^2. \quad (9c)$$

The explicit evaluation of σ_n leads approximately to one of the three values 1, $\frac{1}{2}$, or $\frac{1}{4}$ according to whether the normal mode in question has a wave direction at grazing incidence on none, one, or two wall-pairs. The approximation involved here, in (8), and in evaluating ψ_n as unity for a corner location, is that k_n^2 shall be negligible with respect to ω_n^2. This condition is usually satisfied in practical cases.

Experimental confirmation of these weighting factors is exhibited, in Fig. 2, by two decay curves calculated by the five-group analysis (Case A) and differing only in the use of Eqs. (9a) and (9b) to correspond to the two conditions of microphone placement. A previous suggestion[1] indicating the suitability of a corner location for the microphone is thus supported and the variation of the results from those obtained by conventional techniques is rendered predictable.

IV. Computation of Decay Constants for Individual Modes of Vibration

The decay constant k_n is defined as the exponent in the simple expression for the decay of pressure amplitude, $p_n = P_n \exp(-k_n t)$, and it is now our object to compute this constant in terms of room geometry and some invariant property of the wall surfaces. Two general procedures are available for different, but overlapping, ranges of absorption, and will be designated as the "free wave theory" and the "standing wave theory." The free wave theory is especially applicable for low absorption materials distributed as in Case B and, since it represents a logical extension of the classical treatment of reverberation, it will be discussed first.

Free wave theory

When a free progressive plane wave strikes an absorbing wall at the incident angle θ the pressure reflection coefficient is given by

$$p_{\text{ref}}/p_{\text{inc}} = (Z\cos\theta - \rho c)/(Z\cos\theta + \rho c), \quad (10)$$

in which ρc is the specific "radiation resistance" of the medium and Z is the specific normal impedance of the wall surface (i.e., $Z \equiv$ ratio of pressure to normal component of particle velocity at the wall surface). In evaluating an absorption coefficient from (10) Z is usually restricted to real values, but we may use the reflection coefficient itself for the complex values of wall impedance defined in polar form by $Z/\rho c = \mathbf{z} = \gamma e^{i\phi}$. Let us now direct our attention to some particular mode of vibration having a wave direction making angles θ_x, θ_y, and θ_z with the three axes, and hence having these angles of incidence on the walls perpendicular to the corresponding axes (designated as x, y, and z walls). We may deduce that the waves composing the stationary system will traverse the path length $l_x/\cos\theta_x$ between successive reflections at the x walls, the free path $l_y/\cos\theta_y$ between reflections at the y walls, and so on. It follows that t seconds after sound decay commences the x walls will account for a pressure amplitude reduction factor given by

$$\left[\frac{\mathbf{z}\cos\theta_x - 1}{\mathbf{z}\cos\theta_x + 1}\right]^{\frac{ct\cos\theta_x}{l_x}} = \left[\frac{\mathbf{z}q-1}{\mathbf{z}q+1}\right]^{qct/l}, \quad (11)$$

and that similar reduction factors will be contributed by the other walls. The following approximate identity is then established by expansions in power series, the coefficients of which coincide through the term in $\mathbf{z}^{-2}$;

$$\left[\frac{\mathbf{z}q-1}{\mathbf{z}q+1}\right]^{q} \doteq \frac{\mathbf{z}-1}{\mathbf{z}+1} \equiv R, \text{ when } \mathbf{z}q > 1. \quad (12)$$

The important conclusion may be drawn that *every mode of vibration satisfying* $\mathbf{z}\cos\theta > 1$ *will have the same decay rate as that assigned to a mode whose waves are normally incident on the wall, provided* $\mathbf{z}$ *itself is not too small.* In practice the effect of the restriction $\mathbf{z}\cos\theta > 1$ is to exclude only those modes representing grazing, or nearly grazing, incidence at the wall, and it is this feature which provides the basis for the grouping of normal modes described above in Section I, Case B. Assembling the pressure reduction factors contributed by the three walls we have, for the nth nongrazing mode of vibration,

$$p_n(t)/p_n(0) = R_x^{ct/l_x} \cdot R_y^{ct/l_y} \cdot R_z^{ct/l_z}$$
$$= \exp\frac{ct}{2V}(S_x \log R_x + S_y \log R_y + S_z \log R_z), \quad (13)$$

where $S_x = 2l_y l_z, \cdots, V = l_x l_y l_z$. The similarity between (13) and the classical reverberation formula is striking. It should be emphasized, however, that the normal incidence reflection coefficient is *not* to be used in connection with a *mean* free path, but with l_x, the free path associated with a normally incident wave. The actual change in free path occurring as the direction of the wave is altered is just compensated by the change in reflection coefficient with angle of incidence (except near grazing incidence).

Before identifying the exponent in (13) as a conventional decay constant it is necessary to resolve the apparent dilemma presented in the complex value assumed by the reflection coefficient when the wall impedance is complex. The dilemma can be resolved by writing for $R_x, \cdots$ in (13) its absolute value, and imposing a limitation on the applicability of the free wave theory. Replacement of the complex R by its magnitude rejects, in effect, the information concerning phase shift on reflection at the wall, and hence ignores any shift of the pressure loop of the standing wave system with respect to the wall. Application of the free wave theory should, therefore, be restricted to cases in which this displacement of the pressure loop is small. Morse[7] shows a series of pressure distribution curves from which one may infer that the restriction $|Z/\rho c| \geqslant 15$ is sufficient for this purpose. The same restriction is also sufficient to assure the identity expressed in (12). We may now rewrite (13) as follows:

$$p_n(t)/p_n(0) = \exp(-k_n t)$$

$$= \exp \frac{-c}{2}\left[\frac{\delta_{n_x}}{l_x}+\frac{\delta_{n_y}}{l_y}+\frac{\delta_{n_z}}{l_z}\right]t, \quad (14)$$

where we have used the abbreviation

$$\delta_{n_x} = \log |R_x|^2.$$

(We have suggested that this quantity δ, defined through (14) in terms of k, be called the *damping coefficient*, reserving for k the term *decay constant* in analogy with the *attenuation constant* and *transfer constant* of network theory. The *decay rate* may then designate k expressed in db/sec.)

The modes of vibration at grazing incidence excluded by the restriction on Eq. (12) must now be considered. The erroneous prediction from Eq. (10) for $\theta = 90°$ is avoided by borrowing an evaluation of the ratio of grazing incidence to normal incidence damping coefficients from the standing wave analysis described below. Assuming that these results are available, we may define $\mu_x = \delta_{0x}/\delta_{n_x}$ and introduce the factor μ in the appropriate terms of (14) to form the decay constant for those modes involving grazing incidence at either one or two wall-pairs.

[7] P. M. Morse, J. Acous. Soc. Am. **11**, 55 (1939). We are indebted to Professor Morse for the opportunity to examine and discuss this manuscript in advance of publication.

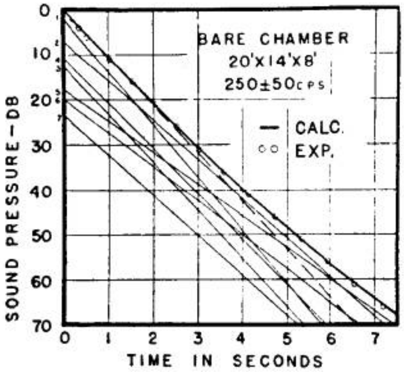

FIG. 3. Comparison between an experimental decay curve and a calculated curve representing the sum of seven decay components. α_n is taken as 0.0096 and the seven components are numbered as they appear in the following complete decay equation:

$$p^2/p_0^2 = 0.633e^{-t/0.37} + 0.201e^{-t/0.499} + 0.0507e^{-t/0.412} + 0.0828e^{-t/0.433} + 0.0172e^{-t/0.617} + 0.0105e^{-t/0.576} + 0.00433e^{-t/0.49}.$$

The straight dashed line is shown for comparison to exhibit the curvature.

We may now, at last, assemble the complete equation of decay. Adopting the seven-group division from I, Case B, we select group populations from (4a), (5), and (6), weighting factors from (9b) and decay constants from (14), to yield finally

$$p^2(t)/p^2(0) \propto \frac{N_n(1)}{k_n^2}\exp(-2k_n t)$$

$$+\sum_{x,y,z} \frac{N_{0x}(1/2)}{\left[k_n-(1-\mu_x)\dfrac{\delta_x}{l_x}\dfrac{c}{4}\right]^2} \times \exp 2\left[-k_n+(1-\mu_x)\frac{\delta_x}{l_x}\frac{c}{4}\right]t$$

$$+\sum_{x,y,z} \frac{N_{0xy}(1/4)}{\left[k_n-(1-\mu_x)\dfrac{c}{4}\dfrac{\delta_x}{l_x}-(1-\mu_y)\dfrac{c}{4}\dfrac{\delta_y}{l_y}\right]^2} \times \exp 2\left[-k_n+(1-\mu_x)\frac{c}{4}\frac{\delta_x}{l_x}+(1-\mu_y)\frac{c}{4}\frac{\delta_y}{l_y}\right]t. \quad (15)$$

The summation symbols indicate that three

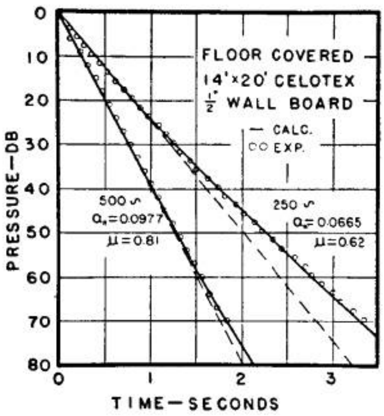

FIG. 4. Calculated and experimental sound decay curves for two frequencies. The values of α_n and μ are computed from the impedance of the material.

terms are to be formed by cyclic advance of the subscripts. The expression can be normalized by dividing through by the sum of the seven coefficients of the exponentials, whence the pressure ratio reduces properly to unity at $t=0$ and the complete decay may be calculated. In Fig. 3 this analysis is applied to precalculate the decay curve for our bare reverberation chamber. The value of the normal incidence absorption, $\alpha_n=1-|R|^2$, is determined by successive trials and the factor μ is taken as 0.5 from the analysis below. Although the necessity of successive trials to determine α_n appears cumbersome the quantity is principally determined by the initial slope and two trials are usually sufficient to fix α_n within one percent of its value. As an alternative procedure the reflection factors, and hence the damping coefficients, may be computed directly if the complex impedance of the wall surfaces is known. This is illustrated in Fig. 4 in which the calculation of two decay curves is based on values of α_n computed from the impedance of the sample. The impedance and the ratio μ were actually deduced from the reverberation data by a method described below, but any other scheme (e.g., model chamber measurements[1]) yielding values for α_n and μ would be equally suitable. Fig. 5 presents an example of this analysis applied to a decay curve obtained in another reverberation chamber. The agreement between calculation and experiment is especially gratifying since this experimental decay curve was published[8] in 1936 as a typical example for which ". . . this curvature should be amenable to appropriate mathematical analysis."

Standing wave theory

The conception and properties of the normal modes of aerial vibration in a room have been used freely in the preceding sections of this paper. We shall now give a brief summary of the mathematical basis for these properties and describe a method of computing the damping coefficients from the boundary conditions.

The initial assumption is the validity of the small-signal wave equation

$$c^2\nabla^2 p=\partial^2 p/\partial t^2. \tag{16}$$

Choosing the origin of coordinates at a corner of the room, we may verify, by direct substitution, that

$$p=P_0\cosh\left[(k_x-j\omega_x)\frac{x}{c}+\psi_x\right]\times\cosh\left[(k_y-j\omega_y)\frac{y}{c}+\psi_y\right]\times\cosh\left[(k_z-j\omega_z)\frac{z}{c}+\psi_z\right]\exp\,(j\omega_n-k_n)t \tag{17}$$

will be a solution of the wave equation and will describe a normal mode of vibration provided the "allowed," or normal, angular frequency ω_n and the decay constant k_n are related to the other k's and ω's by

$$(j\omega_n-k_n)^2=(k_x-j\omega_x)^2+(k_y-j\omega_y)^2+(k_z-j\omega_z)^2. \tag{18}$$

and provided the boundary conditions are satisfied. The real and imaginary parts of (18) provide the following relations

$$\omega_n^2-k_n^2=\omega_x^2+\omega_y^2+\omega_z^2-(k_x^2+k_y^2+k_z^2) \tag{19}$$

$$k_n=k_x\omega_x/\omega_n+k_y\omega_y/\omega_n+k_z\omega_z/\omega_n. \tag{20}$$

We observe that if $k_x=k_y=k_z=0$, then $k_n=0$, the wave motion is undamped, and the "undamped normal frequency" is given by

$$\omega_n^2=\omega_x^2+\omega_y^2+\omega_z^2. \tag{21}$$

[8] F. V. Hunt, J. Acous. Soc. Am. 8, 34 (1936).

This quadratic equation is interpreted geometrically by the normal frequency diagram and we may note that (21) is only slightly affected if

$$k_n \neq 0, \quad (k_x^2+k_y^2+k_z^2) \ll \omega_n^2. \qquad (22)$$

As mentioned before, this condition is nearly always satisfied in practice and we conclude that the effect of boundary absorption is merely to distort the normal frequency diagram without destroying its significance or its general configuration.

A very important additive property of this solution may now be recognized. The three terms comprising k_n through Eq. (20) are each functions of the absorption at one wall-pair alone, except for the factor ω_n. But, if the approximation condition (22) is satisfied, ω_n is not appreciably affected by any of the k's and it follows that *the contribution to the total decay constant arising from absorption at one wall-pair is independent of the absorption at the remaining walls.* The result has already been used in writing the defining Eq. (14) with the tacit implication that δ_x, δ_y, and δ_z are independent, and we may proceed, without loss of generality, to consider in detail the case in which absorption is confined to one wall-pair.

Assuming that absorption occurs only at the x walls ($k_y=k_z=0$) the boundary conditions are introduced by computing the x component of particle velocity, u, from (17) and forming the ratio (exact),

$$(p/u)_x=\rho c\frac{k_n-j\omega_n}{k_x-j\omega_x}\coth\left[(k_x-j\omega_x)\frac{x}{c}+\psi_x\right]. \qquad (23)$$

Substituting two complex x-wall impedances for $(p/u)_{0,\,l_x}$ provides four equations to determine k_x, ω_x and the real and imaginary parts of ψ_x. The effects of absorption at the two opposite walls are not, unfortunately, additive when both walls are highly absorbing and similar. Since Morse[7] has discussed these interaction phenomena in some detail we shall confine the remainder of this discussion to the case in which only one wall is absorbing (i.e., $\psi_x=0$), remarking, however, that the effect of such slight absorption as that of an opposite untreated surface can be added as a correction to the damping coefficient without significant error.

The transcendental equation which must be solved is, therefore,

$$(k_x-j\omega_x)\gamma \exp j\phi = (k_n-j\omega_n)\coth\left[(k_x-j\omega_x)l_x/c\right]. \qquad (24)$$

It is at this point that the present analysis becomes restricted to the case of uniform coverage of each wall surface since it is implicitly assumed, in writing (24), that the wall impedance $Z=\rho c\gamma e^{j\phi}$ is not a function of y or z. It is also assumed in (24) that the normal impedance of the wall is independent of the angle of incidence; that is, the ratio of pressure to the normal component of particle velocity at the wall is assumed to be independent of the direction of the particle velocity in the incident wave. The plausibility of this assumption has been discussed by Monna[9] and by Morse.[7]

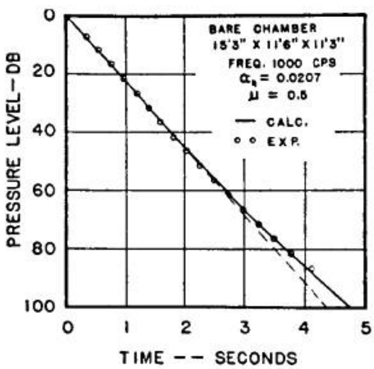

FIG. 5. Theoretical decay curve based on the present analysis and in good agreement with experimental observations reported in 1936.

In order to provide a basis for an experimental study of the dependence of Z on the angle of incidence a solution of (24) must be obtained without invoking the approximation $k_n^2 \ll \omega_n^2$, so that independent measures of k_n and ω_n for a single mode of vibration may yield both magnitude and phase angle of the wall impedance. Such a solution can be obtained from (24) and (18) by a tedious process of power series manipulations. The results for modes of vibration at grazing and nongrazing incidence are, for $\omega l_x/\gamma c \leq 1$:

[9] A. F. Monna, Physica **5**, 129 (1938).