電磁場與電磁波詳解

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前　　言

研習理工的同學，都有一種認識，那就是：一本書的習題往往是該書的精華所在，藉着習題的印證，才能對書中的原理原則澈底的吸收與瞭解。

有鑒於此，曉園出版社特地聘請了許多在本科上具有相當研究與成就的人士，精心出版了一系列的題解叢書，為各該科目的研習，作導介紹與鋪路的工作。

一個問題的解答方法，常因思惟的角度而異。曉園題解叢書，毫疑問的都是經過一番精徵的思考與分析而得。其目的在提供對各該目研讀時的參考與比較；而對於一般的自修者，則有啓發與提示的用。希望讀者能藉着這一系列題解叢書的幫助，而在本身的學問進程上有更上層樓的成就。

期　限　表

请于下列日期前将书还回

电磁场与电磁波详解

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電磁場與電磁波詳解

（目　　錄）

第一章　向　　量

1-1　證明出兩向量　$\boldsymbol{A}=9\boldsymbol{i}+\boldsymbol{j}-6\boldsymbol{k}$ 與 $\boldsymbol{B}=4\boldsymbol{i}-6\boldsymbol{j}+5\boldsymbol{k}$ 是互相垂直的。

解　$\boldsymbol{A}\cdot\boldsymbol{B}=(9\boldsymbol{i}+\boldsymbol{j}-6\boldsymbol{k})\cdot(4\boldsymbol{i}-6\boldsymbol{j}+5\boldsymbol{k})$

$=36-6-30$

$=0$

$\therefore \boldsymbol{A}\perp\boldsymbol{B}$

1-2　證明出兩向量　$\boldsymbol{A}=2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ 與 $\boldsymbol{B}=\boldsymbol{i}-6\boldsymbol{j}+\boldsymbol{k}$ 所夾的角大小爲 $130.5°$。

解　$\boldsymbol{A}\cdot\boldsymbol{B}=(2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})\cdot(\boldsymbol{i}-6\boldsymbol{j}+\boldsymbol{k})$

$=2-18+1$

$=-15$

$|\boldsymbol{A}|=\sqrt{2^2+3^2+1}$

$=\sqrt{14}$

$|\boldsymbol{B}|=\sqrt{1^2+6^2+1^2}$

$=\sqrt{38}$

$$\cos\theta=\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{|A|\,|B|}$$

$$=\frac{-15}{\sqrt{532}}$$

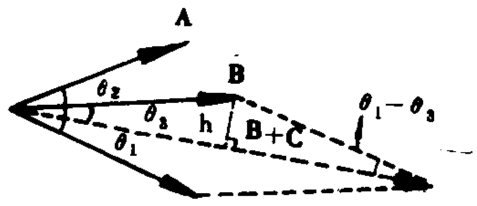
$$\theta=\cos^{-1}\frac{-15}{\sqrt{532}}$$

$=130.5°$

1-3　向量　$\boldsymbol{A}$，$\boldsymbol{B}$，$\boldsymbol{C}$ 共平面，用圖解證明

$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}$

解



$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=|\boldsymbol{A}|\cdot|\boldsymbol{B}+\boldsymbol{C}|\cdot\cos(\theta_2+\theta_3)$

$|\boldsymbol{B}+\boldsymbol{C}|=|\boldsymbol{B}|\cos\theta_3+|\boldsymbol{C}|\cos(\theta_1-\theta_3)$

$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=|\boldsymbol{A}|[|\boldsymbol{B}|\cos\theta_3+|\boldsymbol{C}|\cos(\theta_1-\theta_3)]\cos(\theta_2+\theta_3)$

$=|\boldsymbol{A}||\boldsymbol{B}|[\cos(\theta_3)\cos(\theta_2+\theta_3)]+|\boldsymbol{A}||\boldsymbol{C}|\cos(\theta_1-\theta_3)\cos(\theta_2+\theta_3)$

$=|\boldsymbol{A}||\boldsymbol{B}|[\cos\theta_3\cos(\theta_2+\theta_3)+\sin\theta_3\sin(\theta_2+\theta_3)-\sin\theta_3\sin(\theta_2+\theta_3)]+|\boldsymbol{A}||\boldsymbol{C}|[\cos(\theta_1-\theta_3)\cos(\theta_2+\theta_3)-\sin(\theta_1-\theta_3)\sin(\theta_2+\theta_3)+\sin(\theta_1-\theta_3)\sin(\theta_2+\theta_3)]$

$=|\boldsymbol{A}||\boldsymbol{B}|\cos\theta_2-|\boldsymbol{A}||\boldsymbol{B}|\sin\theta_3\sin(\theta_2+\theta_3)+|\boldsymbol{A}||\boldsymbol{C}|\cos(\theta_1+\theta_2)+|\boldsymbol{A}||\boldsymbol{C}|\sin(\theta_1-\theta_3)\sin(\theta_2+\theta_3)$

$=|\boldsymbol{A}||\boldsymbol{B}|\cos\theta_2+|\boldsymbol{A}||\boldsymbol{C}|\cos(\theta_1+\theta_2)-|\boldsymbol{A}||\boldsymbol{B}|\sin\theta_3\sin(\theta_2+\theta_3)+|\boldsymbol{A}||\boldsymbol{C}|\sin(\theta_1-\theta_3)\sin(\theta_2+\theta_3)$

由圖知

$|\boldsymbol{B}|\sin\theta_3=|\boldsymbol{C}|\sin(\theta_1-\theta_3)$

$\therefore\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=|\boldsymbol{A}||\boldsymbol{B}|\cos\theta_2+|\boldsymbol{A}||\boldsymbol{C}|\cos(\theta_1+\theta_2)$

$=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}$

1-4 假設 $\boldsymbol{A}$ 和 $\boldsymbol{B}$ 爲平行四邊形之兩鄰邊，$\boldsymbol{C}=\boldsymbol{A}+\boldsymbol{B}$ 及 $\boldsymbol{D}=\boldsymbol{A}-\boldsymbol{B}$，爲對角線，而 θ 爲 $\boldsymbol{A}$ 與 $\boldsymbol{B}$ 的夾角。證明 $(C^2+D^2)=2(A^2+B^2)$ 而且 $(C^2-D^2)=4AB\cos\theta$

解 $\boldsymbol{C}=\boldsymbol{A}+\boldsymbol{B}$

$\boldsymbol{D}=\boldsymbol{A}-\boldsymbol{B}$

$C^2=(\boldsymbol{A}+\boldsymbol{B})\cdot(\boldsymbol{A}+\boldsymbol{B})$

$=A^2+B^2+2\boldsymbol{A}\cdot\boldsymbol{B}$

$D^2=(\boldsymbol{A}-\boldsymbol{B})\cdot(\boldsymbol{A}-\boldsymbol{B})$

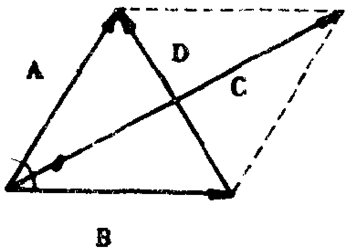
$=A^2+B^2-2\boldsymbol{A}\cdot\boldsymbol{B}$

$C^2+D^2=2(A^2+B^2)$

$C^2-D^2=4\boldsymbol{A}\cdot\boldsymbol{B}$

$=4|\boldsymbol{A}||\boldsymbol{B}|\cos\theta$

$=4AB\cos\theta$



1-5　假設a與b爲xy平面上的兩單位向量。而再設α爲a與x軸之夾角，β爲b與x軸之夾角，因此

$$a = \cos\alpha\, i + \sin\alpha\, j \text{ 及 } b = \cos\beta\, i + \sin\beta\, j$$

試證兩個角之和與差之正弦及餘弦三角關係式是由$a \cdot b$及$a \times b$之解釋得來的。

解　$a \cdot b = \cos(\alpha - \beta)$

但　$a \cdot b = (\cos\alpha\, i + \sin\alpha\, j) \cdot (\cos\beta\, i + \sin\beta\, j)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

於是　$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

令　$\beta' = -\beta$

則　$\cos(\alpha + \beta') = \cos\alpha\cos\beta' - \sin\alpha\sin\beta'$

同理　$a \times b = -\sin(\alpha - \beta)\, k$

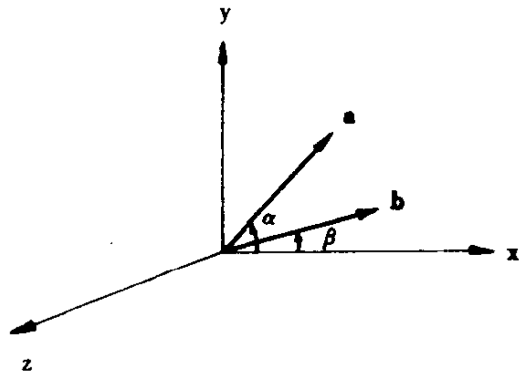
而　$a \times b = \begin{vmatrix} i & j & k \\ \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$

$= k(\cos\alpha\sin\beta - \sin\alpha\cos\beta)$

對照得　$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

令　$\beta' = -\beta$

則　$\sin(\alpha + \beta') = \sin\alpha\cos\beta' + \cos\alpha\sin\beta'$



1-6　試證$(A \times B) \cdot C$這個量爲平行六面體之體積，它的三個邊爲A，B，C，並證明$(A \times B) \cdot C = A \cdot (B \times C)$

解　平行六面體體積

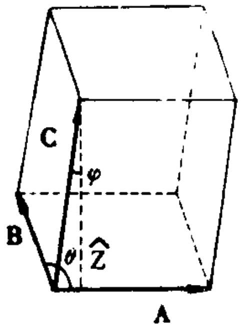
$=$（底面積）$\times$（高）

$= AB\sin\theta \times C\cos\varphi$

$(A \times B) \cdot C = AB\sin\theta\, Z \cdot C$

$= AB\sin\theta\, C\cos\varphi$

$= ABC\sin\theta\cos\varphi$



$\therefore\quad (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}$ 爲平行六面體之體積

同理 $\quad (\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A}$ 亦爲此平行六面體之體積

$\therefore\quad (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = (\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A}$

$= \boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$

1-7 試證 $\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\times\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{C}$

解 $\boldsymbol{A} = A_x\boldsymbol{i}+A_y\boldsymbol{j}+A_z\boldsymbol{k}$

$\boldsymbol{B} = B_x\boldsymbol{i}+B_y\boldsymbol{j}+B_z\boldsymbol{k}$

$\boldsymbol{C} = C_x\boldsymbol{i}+C_y\boldsymbol{j}+C_z\boldsymbol{k}$

$$\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C}) = \begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x+C_x & B_y+C_y & B_z+C_z \end{matrix}$$

$$= (A_yB_z+A_yC_z-A_zB_y-A_zC_y)\boldsymbol{i}+(A_zB_x+A_zC_x-A_xB_z-A_xC_z)\boldsymbol{j}+(A_xB_y+A_xC_y-A_yB_x-A_yC_x)\boldsymbol{k}$$

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z-A_zB_y)\boldsymbol{i}+(A_zB_x-A_xB_z)\boldsymbol{j}+(A_xB_y-A_yB_x)\boldsymbol{k}$$

$$\boldsymbol{A}\times\boldsymbol{C} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (A_yC_z-A_zC_y)\boldsymbol{i}+(A_zC_x-A_xC_z)\boldsymbol{j}+(A_xC_y-A_yC_x)\boldsymbol{k}$$

$$\boldsymbol{A}\times\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{C} = (A_yB_z+A_yC_z-A_zB_y-A_zC_y)\boldsymbol{i}+(A_zB_x+A_zC_x-A_xB_z-A_xC_z)\boldsymbol{j}+(A_xB_y+A_xC_y-A_yB_x-A_yC_x)\boldsymbol{k}$$

$\therefore \boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\times\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{C}$

1-8 試證 $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$

解

$$\boldsymbol{b}\times\boldsymbol{c}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ b_x&b_y&b_z\\ c_x&c_y&c_z\end{vmatrix}$$

$$=(b_yc_z-b_zc_y)\boldsymbol{i}+(b_zc_x-b_xc_z)\boldsymbol{j}+(b_xc_y-b_yc_x)\boldsymbol{k}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ a_x&a_y&a_z\\ b_yc_z-b_zc_y&b_zc_x-b_xc_z&b_xc_y-b_yc_x\end{vmatrix}$$

$$=(a_yb_xc_y-a_yb_yc_x-a_zb_zc_x+a_zb_xc_z)\boldsymbol{i}+(a_zb_yc_z-a_zb_zc_y-a_xb_xc_y+a_xb_yc_x)\boldsymbol{j}+(a_xb_zc_x-a_xb_xc_z-a_yb_yc_z+a_yb_zc_y)\boldsymbol{k}$$

$$=[b_x(a_yc_y+a_zc_z+a_xc_x)-c_x(a_xb_x+a_yb_y+a_zb_z)]\boldsymbol{i}+[b_y(a_xc_x+a_yc_y+a_zc_z)-c_y(a_xb_x+a_yb_y+a_zb_z)]\boldsymbol{j}+[b_z(a_xc_x+a_yc_y+a_zc_z)-c_z(a_xb_x+a_yb_y+a_zb_z)]\boldsymbol{k}$$

$$=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

1-9 如果 $\boldsymbol{r}\cdot\dfrac{d\boldsymbol{r}}{dt}=0$，試示 r 爲常數

解 $\boldsymbol{r}=r\boldsymbol{r}_1$

$$\frac{d\boldsymbol{r}}{dt}=\boldsymbol{r}_1\frac{dr}{dt}+r\frac{d\boldsymbol{r}_1}{dt}$$

$$\boldsymbol{r}\cdot\left(\frac{d\boldsymbol{r}}{dt}\right)=r\boldsymbol{r}_1\cdot\left(\boldsymbol{r}_1\frac{dr}{dt}+r\frac{d\boldsymbol{r}_1}{dt}\right)$$

$$=r\frac{dr}{dt}+r^2\boldsymbol{r}_1\cdot\frac{d\boldsymbol{r}_1}{dt}$$

$$\because\ \boldsymbol{r}_1\cdot\frac{d\boldsymbol{r}_1}{dt}=0$$

$$\therefore\ \boldsymbol{r}_1\cdot\left(\frac{d\boldsymbol{r}}{dt}\right)=r\frac{dr}{dt}=0$$

$r = 0$ 或 $\frac{dr}{dt} = 0$

$\therefore r$ 爲常數

1-10 由一槍發出一顆子彈，它的速度爲 $500\, m/sec$，而且與水平面成 $30°$。

試求在發射後 t 秒，子彈的位置向量 r 速度 v 和加速度 a 爲何；並繪出軌跡圖，並對某一特定時間 t 在圖中表示這三個向量。

解 $\boldsymbol{a} = -9.8\, m/sec\ \boldsymbol{K}$

$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{a}t$

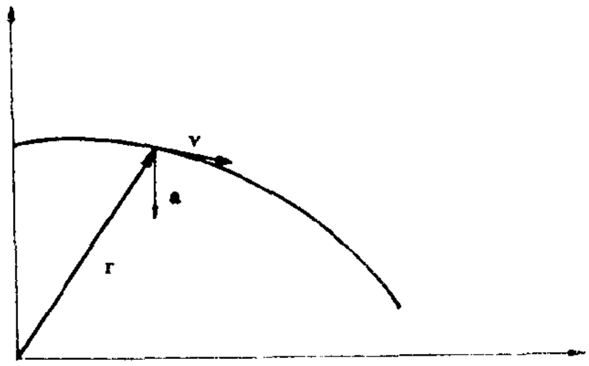
$= 500(\cos 30°\boldsymbol{j} + \sin 30°\boldsymbol{k}) - 9.8t\,\boldsymbol{K}$

$= 250\sqrt{3}\,\boldsymbol{j} + \boldsymbol{K}(250 - 9.8t)$

$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}t^2$

$= \boldsymbol{r}_0 + 500 \cdot (\cos 30°\boldsymbol{j} + \sin 30°\boldsymbol{k})t - \frac{1}{2} \cdot 9.8 \cdot t^2 \boldsymbol{K}$

$= \boldsymbol{r}_0 + 250\sqrt{3}\,t\,\boldsymbol{j} + \boldsymbol{k}(250t - 4.9t^2)$



1-11 如果 $\boldsymbol{r}$ 是由座標原點到任意一點的徑向量，同時 $\boldsymbol{A}$ 爲一常數向量，試證 $\nabla(\boldsymbol{A} \cdot \boldsymbol{r}) = \boldsymbol{A}$

解 $\boldsymbol{A} \cdot \boldsymbol{r} = A_x x + A_y y + A_z z$

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{r}) = \boldsymbol{i}\frac{\partial(\boldsymbol{A} \cdot \boldsymbol{r})}{\partial x} + \boldsymbol{j}\frac{\partial(\boldsymbol{A} \cdot \boldsymbol{r})}{\partial y} + \boldsymbol{k}\frac{\partial(\boldsymbol{A} \cdot \boldsymbol{r})}{\partial z}$$

$$= \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z = \boldsymbol{A}$$

1-12 如圖 1-20 所示，向量 $\boldsymbol{r}$ 是由 $P'(x', y', z')$ 指向點 $P(x, y, z)$

(a)如果 P 點固定而 P' 點可以移動，在這種情況下

試證 $(\frac{1}{r})$ 的梯度爲

$$\nabla'(\frac{1}{r})=\frac{\boldsymbol{r}_1}{r^2}$$

這裡的 $\boldsymbol{r}_1$ 是沿著 r 的單位向量；並證明這是 $(\frac{1}{r})$ 的最大變化率。

(b)反過來，如果 P' 固定而 P 可以移動，試證

$$\nabla(\frac{1}{r})=-\frac{\boldsymbol{r}_1}{r^2}$$

解 (a) $\boldsymbol{r}=(x-x')\boldsymbol{i}+(y-y')\boldsymbol{j}+(z-z')\boldsymbol{k}$

$$\frac{1}{r}=[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}$$

$$=\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$\nabla'(\frac{1}{r})=\frac{\partial}{\partial x'}(\frac{1}{r})\boldsymbol{i}+\frac{\partial}{\partial y'}(\frac{1}{r})\boldsymbol{j}+\frac{\partial}{\partial z'}(\frac{1}{r})\boldsymbol{k}$$

$$=\frac{1}{2}\frac{2(x-x')\boldsymbol{i}+2(y-y')\boldsymbol{j}+2(z-z')\boldsymbol{k}}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

$$=\frac{(x-x')\boldsymbol{i}+(y-y')\boldsymbol{j}+(z-z')\boldsymbol{k}}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

$$=\frac{1}{(x-x')^2+(y-y')^2+(z-z')^2}\times$$

$$\frac{(x-x')\boldsymbol{i}+(y-y')\boldsymbol{j}+(z-z')\boldsymbol{k}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$=\frac{1}{r^2}\boldsymbol{r}_1$$

$$d\left(\frac{1}{r}\right)=\frac{\partial\left(\frac{1}{r}\right)}{\partial x'}\,dx'+\frac{\partial\left(\frac{1}{r}\right)}{\partial y'}\,dy'$$

$$+\frac{\partial\left(\frac{1}{r}\right)}{\partial z'}\,dz'$$

$$=\nabla'\left(\frac{1}{r}\right)\cdot d\boldsymbol{l}'$$

其值要最大，則 $\theta=0$，故 $\nabla'\left(\frac{1}{r}\right)$ 為最大變化率

(b) 同理

$$\nabla\left(\frac{1}{r}\right)=\frac{\partial\left(\frac{1}{r}\right)}{\partial x}\,\boldsymbol{i}+\frac{\partial\left(\frac{1}{r}\right)}{\partial y}\,\boldsymbol{j}+\frac{\partial\left(\frac{1}{r}\right)}{\partial z}\,\boldsymbol{k}$$

$$=-\frac{(x-x')\boldsymbol{i}+(y-y')\boldsymbol{j}+(z-z')\boldsymbol{k}}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

$$=-\frac{1}{(x-x')^2+(y-y')^2+(z-z')^2}\times$$

$$\frac{(x-x')\boldsymbol{i}+(y-y')\boldsymbol{j}+(z-z')\boldsymbol{k}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$=-\frac{1}{r^2}\,\boldsymbol{r}_1$$

1-13 (a)試證 $\nabla\cdot\boldsymbol{r}=3$

(b)對一個半徑為 a 的球面，$\boldsymbol{r}$ 的通量是多少？

解

(a) $\nabla\cdot\boldsymbol{r}=\left(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\right)\cdot(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

(b) $d\Phi = a\boldsymbol{r} \cdot d\boldsymbol{a}$

$$= a\boldsymbol{r}_1 \cdot a^2 sin\theta d\theta d\phi \boldsymbol{r}_1$$

$$= a^3 sin\theta d\theta d\phi$$

$$\Phi = \int_0^{2\pi} d\phi \int_0^{\pi} a^3 sin\theta d\theta d\phi$$

$$= 2\pi a^3 \int_0^{\pi} sin\theta d\theta$$

$$= 2\pi a^3 \times 2$$

$$= 4\pi a^3$$

1-14 若 f 表純量函數，A 表向量函數

$$\nabla \cdot (fA) = f\nabla \cdot A + A \cdot \nabla f$$

解

$$\nabla \cdot (fA) = \frac{\partial fA_x}{\partial x} + \frac{\partial fA_y}{\partial y} + \frac{\partial fA_z}{\partial z}$$

$$= f\frac{\partial A_x}{\partial x} + A_x\frac{\partial f}{\partial x} + f\frac{\partial A_y}{\partial y} + A_y\frac{\partial f}{\partial y} +$$

$$f\frac{\partial A_z}{\partial z} + A_z\frac{\partial f}{\partial z}$$

$$= (f\frac{\partial A_x}{\partial x} + f\frac{\partial A_y}{\partial y} + f\frac{\partial A_z}{\partial z}) + (A_x\frac{\partial f}{\partial x} +$$

$$A_y\frac{\partial f}{\partial y} + A_z\frac{\partial f}{\partial z})$$

$$= f\nabla \cdot A + A \cdot \nabla f$$

1-15 向量 $\boldsymbol{A}=3x\boldsymbol{i}+y\boldsymbol{j}+2z\boldsymbol{k}$ 且 $f=x^2+y^2+z^2$

(a)先求 $f\boldsymbol{A}$，再將之取散度，試證 $\nabla\cdot(f\boldsymbol{A})$ 在點（2，2，2）處為120。

(b)證明 $\nabla\cdot(f\boldsymbol{A})$ 在點（2，2，2）處為120，如今先求 ∇f 與 $\nabla\cdot\boldsymbol{A}$ 再利用前1-14題之恒等式。

(c)若 x，y，z 以米度量，試問 $\nabla\cdot(f\boldsymbol{A})$ 單位為何？

解 (a) $f\boldsymbol{A}=(x^2+y^2+z^2)\cdot(3x\boldsymbol{i}+y\boldsymbol{j}+2z\boldsymbol{k})$

$$=[3x(x^2+y^2+z^2)\boldsymbol{i}+y(x^2+y^2+z^2)\boldsymbol{j}+2z(x^2+y^2+z^2)\boldsymbol{k}]$$

$$\nabla\cdot(f\boldsymbol{A})=\frac{\partial(3x^3+3xy^2+3xz^2)}{\partial x}+\frac{\partial(x^2y+y^3+yz^2)}{\partial y}+\frac{\partial(2zx^2+2y^2z+2z^3)}{\partial z}$$

$$=9x^2+3y^2+3z^2+x^2+3y^2+z^2+2x^2+2y^2+6z^2$$

$\nabla\cdot(f\boldsymbol{A})$ at（2，2，2）處

$$=9\times4+3\times4+3\times4+4+3\times4+4+2\times4+2\times4+6\times4$$

$$=120$$

(b) $\nabla f=\nabla(x^2+y^2+z^2)$

$$=2x\boldsymbol{i}+2y\boldsymbol{j}+2z\boldsymbol{k}$$

$\nabla\cdot\boldsymbol{A}=\nabla\cdot(3x\boldsymbol{i}+y\boldsymbol{j}+2z\boldsymbol{k})$

$$=3+1+2$$

$$=6$$

$\nabla\cdot(f\boldsymbol{A})=f\nabla\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\nabla f$ at（2，2，2）

$f=12$

$\boldsymbol{A}=6\boldsymbol{i}+2\boldsymbol{j}+4\boldsymbol{k}$

$\nabla f=4\boldsymbol{i}+4\boldsymbol{j}+4\boldsymbol{k}$

$\nabla\cdot(f\boldsymbol{A})=12\times6+(6\boldsymbol{i}+2\boldsymbol{j}+4\boldsymbol{k})\cdot(4\boldsymbol{i}+4\boldsymbol{j}+4\boldsymbol{k})$

$$=72+24+8+16$$

$$=120$$

(c) $f\boldsymbol{A}$ 是 m^3 的單位

$\therefore$ $\nabla\cdot(fA)$是m^2的單位。

1-16 有人提出 $\nabla\cdot(A\times\frac{r}{r^3})$恆等於零，只要$A$是常數。此說是否正確？

解 設$A=A_x\,i+A_y\,j+A_z\,k$

$$A\times r=\begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ x & y & z \end{vmatrix}$$

$$=(A_y z-A_z y)\,i+(xA_z-zA_x)\,j+(yA_x-xA_y)\,k$$

$$\frac{A\times r}{r^3}=\frac{(zA_y-yA_z)\,i+(xA_z-zA_x)\,j+(yA_x-xA_y)\,k}{r^3}$$

$$\nabla\cdot\left(\frac{A\times r}{r^3}\right)=(zA_y-yA_z)\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)+(xA_z-zA_x)$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right)+(yA_x-xA_y)\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right)$$

$$=(zA_y-yA_z)\frac{-3x}{r^5}+(xA_z-zA_x)\frac{-3y}{r^5}$$

$$+(yA_x-xA_y)\frac{-3z}{r^5}$$

$$=\frac{3}{r^5}(xyA_z-xzA_y+yzA_x-xyA_z+xzA_y-$$

$$yzA_x)=0$$

此說是正確的。

1-17 向量A之分量爲：

$$A_x=y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\ ,\ A_y=z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\ ,$$

$A_z = x\dfrac{\partial f}{\partial y} - y\dfrac{\partial f}{\partial x}$，此處 f 為 x，y，z 之函數，試證

$A = r \times \nabla f$　，　$A \cdot r = 0$　，　$A \cdot \nabla f = 0$

解

$$r \times \nabla f = \begin{vmatrix} i & j & k \\ x & y & z \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y} \right) i + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z} \right) j + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} \right) k$$

$$= A$$

$$A \cdot r = \left[\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y} \right) i + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z} \right) j + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} \right) k \right] \cdot \left(x i + y j + z k \right)$$

$$= xy\frac{\partial f}{\partial z} - xz\frac{\partial f}{\partial y} + yz\frac{\partial f}{\partial x} - xy\frac{\partial f}{\partial z} + xz\frac{\partial f}{\partial y} - yz\frac{\partial f}{\partial x}$$

$$= 0$$

$$A \cdot \nabla f = \left[\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y} \right) i + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z} \right) j + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} \right) k \right] \cdot \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right)$$

$$= y\frac{\partial f}{\partial x}\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial y}\frac{\partial f}{\partial z} + x\frac{\partial f}{\partial y}\frac{\partial f}{\partial z} - y\frac{\partial f}{\partial x}\frac{\partial f}{\partial z}$$

$$= 0$$

1-18 對 f 為一純量函數，A 為一向量函數言，試證：

$$\nabla\times(fA) = (\nabla f)\times A + f(\nabla\times A)$$

解

$$\nabla\times(fA) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fA_x & fA_y & fA_z \end{vmatrix}$$

$$= \left(\frac{\partial fA_z}{\partial y} - \frac{\partial fA_y}{\partial z}\right)i + \left(\frac{\partial fA_x}{\partial z} - \frac{\partial fA_z}{\partial x}\right)j + \left(\frac{\partial fA_y}{\partial x} - \frac{\partial fA_x}{\partial y}\right)k$$

$$= \left(f\frac{\partial A_z}{\partial y} + A_z\frac{\partial f}{\partial y} - f\frac{\partial A_y}{\partial z} - A_y\frac{\partial f}{\partial z}\right)i + \left(f\frac{\partial A_x}{\partial z} + f\frac{\partial f}{\partial z} - f\frac{\partial A_z}{\partial x} - A_z\frac{\partial f}{\partial x}\right)j + \left(f\frac{\partial A_y}{\partial x} + A_y\frac{\partial f}{\partial x} - f\frac{\partial A_x}{\partial y} - A_x\frac{\partial f}{\partial y}\right)k$$

$$\nabla f\times A = \begin{vmatrix} i & j & k \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$