PAPERS IN ELEMENTARY MATHEMATICS

by W. H. Hsiang

基础数学论文集

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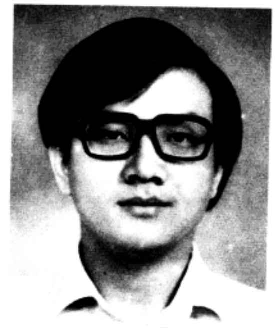
谷超豪题



作者简介

项文涵，祖籍中国浙江省温州瑞安县，1954年出生台湾。1976年毕业于美国柏克莱大学，1980年获美国俄亥俄州立大学数学硕士学位，后在美国路易斯安那州立大学通过博士资格与语言考试。曾在美国俄亥俄州立大学数学系任教，台湾淡江大学数学系讲师。1986年移居加拿大，即从事著书及研究工作。1992年应中国科技大学聘任为数学系教授，但因病未能成行，旋即不幸逝世于加拿大，年仅38岁。其遗作《中级基础数学》、《基础数学论文集》已正式出版，尚有40余篇论文未出版。

作 者 自 序

这本书集合了作者十多年来未正式发表的二十三篇基础数学论文。

所谓基础数学，作者认为，它包括三角与解析几何、高等代数 (即方程式论、差分方程式、数的基本性质等) 、微积分、矩阵、线性代数、实、复变分析与拓扑。前二者是中学教材，中间三者是大学一、二年级理工科的必修课程，最后二者也不超过大学三年级的范围。尤其本书的论文偏重技巧与方法，所需的观念并不深奥。

古人说：“文章千古事，得失寸心间”。作者就是以这种严谨慎重的态度来写数学论文的。每一个细节都经过验检，观念的衔接也不含糊。许多人认为，数学论文既少人阅读，又没什么人了解，何必如此仔细费劲，只要有一个形式即可。如果拿此类论文仔细阅读，读者会发现它们岂仅是“似是而非”, 根本是错误百出。作者虽只是一个渺小无名的数学工作者，但仍相信数学不是神秘主义，不是艺术。除了形式外，它仍能配合其他知识为人类造福谋利。作者就是努力地将数学的实用性发掘出来。

项文涵谨识

Contents (目　次)

Chapter I. HIGHER ALGEBRA

本章包含高等代数的四篇论文。高等代数泛指抽象代数 (线性代数，群，环，体等) 以下的各个代数专题。主要包括解方程式 (一元高次方程式，多元联立方程式等), 差分 (Differences), 解差分方程式 (Difference Equations), 求 (有限或无穷) 级数的和等。因为第二章专门讨论方程式，所以本章即囊括不列入第二章的所有高等代数的论文。

第一篇论文 "Applications of de Moivre's formula", 是利用棣美弗公式来求几个关于正切函数 (Tangent function) 与余切函数 (Cotangent function) 的有限和。说得更详尽些，比较等式 $\cos n\theta + i\sin n\theta = \cos^n\theta(1+i\tan\theta)^n = \sin^n\theta(i+\cot\theta)^n$ 的实部与虚部 (当然 $(1+i\tan\theta)^n$ 与 $(i+\cot\theta)^n$ 必须展开), 且考虑 n 为正偶数或正奇数，作者得到八个关于正切函数与余切函数的等式，如 $\sum\limits_{k=1}^{n-1}\tan^2\frac{k\pi}{2n} = \sum\limits_{k=1}^{n}\cot^2\frac{k\pi}{2n} = \frac{(n-1)(2n-1)}{3}$ 对于所有正整数 $n \geq 2$ 。更具体的例子，如 $\tan\frac{\pi}{5}$ 是方程式 $y^6 - 12y^4 + 25y^2 - 10 = 0$ 的根可得到证明。

第二篇论文 "Simple proof of two combinatorial identities" 是组合等式 $\sum\limits_{m=k}^{[\frac{n}{2}]}\binom{n}{2m}\binom{m}{k} = 2^{n-2k-1}\frac{n}{n-k}\binom{n-k}{k}$, $k = 0,1,2,\cdots,[\frac{n}{2}]$ 与 $\sum\limits_{m=k}^{[\frac{n-1}{2}]}\binom{n}{2m+1}\binom{m}{k} = 2^{n-2k-1}\binom{n-k-1}{k}$, $k = 0,1,2,\cdots,[\frac{n-1}{2}]$ 的简单证明。前者是 $\cos n\theta$ 表为 $\cos\theta$ 的多项式 (Polynomial) 时的系数，而后者是 $\frac{\sin n\theta}{\sin\theta}$ 表为 $\cos\theta$ 的多项式时的系数。两个等式的证明素来困难。在这篇论文中，作者利用解巡回定义方程式 (Recursive equation) $C_{nk} = 2C_{n-1,k} - C_{n-2,k-1}$, $k,n \in \mathbb{Z}_+ \cup \{0\}$ 满足两组不同的初始条件 (Initial conditions), 轻易证明了这两个等式。所指方程式的解即称为双重注码序列 $\{C_{nk}\}_{n,k=0}^{+\infty}$ (Doubly indexed sequence) 。顺带一提的是第一个等式在解方程式时亦甚有用。一元高次方程式称为回文方程式 (Palindromic or reciprocal equation), 如果其系数对称于中间项。在些情况下，我们可令 $y = x + \frac{1}{x}$, 那么 $x^n + \frac{1}{x^n}$ 可表为 y 的多项式，其系数即为第一个组合等式。

第三篇论文 "Cauchy's condensation test for convergence of infinite series" 是讨论一检测正项递减无穷级数 (Decreasing infinite series with positive terms) 收敛的方法。它能将所予的级数化为一几何级数。原级数与新级数同为收敛或发散。较具体的例子是讨论调和级数 (Harmonic series) 如 $\sum\limits_{n=1}^{+\infty}\frac{1}{n^\alpha}$ 与 $\sum\limits_{n=2}^{+\infty}\frac{1}{n(\log n)^\alpha}$ 的敛散。

第四篇论文 "Simple construction of non-summable (H,m) infinite series which are $(H,m+1)$ summable" 是讨论无穷级数的可和性。一无穷级数 $\sum\limits_{n=0}^{+\infty}a_n$ 为发散，如果其前 n 项和 $S_n = \sum\limits_{k=0}^{n}a_k$ 不收敛。如果调整 $\{S_n\}_{n=0}^{+\infty}$ 而使其收敛，那么就称为一可和性 (Summability) 的方法。在这篇论文中，我们令 $\sigma_n^0 = S_n$ 而巡回定义 $\sigma_n^m = \frac{1}{n+1}\sum\limits_{k=0}^{n}\sigma_k^{m-1}$, $m = 1,2,\cdots$ 。如果 $\{\sigma_n^m\}_{n=0}^{+\infty}$ 收敛到 σ,

那么原级数 $\sum_{n=0}^{+\infty} a_n$ 称为第 m 阶赫尔德性可和，其和为 σ ($\sum_{n=0}^{+\infty} a_n$ is said to be (H,m) summable to σ) 。本文提供一简单方法来建立一无穷级数其为第 $m+1$ 阶赫尔德性可和，但不为第 m 阶赫尔德性可和。

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I:1.1,3-6.

APPLICATIONS OF DE MOIVRE'S FORMULA

Abstract

By equating $\cos n\theta + i \sin n\theta = \cos^n \theta (1 + i \tan \theta)^n = \sin^n \theta (i + \cot \theta)^n$, this paper derives eight finite sums of tangent functions and cotangent functions. Most results are believed to be new. By using the same techniaue, two identities of L. Euler can be proved.

We derive the following sums of tangent functions and cotangent functions. Let $\boldsymbol{Z}_+$ and $\boldsymbol{Z}$ be the sets of all positive integers and integers, respectively.

§1. Summation of Tangent Functions

Let $m = \boldsymbol{Z}_+$. Then

$$\cos m\theta + i \sin m\theta = (\cos \theta + i \sin \theta)^m = \cos^m (1 + i \tan \theta)^m = \cos^m \theta \sum_{k=0}^{m} \binom{m}{k} (i \tan \theta)^k$$

implies

$$\cos m\theta = \cos^m \theta \sum_{k=0}^{[\frac{m}{2}]} (-1)^k \binom{m}{2k} \tan^{2k} \theta, \tag{A}$$

$$\sin m\theta = \cos^m \theta \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{m}{2k+1} \tan^{2k+1} \theta$$

$$= \cos^m \theta \tan \theta \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{m}{2k+1} \tan^{2k} \theta. \tag{B}$$

(1) $\sum\limits_{k=0}^{n-1} \tan^2 \frac{(2k+1)}{4n+2} = n(2n-1)/3$ for any $n \in \boldsymbol{Z}_+$: Let $m = 2n+1$ in (A). Then

$$\cos(2n+1)\theta = \cos^{2n+1} \theta \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} \tan^{2k} \theta = \cos^{2n+1} \theta P(\tan^2 \theta)$$

with $P(x) = \sum\limits_{k=0}^{n} (-1)^k \binom{2n+1}{2k} x^k$. $P(x)$ is a polynomial with degree n and has n zeros. We claim that the zeros of $P(x)$ are $\tan^2 \frac{(2k+1)\pi}{4n+2}$ for $k = 0, 1, 2, \cdots, n-1$. Let $0 < \theta < \frac{1}{2}\pi$, then $\cos(2n+1)\theta = 0$ iff $P(\tan^2 \theta) = 0$ iff $(2n+1)\theta = \frac{2k+1}{2}\pi$ for some $k \in \boldsymbol{Z}$. Thus $\theta = \frac{(2k+1)\pi}{4n+2}$ for $k = 0, 1, 2, \cdots, n-1$. Hence $P(x) = 0$ iff $x = \tan^2 \frac{(2k+1)\pi}{4n+2}$ for $k = 0, 1, 2, \cdots, n-1$ is true, and the claim is proved. This imples

$$\sum_{k=0}^{n-1} \tan^2 \frac{(2k+1)\pi}{4n+2} = \binom{2n+1}{2n-1} \Big/ \binom{2n+1}{2n} = n(2n-1)/3$$

for any $n \in \boldsymbol{Z}_+$.

Example 1. For $n = 2$, we can prove that $y = \tan \frac{\pi}{10}$ satisfies $5y^4 - 10y^2 + 1 = 0$.

(2) $\sum_{k=0}^{n-1} \tan^2 \frac{(2k+1)}{4n} = n(2n-1)$ for any $n \in \mathbb{Z}_+$: Let $m = 2n$ in (A). Then

$$\cos 2n\theta = \cos^{2n}\theta \sum_{k=0}^{n}(-1)^k \binom{2n}{2k} \tan^{2k}\theta = \cos^{2n}\theta P(\tan^2\theta)$$

with $P(x) = \sum_{k=0}^{n}(-1)^k\binom{2n}{2k}x^k$. Let $0 < \theta < \frac{\pi}{2}$. Then $\cos 2n\theta = 0$ iff $\theta = \frac{2k+1}{4n}\pi$ for $k = 0, 1, 2, \cdots, n-1$ is true. Thus $P(x)$ has zeros $x = \tan^2\frac{(2k+1)\pi}{4n}$ for $k = 0, 1, 2, \cdots, n-1$. This imples $\sum_{k=0}^{n-1}\tan^2\frac{(2k+1)\pi}{4n} = \binom{2n}{2n-2}\Big/\binom{2n}{2n} = n(2n-1)$ for any $n \in \mathbb{Z}_+$.

Example 2. Let $n = 2$. Then $\tan^2\frac{\pi}{8} + (\tan^2\frac{\pi}{8})^{-1} = 6$ implies $\tan^4\frac{\pi}{8} - 6\tan^2\frac{\pi}{8} + 1 = 0$ and $\tan^3\frac{\pi}{8} = 3 - 2\sqrt{2}$. Hence $\tan\frac{\pi}{8} = \sqrt{2} - 1$ is obtained.

(3) $\sum_{k=1}^{n-1}\tan^2\frac{k\pi}{2n+1} = n(2n+1)$ for any $n \in \mathbb{Z}_+$: Let $m = 2n+1$ in (B). Then

$$\sin(2n+1)\theta = \cos^{2n+1}\theta\tan\theta\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}\tan^{2k}\theta = \cos^{2n+1}\theta\tan\theta P(\tan^2\theta)$$

with $P(x) = \sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}x^k$. Let $0 < \theta < \frac{\pi}{2}$. Then $\sin(2n+1)6 = 0$ iff $\theta = \frac{k\pi}{2n+1}$ for $k = 1, 2, \cdots, n$ is true. Thus $P(x)$ has zeros $x = \tan^2\frac{k\pi}{2n+1}$ for $k = 1, 2, \cdots, n$. This implies $\sum_{k=0}^{n-1}\tan^2\frac{k\pi}{2n+1} = \binom{2n+1}{2n-1}\Big/\binom{2n+1}{2n+1} = n(2n+1)$ for any $n \in \mathbb{Z}_+$.

(4) $\sum_{k=1}^{n-1}\tan^2\frac{k\pi}{2n} = (n-1)(2n-1)/3$ for any $n \in \mathbb{Z}_+, n \geq 2$: Let $m = 2n$ in (B). Then

$$\sin 2n\theta = \cos^{2n}\theta\tan\theta\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}\tan^{2k}\theta = \cos^{2n}\theta\tan\theta P(\tan^2\theta)$$

with $P(x) = \sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}x^k$. Let $0 < \theta < \frac{\pi}{2}$. Then $\sin 2n\theta = 0$ iff $\theta = \frac{k\pi}{2n}$ for $k = 1, 2, \cdots, n-1$ is true. Thus $P(x)$ has zeros $x = \tan^2\frac{k\pi}{2n}$ for $k = 1, 2, \cdots, n-1$. This imples $\sum_{k=1}^{n-1}\tan^2\frac{k\pi}{2n} = \binom{2n}{2n-3}\Big/\binom{2n}{2n-1} = (n-1)(2n-1)/3$ for any $n \in \mathbb{Z}_+$ with $n \geq 2$.

§2. Summation of Cotangent Functions

Let $n \in \mathbb{Z}_+$. Then

$$\cos m\theta + \sin m\theta = (\cos\theta + i\sin\theta)^m = \sin^m\theta(i + \cot\theta)^m = \sin^m\theta\sum_{k=0}^{m}\binom{m}{k}i^k\cot^{m-k}\theta$$

implies

$$\cos m\theta = \sin^m\theta\sum_{k=0}^{[\frac{m}{2}]}(-1)^k\binom{m}{2k}\cot^{m-2k}\theta, \tag{C}$$

$$\sin m\theta = \sin^m\theta\sum_{k=0}^{[\frac{m-1}{2}]}(-1)^k\binom{m}{2k+1}\cot^{m-2k-1}\theta. \tag{D}$$

(5) $\sum\limits_{k=0}^{n-1} \cot^2 \frac{(2k+1)\pi}{4n+2} = n(2n+1)$ for any $n \in \boldsymbol{Z}_+$: Let $m = 2n+1$ in (C). Then

$$\begin{aligned}\cos(2n+1)\theta &= \sin^{2n+1}\theta \sum_{k=0}^{n}(-1)^k \binom{2n+1}{2k} \cot^{2(n-k)+1}\theta \\ &= \sin^{2n+1}\theta \cot\theta \sum_{k=0}^{n}(-1)^k \binom{2n+1}{2k} \cot^{2(n-k)}\theta \\ &= \sin^{2n+1}\theta \cot\theta P(\cot^2\theta)\end{aligned}$$

with $P(x) = \sum\limits_{k=0}^{n}(-1)^k\binom{2n+1}{2k}x^{n-k}$. Let $0 < \theta < \frac{\pi}{2}$. Then $\cos(2n+1)\theta = 0$ iff $\theta = \frac{(2k+1)\pi}{4n+2}$ for $k = 0, 1, 2, \cdots, n-1$ is true. Thus $P(x)$ has zeros $x = \cot^2 \frac{(2k+1)\pi}{4n+2}$ for $k = 0, 1, 2, \cdots, n-1$. This imples $\sum\limits_{k=0}^{n-1} \cot^2 \frac{(2k+1)\pi}{4n+2} = \binom{2n+1}{2}\Big/\binom{2n+1}{0} = n(2n+1)$ for any $n \in \boldsymbol{Z}_+$.

(6) $\sum\limits_{k=0}^{n-1} \cot^2 \frac{(2k+1)\pi}{4n} = n(2n-1)$ for any $n \in \boldsymbol{Z}_+$: Let $m = 2n$ in (C). Then

$$\cos 2n\theta = \sin^{2n}\theta \sum_{k=0}^{n}(-1)^k\binom{2n}{2k}\cot^{2(n-k)}\theta = \sin^{2n}\theta P(\cot^2\theta)$$

with $P(x) = \sum\limits_{k=0}^{n}(-1)^k\binom{2n}{2k}x^{n-k}$. Let $0 < \theta < \frac{\pi}{2}$. Then $\cos 2n\theta = 0$ iff $\theta = \frac{(2k+1)\pi}{4n}$ for $k = 0, 1, 2, \cdots, n-1$ is true. Thus $P(x)$ has zeros $x = \cot^2 \frac{(2k+1)\pi}{4n}$ for $k = 0, 1, 2, \cdots, n-1$. This imples $\sum\limits_{k=0}^{n-1} \cot^2 \frac{(2k+1)\pi}{4n} = \binom{2n}{2}\Big/\binom{2n}{0} = n(2n-1)$ for any $n \in \boldsymbol{Z}_+$.

Example 3. Let $n = 2$. Then $\cot^2 \frac{\pi}{8} + (\cot^2 \frac{\pi}{8})^{-1} = 0$ implies $\cot^4 \frac{\pi}{8} - 6\cot^2 \frac{\pi}{8} + 1 = 0$ and $\cot^2 \frac{\pi}{8} = 3 + \sqrt{2}$. Hence $\cot \frac{\pi}{8} = \sqrt{2} + 1$ is obtained.

(7) $\sum\limits_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = n(2n-1)/3$ for any $n \in \boldsymbol{Z}_+$: Let $m = 2n+1$ in (D). Then

$$\sin(2n+1)\theta = \sin^{2n+1}\theta \sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}\cot^{2(n-k)}\theta = \sin^{2n+1}\theta P(\cot^2\theta)$$

with $P(x) = \sum\limits_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}x^{n-k}$. Let $0 < \theta < \frac{\pi}{2}$. Then $\sin(2n+1)\theta = 0$ iff $\theta = \frac{k\pi}{2n+1}$ for $k = 1, 2, \cdots, n$ is true. Thus $P(x)$ has zeros $x = \cot^2 \frac{k\pi}{2n+1}$ for $k = 1, 2, \cdots, n$. This implies $\sum\limits_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = \binom{2n+1}{3}\Big/\binom{2n+1}{1} = n(2n-1)/3$ for any $n \in \boldsymbol{Z}_+$. This identity is known.

Example 4. Let $n = 2$. Then $\cot^2 \frac{\pi}{5} + \cot^2 \frac{2\pi}{5} = 2$ implies $y^2 + \left(\frac{y^2-1}{2y}\right)^2 = 2$ and $5y^4 - 10y^2 + 1 = 0$, where $y = \cot \frac{\pi}{5}$.

(8) $\sum\limits_{k=1}^{n-1} \cot^2 \frac{k\pi}{2n} = (n-1)(2n-1)/3$ for any $n = \boldsymbol{Z}_+,\ n \geq 2$: Let $m = 2n$ in (D). Then

$$\begin{aligned}\sin 2n\theta &= \sin^{2n}\theta \sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}\cot^{2(n-k)-1}\theta \\ &= \sin^{2n}\theta\cot\theta \sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}\cot^{2(n-1-k)}\theta = \sin^{2n}\theta\cot\theta P(\cot^2\theta)\end{aligned}$$

with $P(x) = \sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1}x^{n-1-k}$. Let $0 < \theta < \frac{\pi}{2}$. Then $\sin 2n\theta = 0$ iff $\theta = \frac{k\pi}{2n}$ for $k = 1, 2, \cdots, n-1$ is true. Thus $P(x)$ has zeros $x = \cot^2\frac{k\pi}{2n}$ for $k = 1, 2, \cdots, n-1$. This implies $\sum_{k=1}^{n-1}\cot^2\frac{k\pi}{2n} = \binom{2n}{3}/\binom{2n}{1} = (n-1)(2n-1)/3$ for any $n \in \boldsymbol{Z}_+$ with $n \geq 2$.

From the above identities, we summarize that

$$\sum_{k=0}^{n-1}\tan^2\frac{(2k+1)\pi}{4n+2} = \sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1} = n(2n-1)/3,$$

$$\sum_{k=0}^{n-1}\tan^2\frac{(2k+1)\pi}{4n} = \sum_{k=0}^{n-1}\cot^2\frac{(2k+1)\pi}{4n} = n(2n-1),$$

$$\sum_{k=1}^{n}\tan^2\frac{k\pi}{2n+1} = \sum_{k=0}^{n-1}\cot^2\frac{(2k+1)\pi}{4n+2} = n(2n+1) \text{ for any } n \in \boldsymbol{Z}_+,$$

$$\sum_{k=1}^{n-1}\tan^2\frac{k\pi}{2n} = \sum_{k=1}^{n}\cot^2\frac{k\pi}{2n} = (n-1)(2n-1)/3 \quad \text{for} \quad n \geq 2.$$

By using the same technique, we can prove the following sophisticated identity which is due to L. Euler.

(9) $\sum_{k=0}^{m-1}\cot(\frac{k\pi}{m}+\theta) = m\cot(m\theta)$ fcr any $m \in \boldsymbol{Z}_+$ such that $m\theta$ is not an integer multiple of π : From $\cos(m\theta) \pm i\sin(m\theta) = (\cos\theta \pm i\sin\theta)^m$, we obtain

$$2\cos(m\theta) = (\cos\theta + i\sin\theta)^m + (\cos\theta - i\sin\theta)^m$$

and $2i\sin(m\theta) = (\cos\theta + i\sin\theta)^m - (\cos\theta - i\sin\theta)^m$. Division then yields

$$\frac{1}{i}\cot(m\theta) = \frac{\{(\cot\theta + i)^m + (\cot\theta - i)^m\}}{\{(\cot\theta + i)^m - (\cot\theta - i)^m\}} = \frac{2\sum_{k=0}^{[m/2]}\binom{m}{2k}(-1)^k\cot^{m-2k}\theta}{2i\sum_{k=0}^{[\frac{m-1}{2}]}\binom{m}{2k+1}(-1)^k\cot^{m-2k-1}\theta},$$

that is

$$\cot(m\theta) = \frac{\left\{\cot^m\theta - \binom{m}{2}\cot^{m-2}\theta + \binom{m}{4}\cot^{m-4}\theta - \cdots\right\}}{\left\{m\cot^{m-1}\theta - \binom{m}{3}\cot^{m-3}\theta + \binom{m}{5}\cot^{m-5}\theta - \cdots\right\}},$$

or $x^m - mx^{m-1}\cot(m\theta) - \binom{m}{2}x^{m-2} + \binom{m}{3}x^{m-3}\cot(m\theta) + \binom{m}{4}x^{m-4} - \cdots = 0$ with $x = \cot\theta$.

The last equation in x has degree m, and is also true for $x = \cot(\theta + \frac{k\pi}{m})$ for any $k \in \boldsymbol{Z}_+$ since $\cot(m\theta + k\pi) = \cot(m\theta)$. Hence the m roots of this equation are $\cot(\theta + \frac{k\pi}{m})$ for $k = 0, 1, 2, \cdots, m-1$. This implies $\sum_{k=0}^{m-1}\cot(\theta + \frac{k\pi}{m}) = m\cot(m\theta)$.

(10) $\sum_{k=0}^{m-1}\csc^2(\theta + \frac{k\pi}{m}) = m^2\csc^2(m\theta)$ for any $m \in \boldsymbol{Z}_+$ such that $m\theta$ is not an integer multiple of π : This identity is obtained from (9) by differentiating w.r.t. θ.

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SIMPLE PROOF OF TWO COMBINATORIAL IDENTITIES

Abstract

For any positive integer n, the identities $\sum_{m=k}^{[\frac{n}{2}]} \binom{n}{2m}\binom{m}{k} = 2^{n-2k-1}\frac{n}{n-k}\binom{n-k}{k}$ for $k = 0, 1, 2, \cdots, [\frac{n}{2}]$, and $\sum_{m=k}^{[\frac{n-1}{2}]} \binom{n}{2m+1}\binom{m}{k} = 2^{n-2k-1}\binom{n-k-1}{k}$ for $k = 0, 1, 2, \cdots, [\frac{n-1}{2}]$ are proved by considering the recursive equation $c_{nk} = 2c_{n-1,k} - c_{n-2,k-1}$ which are satisfied by two doubly indexed sequences $\{c_{nk}\}_{n,k=0}^{+\infty}$ under different initial conditions.

The fascinating identities

$$\sum_{m=k}^{[\frac{n}{2}]} \binom{n}{2m}\binom{m}{k} = 2^{n-2k-1}\frac{n}{n-k}\binom{n-k}{k} \quad \text{for } k = 0, 1, 2, \cdots, [\frac{n}{2}]$$

and

$$\sum_{m=k}^{[\frac{n-1}{2}]} \binom{n}{2m+1}\binom{m}{k} = 2^{n-2k-1}\binom{n-k-1}{k} \quad \text{for } k = 0, 1, 2, \cdots, [\frac{n-1}{2}]$$

are difficultly proved, where n is a positive integer. In [1], the proof is by induction and assumes the validity of both identities in inductive hypothesis. The proof of these identities here is simple, direct and without induction. However, the analysis at the end of [3] is inspiring. Let $\mathbb{Z}_+$ be the set of positive integers.

§1. Two Interesting Doubly Indexed Sequences

We construct two doubly indexed sequences $\{c_{nk}\}_{n,k=0}^{+\infty}$ satisfying the recursive equation $c_{nk} = 2c_{n-1,k} - c_{n-2,k-1}$ with different initial conditions.

The first one $\{a_{nk}\}_{n,k=0}^{+\infty}$ is defined by $a_{nk} = 2a_{n-1,k} - a_{n-2,k-1}$ for $n \in \mathbb{Z}_+,\ k \in \mathbb{Z}_+ \cup \{0\}$ except $a_{-1,-1} = a_{00} = 1, a_{n,-1} = a_{-1,k} = 0$ for all n's and k's, and $a_{nk} = 0$ for $n < 2k$. Part of this sequence is illustrated in Table 1.

$n \setminus k$	-1	0	1	2	3	4
-1	①					
0		①				
1		1			0	
2		2	-1			
3		4	-3			
4	0	8	-8	1		
5		16	-20	5		
6		32	-48	18	-1	
7		64	-112	56	-7	
8		128	-256	160	-32	1

Table 1

$n \setminus k$	-1	0	1	2	3	4
-1	(−1)					
0		⓪				
1		1			0	
2		2				
3		4	-1			
4	0	8	-4			
5		16	-12	1		
6		32	-32	6		
7		64	-80	24	-1	0
8		128	-192	80	-8	0

Table 2

For finding the general formula of a_{nk}, an analysis is given at the end of [3]. We note

$$a_{nk} = \begin{cases} (-1)^k 2^{n-2k-1} \frac{n}{n-k}\binom{n-k}{k} & \text{for } n \in \boldsymbol{Z}_+ \text{ and } k \in \boldsymbol{Z}_+ \cup \{0\} \text{ with } n \geq 2k, \\ 0, & \text{otherwise.} \end{cases}$$

The second sequence $\{b_{nk}\}_{n,k=0}^{+\infty}$ is defined by

$$b_{nk} = 2b_{n-1,k} - b_{n-2,k-1} \quad \text{for} \quad n \in \boldsymbol{Z}_+,\ k \in \boldsymbol{Z}_+ \cup \{0\}$$

except

$$b_{-1,-1} = -1,\ b_{00} = b_{n,-1} = b_{-1,k} = 0$$

for all n's and k's, and $b_{nk} = 0$ for $n - 1 < 2k$. Part of this sequence is illustrated in Table 2. For finding the general formula, we make the following analysis.

$k=0$: n	b_{n0}
1	1
2	2
3	4
4	8
5	16
6	32
7	64
8	128

$k=1$: n	$-b_{n1}$
3	$1 = \frac{1}{2} \cdot 2$
4	$4 = 1 \cdot 4$
5	$12 = 2 \cdot 6$
6	$32 = 4 \cdot 8$
7	$80 = 8 \cdot 10$
8	$192 = 16 \cdot 12$

$k=2$: n	b_{n2}
5	$1 = \frac{1}{2} \cdot 2$
6	$6 = 1 \cdot 6$
7	$24 = 2 \cdot 12$
8	$80 = 4 \cdot 20$

$$b_{n0} = 2^{n-1} \text{ for } n \geq 1$$

$$\begin{aligned} b_{n1} &= -2^{n-4}(2n-4) \\ &= -2^{n-3}(n-2) \\ &\text{for } n \geq 3 \end{aligned}$$

$$\begin{aligned} b_{n2} &= 2^{n-6}(n^2 - 7n + 12) \\ &= 2^{n-6}(n-3)(n-4) \\ &\text{for } n \geq 5 \end{aligned}$$

Hence

$$b_{nk} = \begin{cases} (-1)^k 2^{n-2k-1}\binom{n-k-1}{k} & \text{for any } n \in \boldsymbol{Z}_+ \text{ and } k \in \boldsymbol{Z}_+ \cup \{0\} \text{ with } n-1 \geq 2k, \\ 0, & \text{otherwise.} \end{cases}$$

is obtained.

§2. Where do These Sequences Appear ?

Let $\alpha = \cos\theta, \beta = \sin\theta, n \in \boldsymbol{Z}_+$; then

$$\begin{aligned} \cos n\theta + i \sin n\theta &= (\cos\theta + i\sin\theta)^n \\ &= \sum_{m=0}^{n} \binom{n}{m} \cos^{n-m}\theta (i \sin\theta)^m \end{aligned}$$

implies

$$\cos n\theta = \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \binom{n}{2m} \cos^{n-2m}\theta \sin^{2m}\theta$$

and

$$\sin n\theta = \sum_{m=0}^{[\frac{n-1}{2}]} (-1)^m \binom{n}{2m+1} \cos^{n-2m-1}\theta \sin^{2m+1}\theta.$$

Hence

$$\begin{aligned}
\cos n\theta &= \sum_{m=0}^{[\frac{n}{2}]}(-1)^m\binom{n}{2m}\alpha^{n-2m}(1-\alpha^2)^m \\
&= \sum_{m=0}^{[\frac{n}{2}]}(-1)^m\binom{n}{2m}\alpha^{n-2m}\Big\{\sum_{k=0}^{m}\binom{m}{k}(-\alpha^2)^{m-k}\Big\} \\
&= \sum_{m=0}^{[\frac{n}{2}]}\sum_{k=0}^{m}(-1)^k\binom{n}{2m}\binom{m}{k}\alpha^{n-2k} \\
&= \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\alpha^{n-2k}\Big\{\sum_{m=0}^{[\frac{n}{2}]}\binom{n}{2m}\binom{m}{k}\Big\} \\
&= \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\alpha^{n-2k}\Big\{\sum_{m=k}^{[\frac{n}{2}]}\binom{n}{2m}\binom{m}{k}\Big\} \\
&= \sum_{k=0}^{[\frac{n}{2}]}a_{nk}\alpha^{n-2k},
\end{aligned}$$

where $\binom{u}{v}=0$ for $u<v$ and

$$a_{nk}=(-1)^k\sum_{m=k}^{[\frac{n}{2}]}\binom{n}{2m}\binom{m}{k}$$

is the coefficient of α^{n-2k}. $a_{-1,-1}=a_{00}=1,\ a_{n,-1}=a_{-1,k}=0$ for $n,\ k\ge 0$ and $a_{nk}=0$ for $n<2k$ is obvious.

On the other hand,

$$\begin{aligned}
\frac{\sin n\theta}{\sin\theta} &= \sum_{m=0}^{[\frac{n-1}{2}]}(-1)^m\binom{n}{2m+1}\alpha^{n-2m-1}(1-\alpha^2)^m \\
&= \sum_{m=0}^{[\frac{n-1}{2}]}(-1)^m\binom{n}{2m+1}\alpha^{n-2m-1}\Big\{\sum_{k=0}^{m}\binom{m}{k}(-\alpha^2)^{m-k}\Big\} \\
&= \sum_{m=0}^{[\frac{n-1}{2}]}\sum_{k=0}^{m}(-1)^k\binom{n}{2m+1}\binom{m}{k}\alpha^{n-2k-1} \\
&= \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\alpha^{n-2k-1}\Big\{\sum_{m=k}^{[\frac{n-1}{2}]}\binom{n}{2m+1}\binom{m}{k}\Big\} \\
&= \sum_{k=0}^{[\frac{n-1}{2}]}b_{nk}\alpha^{n-2k-1}
\end{aligned}$$

indicates that $\frac{\sin n\theta}{\sin\theta}$ is a polynomial in $\cos\theta$ with degree $n-1$, where

$$b_{nk}=(-1)^k\sum_{m=k}^{[\frac{n-1}{2}]}\binom{n}{2m+1}\binom{m}{k}$$

is the coefficient of α^{n-2k-1}. $b_{-1,-1}=-1, b_{00}=b_{n,-1}=b_{-1,k}=0$ for $n,k\ge 0$ and $b_{nk}=0$

for $n-1<2k$ are also obvious. We note that

$$\sin n\theta = 2\cos(n-1)\theta\sin\theta + \sin(n-2)\theta$$

implies

$$\frac{\sin n\theta}{\sin\theta} = 2\cos(n-1)\theta + \frac{\sin(n-2)\theta}{\sin\theta}.$$

This can express $\frac{\sin n\theta}{\sin\theta}$ as polynomials in $\cos\theta$ recursively.

We next prove that a_{nk} and b_{nk} in §1 and §2 are identical. We note

$$\binom{u}{v} = \binom{u-1}{v} + \binom{u-1}{v-1}.$$

§3. Proof of $\sum_{m=k}^{[\frac{n}{2}]} \binom{n}{2m}\binom{m}{k} = 2^{n-2k-1}\frac{n}{n-k}\binom{n-k}{k}$

Let a_{nk} be defined in §2.

Lemma 1. *Let $n\in\mathbb{Z}_+$ and $k\in\mathbb{Z}_+\cup\{0\}$ satisfy $n\geq 2k$. We assume $\binom{u}{v}=0$ for $v<0$.*

(i) $a_{nk} - a_{n-1,k} = (-1)^k \sum_{m=k}^{[\frac{n}{2}]} \binom{n-1}{2m-1}\binom{m}{k}$ *is true.*

(ii) $a_{nk} - a_{n,k-1} = (-1)^k \sum_{m=k-1}^{[\frac{n}{2}]} \binom{n}{2m}\binom{m+1}{k}$ *is true.*

Proof. (i) If n is even, then

$$\begin{aligned} a_{nk} - a_{n-1,k} &= (-1)^k \sum_{m=k}^{\frac{n}{2}} \binom{n}{2m}\binom{m}{k} - (-1)^k \sum_{m=k}^{\frac{n-2}{2}} \binom{n-1}{2m}\binom{m}{k} \\ &= (-1)^k \left\{ \sum_{m=k}^{\frac{n-2}{2}} \left[\binom{n}{2m} - \binom{n-1}{2m}\right]\binom{m}{k} + \binom{\frac{1}{2}n}{k} \right\} \\ &= (-1)^k \left\{ \sum_{m=k}^{\frac{n-2}{2}} \binom{n-1}{2m-1}\binom{m}{k} + \binom{\frac{1}{2}n}{k} \right\} \\ &= (-1)^k \sum_{m=k}^{\frac{n}{2}} \binom{n-1}{2m-1}\binom{m}{k} \end{aligned}$$

is true. If n is odd, then

$$\begin{aligned} a_{nk} - a_{n-1,k} &= (-1)^k \sum_{m=k}^{\frac{n-1}{2}} \binom{n}{2m}\binom{m}{k} - (-1)^k \sum_{m=k}^{\frac{n-1}{2}} \binom{n-1}{2m}\binom{m}{k} \\ &= (-1)^k \sum_{m=k}^{\frac{n-1}{2}} \binom{n-1}{2m-1}\binom{m}{k} \end{aligned}$$

is true.