陈国旺论文集

Selected Papers of Chen Guowang

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前　言

陈国旺教授1935年6月4日出生于河北万全县，1957年毕业于北京大学数学力学系，同年赴郑州大学数学系任教至今，在此期间于1960年2月被选派为国家公派出国留学人员赴北京外国语学院学习俄语，同年10月赴捷克斯洛伐克查理士大学数学研究所攻读副博士学位，于1964年10月在该校获副博士学位并学成回国。他多年来一直从事应用数学和非线性偏微分方程的教学和科研工作。自1978年起，在郑州大学数学系主持一个非线性偏微分方程讨论班，培养了一批从事非线性偏微分方程研究的博士、硕士和青年教师。陈国旺教授参与了《偏微分方程》（英文版）杂志的创办，任副主编之一并负责编辑部日常工作，该杂志对我国偏微分方程的发展及与国外的学术交流起到了积极作用。

陈国旺教授1989年被评为河南省优秀教师，1992年开始享受政府特殊津贴，1993年被命名为河南省优秀专家，1997年被评为河南省优秀科技期刊出版工作者。

陈国旺教授在教学和科研工作岗位上辛勤耕耘了48年，传道、授业、解惑，从本科生到博士研究生，培养了许多人才。本论文集收集了他在不同时期本人及与他人合作的论文26篇，它们真实地勾画出他的科研工作的轨迹。值此论文集出版之际，恰逢他七十岁生日，特向他致以衷心的祝贺。

学生：苗长兴 邢家省 江成顺 杨志坚
赵占才 王书彬 张宏伟 王艳萍

二零零五年五月一日

Foreward

Prof. Chen Guowang was born on 1935.06.04 in Wanquan County, Hebei province, P.R. China. He graduated from Department of Mathematics and Mechanics of Peking University in 1957. Since then he has taught in Department of Mathematics of Zhengzhou University. During this course he was chosen and sent to study abroad by state in Feb. 1960 and first to learn Russian in Beijing Foreign Languages Institute. In October of the same year he studied for an associate doctorate at Institute of Mathematics of Charles University, Czechoslovakia, and in October 1964 he obtained the degree there and returned home. Since 1978, he has been in charge of a discussion class for nonlinear partial differential equations in Department of Mathematics of Zhengzhou University and trained a series of Doctors, Masters and young teachers engaged in the study of partial differential equations. He also participated the start of "Journal of Partial Differential Equations" (English edition) and holds one of the deputy editors-in-chief of the journal taking care of routine matters. This journal has played important part with respect to the development of partial differential equations in our country and to the academic exchange with foreign countries.

Prof. Chen was chosen as a good teacher of Henan province in 1989 and has enjoyed special subsidy given by the state since 1992; in 1993 he was named as a good specialist of Henan province and in 1997 was chosen as a publisher of good scientific and technological journal of Henan province.

He has diligently worked for 48 years at the teaching and scientific research post, passes knowledge, teaches and explains difficulties and cultivated many men of ability from graduates to doctorial postgraduates.

In this collection, 26 papers written by him and co-written by him with other scholars in different times are included, which really reflects the tracks of his scientific researching works.

On the occasion of the publication of this collection and the time when it happens to be his 70th birthday, we, his students, heartedly extend our cordial greetings to him.

Miao Changxing　Xing Jiasheng　Jiang Chengshun
Yang Zhijian　Zhao Zhancai　Wang Shubin
Zhang Hongwei　Wang Yanping

May 1. 2005

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GENERALIZATION OF STEFFENSEN'S METHOD FOR OPERATOR EQUATIONS IN BANACH SPACE *

Chen Kuo-wang, Praha

1 Introduction

In this paper the Steffensen's method of solution of non-linear equations ([1], Appendix 5) is generalized for solution of non-linear equations in Banach space. Here I use the Schmidt's concept of the divided difference, introduced in [2(I)]; partly, I have made use of this work of his in methodological respect (in particular, paragraph 4), too.

Steffensen's method is an iterative method based on alternate performance of one step of the succesive approximation and one step of the method regula falsi. If we denote the initial approximation by x_0, then the iterative formula for the calculation of the roots of the equation $x = f(x)$ is either

$$x_{n+1} = f(x_n) + \delta f[f(x_n), x_n](x_{n+1} - x_n)$$

or

$$x_{n+1} = f[f(x_n)] + \delta f[f(x_n), x_n][x_{n+1} - f(x_n)],$$

where

$$\delta f[f(x_n), x_n] = \frac{f[f(x_n)] - f(x_n)}{f(x_n) - x_n}.$$

Both formulae are equivalent in the sense that they give the same sequence $\{x_n\}$ when beginning with the same x_0 . In the generalization presented here, it is possible to solve the equation $x = Fx$ by the analogical iterations (2.4) and (2.5) which are again equivalent in the same sense. Therefore, the sufficient conditions for the convergence of any of both sequences defined by the formulae (2.4) and (2.5) are sufficient even for the convergence of the other sequence. The formula (2.5) is simpler for the practical calculation. In spite of that, I shall deal further with formula (2.4), because in this way I have been successfull in obtaining less restrictive sufficient conditions for the convergence.

*Commentationes Mathematicae Universitatis Carolinae, Vol. 5, No.2,1964, 47-77.

In the work [2(I)], J.W.Schmidt studies the solution of the equation $x = Fx$ by means of method applying the iterative process

$$x_{n+1} = Fx_n + \delta F(x_n, x_{n-1})(x_{n+1} - x_n),$$

calling it the Steffenson's method ([2(I)], method (2.9) on p.2; conditions of convergence stated in Theorem 4.1. on p.7). However, this process is quite different from the iterative process (2.5), being, essentially, a modification of the secant method ([1] , Chapter 3, paragraph 9). Its convergence is of an other character than convergence of the process (2.5), as it is easy to see when compared the Schmidt's estimates of errors ([2(I)], (4.1)) with these contained in this paper. See also numerical example in paragraph 3.

The general results of this paper are presented in paragraph 2. Applications of the general theorems on systems of non-linear equations and on non-linear integral equations are stated in paragraphs 3 and 4.

2 Theorems of convergence and uniqueness

We shall use the following denotation: R is a Banach space, F a non -linear operator mapping R into R. The symbol $\delta F(u, v)$ will denote the divided difference of the operator F. This concept, introduced by Schmidt [2] under the title Steigung, is defined as follows. We shall say that the operator F has a divided difference $\delta F(u, v)$ in the space R, when there exist two non-negative numbers a, b such that for every two elements u, v from R there exists a linear bounded operator $\delta F(u, v)$ on R, satisfying the inequality

$$Fu - Fv = \delta F(u, v)(u - v), \tag{2.1}$$

$$||\delta F(u, v) - \delta F(v, w)|| \leq a||u - w|| + b||u - v|| + b||v - w||. \tag{2.2}$$

Let an equation

$$x = Fx \tag{2.3}$$

be given; to solve equation (2.3) we use the iterative processes

$$x_{n+1} = F^2 x_n + \delta F(Fx_n, x_n)(x_{n+1} - Fx_n) \quad (n = 0, 1, 2, \cdots), \tag{2.4}$$

$$x_{n+1} = Fx_n + \delta F(Fx_n, x_n)(x_{n+1} - x_n) \quad (n = 0, 1, 2, \cdots). \tag{2.5}$$

Lemma Iterative processes (2.4) and (2.5) are equivalent in the following sense: Let x_0 be an arbitrary element from R . If the elements of either of the two sequences $x_0, x_1, \cdots, x_n$ defined by the process (2.4); $x'_0, x'_1, \cdots, x'_n$, $(x'_0 = x_0)$ defined by the process

(2.5) are defined, then the ones of the other sequence are defined as well and the equalities $x_i = x_i'$, $i = 1, 2, \cdots, n$ hold.

Proof The proof of this lemma may be achieved by means of full induction, as is easily seen from that, when subtracting the identity

$$\phi = F^2 x_n - F x_n - \delta F(F x_n, x_n)(F x_n - x_n)$$

from (2.4), we get (2.5).

Theorem 1 Let F be an operator which has the divided difference. Let the following conditions be fulfilled:

1) There exists a number $\lambda > 0$ such that inequality

$$\|Fu - Fv\| \le \lambda \|u - v\| \tag{2.6}$$

holds for two arbitrary elements u, v from R.

2) The inequality

$$\|\delta F(F x_0, x_0)\| = d_0 < 1 \tag{2.7}$$

holds for the fixed element $x_0 \in R$.

3) The element x_1 is defined by (2.4) and there exists a real number t $(0 < t < 1)$ such that

$$h = h(t) = \frac{[(a+b) + 2bt]t}{1-t} \|x_1 - x_0\| < 1, \tag{2.8}$$

$$d_0 + [(a+b)(1+\lambda) + 4b][1 + \sigma(h)]\|x_1 - x_0\| \le t < 1, \tag{2.9}$$

where

$$\sigma(h) = \sum_{k=1}^{\infty} h^{2^k - 1}.$$

Then the equation (2.3) has a solution x^* in the sphere

$$D = \{x \in R, \|x - x_1\| \le h[1 + \sigma(h^2)]\|x_1 - x_0\|\}. \tag{2.10}$$

The sequences $\{x_n\}$ defined by equalities (2.4) or (2.5) converge in the norm of R to the solution x^* of (2.3) and the error $\|x^* - x_n\|$ of the approximation x_n satisfies

$$\|x^* - x_n\| \le h^{2^n - 1}[1 + \sigma(h^{2^n})]\|x_n - x_{n-1}\|, n = 1, 2, \cdots, \tag{2.11}$$

$$\|x^* - x_n\| \le h^{2^n - 1}[1 + \sigma(h^{2^n})]\|x_1 - x_0\|, \quad n = 1, 2, \cdots. \tag{2.12}$$

Proof Let us put

$$\|x_{n+1} - x_n\| = \mu_n,$$
$$\|\delta F(F x_n, x_n)\| = d_n, \quad n = 0, 1, 2, \cdots.$$

First of all, we shall show that the following inequalities

$$\|x_{n+1} - Fx_n\| \le d_n\mu_n, \tag{2.13}$$

$$\|Fx_{n+1} - Fx_n\| \le \lambda\mu_n, \tag{2.14}$$

$$\|x_n - Fx_n\| \le d_n\mu_n + \mu_n, \tag{2.15}$$

$$\|Fx_{n+1} - x_{n+1}\| \le \lambda\mu_n + d_n\mu_n \tag{2.16}$$

are fulfilled.

We have

$$\begin{aligned}\|x_{n+1} - Fx_n\| &= \|F^2x_n - Fx_n + \delta F(Fx_n, x_n)(x_{n+1} - Fx_n)\| \\ &= \|\delta F(Fx_n, x_n)(Fx_n - x_n) + \delta F(Fx_n, x_n)(x_{n+1} - Fx_n)\| \\ &= \|\delta F(Fx_n, x_n)(x_{n+1} - x_n)\| \le d_n\mu_n. \end{aligned}$$

The correctness of inequalities (2.14),(2.15) and (2.16) can be easily verified.

We Prove the following inequalities:

$$\mu_{n+1} \le d_{n+1}\mu_{n+1} + [(a+b) + 2bd_n]d_n\mu_n^2, \tag{2.17}$$

$$d_{n+1} \le d_n + [(a+b)(1+\lambda) + 4bd_n)\mu_n, \quad n = 0, 1, 2, \cdots. \tag{2.18}$$

a) In the expression $\mu_{n+1} = \|x_{n+2} - x_{n+1}\|$, we replace x_{n+2} and x_{n+1} according to the formule (2.4); adding $-Fx_{n+1} + Fx_{n+1}$ and using formula (2.1) for the differences $F^2x_{n+1} - Fx_{n+1}$, $Fx_{n+1} - F^2x_n$, we get

$$\mu_{n+1} = \|\delta F(Fx_{n+1}, x_{n+1})(x_{n+2} - x_{n+1}) + [\delta F(x_{n+1}, Fx_n) - \delta F(Fx_n, x_n)](x_{n+1} - Fx_n)\|.$$

Using the inequalities (2.2),(2.13) and (2.15) we obtain (2.17).

b) By means of the triangle inequality, we get

$$\begin{aligned} d_{n+1} \le & d_n + \|\delta F(Fx_{n+1}, x_{n+1}) - \delta F(x_{n+1}, Fx_n)\| \\ & + \|\delta F(x_{n+1}, Fx_n) - \delta F(Fx_n, x_n)\|. \end{aligned}$$

By (2.2)

$$\begin{aligned} d_{n+1} \le & d_n + a\|Fx_{n+1} - Fx_n\| + b\|Fx_{n+1} - x_{n+1}\| \\ & + 2b\|x_{n+1} - Fx_n\| + b\|Fx_n - x_n\| + a\mu_n. \end{aligned}$$

The formula (2.18) follows at once from (2.13),(2.14),(2.15) and (2.16).

Now, by means of full induction we shall prove the following relations:

a) $d_n \leq d_0 + [(a+b)(1+\lambda)+4b][1+\sigma_{n-1}(h)] \quad \mu_0 < t < 1,$

b) $\mu_n < h^{2^n-1}\mu_0,$

c) $x_n \in D,$

where

$$\sigma_n(h) = \sum_{k=1}^{n} h^{2^k-1},$$
$$\sigma_0(h) = 0 \quad \text{for} \quad n \geq 1.$$

1) Let us put $n = 1$, then we get for $n = 0$ from (2.18) and (2.7), (2.9) the inequality

$$d_1 \leq d_0 + [(a+b)(1+\lambda)+4b]\mu_0 < t < 1.$$

Hence, the inequality a) holds for $n = 1$.

Similarly, from (2.17) we have

$$\mu_1 \leq d_1\mu_1 + [(a+b)+2bd_0]d_0\mu_0^2.$$

From (2.8) and in view of that the inequality $0 < d_1 < t < 1$ holds, we get

$$\mu_1 < \frac{[(a+b)+2bt]t}{1-t}\mu_0^2 = h\mu_0.$$

2) Let the inequalities a),b) be fulfilled for n. According to (2.18), it follows that

$$\begin{aligned} d_{n+1} \leq & d_0 + [(a+b)(1+\lambda)+4b][1+\sigma_{n-1}(h)]\mu_0 \\ & + [(a+b)(1+\lambda)+4bd_n]h^{2^n-1}\mu_0. \end{aligned}$$

Because $d_n < 1$ and $\sigma_{n-1}(h) + h^{2^n-1} = \sigma_n(h) < \sigma(h)$, we obtain relations a) for the index $n+1$, considering supposition (2.9).

Because $d_{n+1} < t < 1$, it follows from (2.17) that

$$\mu_{n+1} < \frac{[(a+b)+2bt]t}{1-t}\mu_n^2 < \frac{[(a+b)+2bt]t}{1-t}\mu_0^2 h^{2^{n+1}-2}.$$

From (2.8) it follows that

$$\mu_{n+1} < h^{2^{n+1}-1}\mu_0.$$

Further, from b) we have

$$\begin{aligned} \|x_{n+1} - x_1\| \leq & \mu_1 + \mu_2 + \cdots + \mu_n \\ & < h[1+\sigma_{n-1}(h^2)]\mu_0 < h[1+\sigma(h^2)]\mu_0. \quad n = 1, 2, \cdots. \end{aligned}$$

Therefore,$x_n \in D$.

From the inequality

$$d_n = \|\delta F(Fx_n, x_n)\| < t < 1$$

it follows that the operators $I - \delta F(Fx_n, x_n)$, $n = 0, 1, \cdots$ have the inverse operators. Therefore, the sequence $\{x_n\}_0^\infty$ is defined by (2.4). Consequently, the inequality

$$\begin{aligned}
&\|x_{m+n} - x_n\| \le \mu_{n+m-1} + \mu_{n+m-2} + \cdots + \mu_{n+1} + \mu_n \\
<&(1 + h^{2^{n+1}-2^n} + \cdots + h^{2^{n+m-2}-2^n} + h^{2^{n+m-1}-2^n})h^{2^n-1}\mu_0 \\
=&h^{2^n-1}\left[1 + \sum_{k=1}^{m-1}(h^{2^n})^{2^k-1}\right]\mu_0 = h^{2^n-1}[1 + \sigma_{m-1}(h^{2^n})]\mu_0.
\end{aligned}$$

holds for arbitrary $m > n \ge n_0 > 1$.

From here it follows that $\{x_n\}$ is a fundamental sequence. R being a complete space, the sequence $\{x_n\}$ possesses the limit element x^*. Of course, $x^* \in D$.

We shall prove the inequality (2.11). Let us denote

$$q = \frac{h}{\mu_0} = \frac{[(a+b) + 2bt]t}{1-t}.$$

Because $d_n < t < 1$ for arbitrary n, we have

$$\mu_{n+1} \le q\mu_n^2. \tag{2.19}$$

From here it is easy to show that

$$\mu_{n+k} \le q^{2^k-1}\mu_n^{2^k}, \quad n = 0, 1, \cdots, \ k = 1, 2, \cdots. \tag{2.20}$$

Then it follows

$$\begin{aligned}
&\|x_{n+m-1} - x_n\| \le \mu_{n+m} + \cdots + \mu_n \\
\le&(q^{2^m-1}\mu_n^{2^m} + q^{2^{m-1}-1}\mu_n^{2^{m-1}} + \cdots + q\mu_n^2 + \mu_n) \\
=&[1 + \sigma_m(q\mu_n)]\mu_n, \quad n = 0, 1, 2, \cdots, \quad m = 1, 2, \cdots.
\end{aligned}$$

In view of the relations

$$\begin{aligned}
&q\mu_n \le qh^{2^n-1}\mu_0 = h^{2^n}, \\
&\mu_n \le q\mu_{n-1}^2 \le q\mu_0 h^{2^{n-1}-1}\mu_{n-1} = h^{2^{n-1}}\mu_{n-1}.
\end{aligned}$$

and the above inequality, it follows that

$$\|x_{n+m+1} - x_n\| \le h^{2^{n-1}}[1 + \sigma_m(h^{2^n})]\mu_{n-1}.$$

Hence for $m \to \infty$ we obtain the estimate (2.11).

The proof of the estimate (2.12) from (2.11) using the inequality b) $\mu_{n-1} \le h^{2^{n-1}-1}\mu_0$ is obvious.

We shall prove that x^* satisfies the equation (2.3). First of all we have

$$||x^* - Fx^*|| \le ||x^* - x_n|| + ||x_n - Fx^*||.$$

In the expression $||x_n - Fx^*||$, we replace x_n acording to the formula (2.4); we add $-Fx_{n-1} + Fx_{n-1}$ and we use formula (2.1) for the difference $F^2x_{n-1} - Fx_{n-1}$. We get

$$||x_n - Fx^*|| \le d_{n-1}||x_n - x_{n-1}|| + ||Fx_{n-1} - Fx^*||.$$

Because $x_{n-1} \in D, x^* \in D$ and $d_{n-1} < t < 1$, we obtain by (2.6)

$$||x^* - Fx^*|| \le ||x^* - x_n|| + t||x_n - x_{n-1}|| + \lambda||x_{n-1} - x^*||.$$

Hence, for $n \to \infty$, we get $||x^* - Fx^*|| \le 0$. Therefore

$$x^* = Fx^*.$$

This completes the proof.

Theorem 2. If the assumptions of Theorem 1 are fulfilled and if $0 < \lambda < 1$ holds, then the equation (2.3) has a unique solution in the sphere D.

Further the inequalities

$$(2.21)\qquad ||x^* - x_n|| \le \frac{[(a+b)+2bt]t}{1-\lambda}||x_n - x_{n-1}||^2, \quad n = 1, 2, \cdots,$$

$$(2.22)\qquad ||x^* - x_n|| \le \frac{[(a+b)+2bt]t}{1-\lambda}h^{2^n-2}||x_1 - x_0||, \quad n = 1, 2, \cdots$$

hold.

Proof. Let us assume that the equation (2.3) has two different solutions x^*, $\tilde{x}$ in the sphere D. Then we get

$$||x^* - \tilde{x}|| = ||Fx^* - F\tilde{x}|| \le \lambda||x^* - \tilde{x}|| < ||x^* - \tilde{x}||.$$

This is a contradiction showing that $x^* = \tilde{x}$.

We shall now prove the estimate (2.21). Let $n \ge 1$. We replace x^* in the expression $||x^* - x_n||$ by Fx^* and we replace x_n according to (2.4); adding $-Fx_n + Fx_n$ and using (2.1) for the difference $Fx_n - F^2x_{n-1}$, we get

$$(2.23)\quad ||x^* - x_n|| = ||Fx^* - Fx_n + [\sigma F(x_n, Fx_{n-1}) - \delta F(Fx_{n-1}, x_{n-1})](x_n - Fx_{n-1})||.$$

We shall use the triangle inequality for the norm of the difference in the square brackets. We shall use the inequality (2.2) and for the difference $Fx^* - Fx_n$ (2.6).

After a slight modification we obtain

$$\|x^* - x_n\| \le \lambda\|x^* - x_n\| + [(a+b) + 2bt]t\|x_n - x_{n-1}\|^2.$$

Because $0 < \lambda < 1$, it follows that

$$\|x^* - x_n\| \le \frac{[(a+b) + 2bt]t}{1-\lambda}\|x_n - x_{n-1}\|^2.$$

From the inequality $\|x_n - x_{n-1}\| \le h^{2^{n-1}-1}\|x_1 - x_0\|$ and (2.21), we get the estimate (2.22).

Theorem 3. Let F be an operator which has the divided difference. Let the following conditions be fulfilled:

1) The inequality

$$\|\delta F(Fx_0, x_0)\| = d_0 < 1 \tag{2.24}$$

holds for the fixed element $x_0 \in R$.

2) The element x_1 is defined by (2.4) and there exists a real number t $(0 < t < 1)$ such that

$$h = h(t) = \frac{[(a+b) + 2bt]t}{1-t}\|x_1 - x_0\| < 1, \tag{2.25}$$

$$d_0 + (a+b)(a+7b)(1+\sigma(h^2)]\|x_1 - x_0\|^2 \tag{2.26}$$

$$+2(a+3b)[1+\sigma(h)]\|x_1 - x_0\| \le t < 1.$$

Then the equation (2.3) has a solution x^* in the sphere

$$D = \{x \in R, \|x - x_1\| \le h[1+\sigma(h^2)]\|x_1 - x_0\|\}.$$

The sequences $\{x_n\}$ defined by formulae (2.4) or (2.5) are convergent in the norm of R to the solution x^* of (2.3) and the error $\|x^* - x_n\|$ of the approximation x_n satisfies

$$\|x^* - x_n\| \le h^{2^{n-1}}[1+\sigma(h^{2^n})]\|x_n - x_{n-1}\|, \tag{2.27}$$

$$\|x^* - x_n\| \le h^{2^n-1}[1+\sigma(h^{2^n})]\|x_1 - x_0\|, \tag{2.28}$$

$$\|x^* - x_n\| \le \frac{\{h^{2^n}(a+b)[1+\sigma(h^{2^n})]^2 + [(a+b) + 2bt]t\}}{1-t}\|x_n - x_{n-1}\|^2, \tag{2.29}$$

$$\|x^* - x_n\| \le \frac{h^{2^{n+1}-2}(a+b)[1+\sigma(h^{2^n})]}{1-t}\|x_1 - x_0\|^2 + h^{2^n-1}\|x_1 - x_0\|, \quad n = 1, 2. \cdots. \tag{2.30}$$

Proof. The main idea of the proof is the same as in Theorem 1. The formulae

$$\mu_{n+1} \le d_{n+1}\mu_{n+1} + [(a+b) + 2bd_n]d_n\mu_n^2, \tag{2.31}$$

$$d_{n+1} \le d_n + 2(a+3b)\mu_n + (a+b)(a+7b)\mu_n^2, \; n = 0, 1, 2, \cdots \tag{2.32}$$

play here the role of formulae (2.17) and (2.18).

The first formula is the same as formula (2.17), the second one will be proved in the following way.

By means of the triangle ineqauality, and using the inequalities (2.2), (2.13) and

$$\|Fx_{n+1} - x_{n+1}\| \le \|Fx_{n+1} - Fx_n\| + \|Fx_n - x_{n+1}\|,$$
$$\|Fx_n - x_n\| \le \|Fx_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

we get

$$\begin{aligned} d_{n+1} \le & d_n + \|\delta F(Fx_{n+1}, x_{n+1}) - \delta F(x_{n+1}, Fx_n)\| + \|\delta F(x_{n+1}, Fx_n) \\ & - \delta F(Fx_n, x_n)\| \\ \le & d_n + (a + b + 4bd_n)\mu_n + (a+b)\|\|\|Fx_{n+1} - Fx_n\|. \end{aligned}$$

Now

$$\begin{aligned} & \|Fx_{n+1} - Fx_n\| \le \|\delta F(x_{n+1}, x_n) - \delta F(x_n, Fx_n) + \delta F(x_n, Fx_n) \\ & - \delta F(Fx_n, x_n) + \delta F(Fx_n, x_n)\|\mu_n \\ \le & [\|\delta F(x_{n+1}, x_n) - \delta F(x_n, Fx_n)\| + \|\delta F(x_n, Fx_n) - \delta F(Fx_n, x_n)\| + d_n]\mu_n \\ \le & [(a+3b)d_n\mu_n + 4b\mu_n + d_n]\mu_n. \end{aligned}$$

From the above inequality we obtain (2.32).

We shall now prove that the relations

(2.33) a) $d_n \le d_0 + (a+b)(a+7b)[1 + \sigma_{n-1}(h^2)]\mu_0^2 + 2(a+3b)[1 + \sigma_{n-1}(h)]\mu_0 < t < 1,$

(2.34) b) $\mu_n \le h^{2^n - 1}\mu_0,$

c) $x_n \in D$

hold for $n = 1, 2, \cdots$. For $n = 1$ it is evident.

Let us suppose that a),b) hold for n. According to (2.32) we have

$$\begin{aligned} & d_{n+1} \le d_0 + (a+b)(a+7b)[1 + \sigma_{n-1}(h^2)]\mu_0^2 + 2(a+3b)[1 + \sigma_{n-1}(h)]\mu_0 \\ & + 2(a+3b)h^{2^n-1}\mu_0 + (a+b)(a+7b)h^{2^{n+1}-2}\mu_0^2 \\ \le & d_0 + (a+b)(a+7b), \\ & [1 + \sigma_n(h^2)]\mu_0^2 + 2(a+3b)[1 + \sigma_n(h)]\mu_0 < t < 1. \end{aligned}$$

The proof of all the assertions of theorem except that $Fx^* = x^*$ and except the proofs of the inequalities (2.29) and (2.30) will be made in the same manner as in Theorem 1.