多复变函数值分布论

Value Distribution Theory in Several Complex Variables

by
Wilhelm Stoll
〔美〕维尔海姆·斯铎尔 著

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Preface

This monograph is an expanded version of a lecture series which I began at Fudan University and continued at the University of Science and Technology, Hefei, Anhui Province, in June 1986. The two Main Theorems of Nevanlinna are derived for meromorphic maps form parabolic manifolds into complex projective space. Since not every detail is proved, this monograph has a bit the character of an outline.

Let M be a connected complex manifold of dimension m. Let V be a complex vector space of dimension $n + 1 > 2$. Let $\mathbb{P}(V)$ be the associated projective space of V. Let $f : M \to \mathbb{P}(V)$ be a meromorphic map. Cartan [8] was the first to study Nevanlinna theory for this situation. In the case $M = \mathbb{C}$ he proved the two Main Theorems using the lemma of the logarithmic derivative. Independently Ahlfors [1] gave a deep alternative proof avoiding the lemma of the logarithmic derivative. Thus Weyl-Weyl [112] were able to lift the theory to Riemann surfaces and I [80], [81] extended their work to Kaehler manifolds M. In the case $M = \mathbb{C}$, Cowen and Griffiths [16] replaced Ahlfors' integral averaging method by curvature considerations. Combining these results and modifying them as needed Wong [113] established the theory on parabolic manifolds M. Here we will follow Wong's method.

In 1986, Steinmetz [77] solved Nevanlinna's conjecture concerning "small" target functions. While at Fudan and in Hefei, I found a proof of a defect relation for holomorphic curves intersecting slowly moving target hyperplanes. However the defect bound was not sharp (Stoll [104]). At this time, I met Ru Min a young bright graduate student in Shanghai. I was able to bring him to Notre Dame. Within three years, he found the way to a proof of the defect relation which gives the exact bound (Ru-Stoll [59], [60]). The moving target results are stated in this monograph, but the proofs are omitted.

The value distribution theory of meromorphic functions of one complex variable contains many beautiful results involving the derivative of the function. However meromorphic maps into projective space have no derivative. The associated maps are no substitute. Thus in section 5, I reformulate the value distribution theory of meromorphic maps into projective space into a value distribution theory of meromorphic vector functions which permits derivatives and partial derivatives of any order. Naturally since each meromorphic vector function is also a meromorphic map Nevanlinna's two Main Theorems still are true. However, I hope that future investigators will be able to derive Second Main Theorems and Defect Relations involving derivatives in an imaginative extension of the one variable theory. With this goal in mind, I draw the readers attention to this section.

The organization of the manuscript is evident from tbe table of content. In order to facilitate the reading, a table of Latin, German and Greek letters is placed at the end of the book. The reference lists considerably more publications than are cited in the manuscript. Thus the reader can easily identify additional literature on the subject matter.

Any account of the 1986 visit to China has to record my wife Marilyn's marathon piano teaching performance at the Anhui Fine Art Institute, 9 hours a day seven days a week from June 8 to July 4 punctured only by very few breaks. One such day "off" was a "Betriebsausflug" with cars to Lake Chow Hu. We crossed the lake to an island on the roof of a boat driven by an ancient motor, climbed a steep mountain in the hot morning sun toward a pagoda, crossed the lake again and visited an old monastery where Marilyn got her future read. After a good meal in the village we returned to continue our duties. Marilyn was the first foreigner to teach at the Anhui Fine Art Institute. and her extraordinary performance left a lasting tinpression. Thus this monograph is properly dedicated to her.

Professor Gong Shen invited us to China and organized the visit. Professor Chen Zhi-hua arranged lectures at Jiao Tong University and East China Normal University. The first part of the lecture series at Fudan was part of the Summer Institute. Professor Chen Huai-Hui invited me to a colloquium at Nanjing Normal University. Professor Gong Shen and Professor and Mrs. Yin Weiping were most helpful while we were in Hefei in particular also in connection with the Art Institute. Professor Yang Lo invited me for one week at the Mathematics Institute. There and at Beijing University, I gave a lecture at each institution on my new result on moving targets for holomorphic curves which I obtained at Fudan and in Hefei (Stoll [104]). At Beijing University I met Professor Chuang Chi-tai, who contributed much to the moving target problem.

After our return nothing was done to publish the lecture series, until Professor Yang Lo visited the University of Notre Dame during the Academic Year 1989/90. He arranged for the publication of an expanded version with the Shandong press. Also he requested that some basic concepts and facts of several complex variables be explained in an appendix for the benefit of experts more familiar with the theory of one complex variable. Later the requirement was softened to "foot notes" which you find now at the end of the text but which in part still are quite long. Unfortunately, other commitments delayed the production of the manuscript again.

I thank all those who made this visit and this publication possible, in particular I am most grateful to Professors Gong Shen and Yang Lo. On the technical side, I thank the secretarial staff of the Notre Dame Mathematics department who word processed the manuscript. In particular I am grateful to Karen Jacobs and Joan Hoerstman who processed the lions share and who patiently made innumerable corrections. I thank my student George Ashline for reading the first proofs. I thank Mr. Tom Ringenberg of the Notre Dame Press for his advice.

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Again I thank you all.

Wilhelm Stoll
November 1993
University of Notre Dame
Notre Dame, IN 46556, USA

CONTENTS

1. A SKETCH OF ONE VARIABLE VALUE DISTRIBUTION THEORY

The Nevanlinna theory of meromorphic functions on the complex plane $\mathbb{C}$ will be outlined in the terminology common to several complex variables. For $0 < r \in \mathbb{R}$ define

$$\mathbb{C}[r] = \{z \in \mathbb{C} \mid |z| \le r\} \qquad \mathbb{C}(r) = \{z \in \mathbb{C} \mid |z| < r\} \quad (1.1)$$

$$\mathbb{C}\langle r\rangle = \{z \in \mathbb{C} \mid |z| = r\} \qquad \mathbb{C}_* = \mathbb{C} - \{0\}. \quad (1.2)$$

An exhaustion $\tau : \mathbb{C} \to \mathbb{R}$ of $\mathbb{C}$ is defined by $\tau(z) = |z|^2$ for all $z \in \mathbb{C}$. On each complex manifold the exterior derivative d operating on differential forms[1] splits into $d = \partial + \overline{\partial}$ and twists to

$$d^c = \frac{i}{4\pi}(\overline{\partial} - \partial). \quad (1.3)$$

Then $\upsilon = dd^c\tau > 0$ is the standard Kaehler form on $\mathbb{C}$. Put $\sigma = d^c \log \tau$. Then $d\sigma = 0$. For $r > 0$ orient the circle $\mathbb{C}\langle r\rangle$ counterclockwise. Then[2]

$$\int_{\mathbb{C}\langle r\rangle} \sigma = 1. \quad (1.4)$$

The Riemann sphere $S = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 | \xi^2 + \eta^2 + \zeta^2 = 1/4\}$ is realized as an Euclidean sphere of diameter 1 and center 0 in $\mathbb{R}^3$. Let Ω be the rotation invariant measure on S with total measure

$$\int_S \Omega = 1. \quad (1.5)$$

Let $\Vert w \wedge a \Vert$ be the chordale distance from $w \in S$ to $a \in S$. If $a \in S$ is fixed, and if $w \in S - \{a\}$ is variable, then $0 < \Vert w \wedge a \Vert \le 1$ and

$$\Omega(w) = -dd^c \log \Vert w \wedge a \Vert^2. \quad (1.6)$$

By means of the stereographic projection, $p : S \to \overline{\mathbb{C}}$, the Riemann sphere can be identified with the point compactification $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$. This identification yields

$$\Omega = dd^c \log(1 + \tau) = \frac{v}{(1+\tau)^2} \text{ on } \mathbb{C}. \qquad (1.7)$$

If $w \in \mathbb{C}$ and $a \in \mathbb{C}$, then

$$[\![w \curlywedge a]\!] = \frac{|w - a|}{\sqrt{1 + |w|^2}\sqrt{1 + |a|^2}} \qquad (1.8)$$

$$[\![w \curlywedge \infty]\!] = \frac{1}{\sqrt{1 + |w|^2}} \qquad [\![\infty \curlywedge \infty]\!] = 0. \qquad (1.9)$$

On the two dimensional complex vector space $\mathbb{C}^2 = \mathbb{C} x \mathbb{C}$ define the following products for $\mathfrak{z} = (z_0, z_1)$ and $\mathfrak{w} = (w_0, w_1)$ in $\mathbb{C}^2$

$$\langle \mathfrak{z}, \mathfrak{w} \rangle = z_0 w_0 + z_1 w_1 \qquad \text{(inner product)} \qquad (1.10)$$

$$\mathfrak{z} \wedge \mathfrak{w} = z_0 w_1 - z_1 w_0 \qquad \text{(exterior product)} \qquad (1.11)$$

$$(\mathfrak{z} | \mathfrak{w}) = z_0 \bar{w}_0 + z_1 \bar{w}_1 \qquad \text{(hermitian product)} \qquad (1.12)$$

$$\|\mathfrak{z}\| = \sqrt{(\mathfrak{z}|\mathfrak{z})} \qquad \text{(norm).} \qquad (1.13)$$

An exhaustion $\bar{\tau} : \mathbb{C}^2 \to \mathbb{R}$ is defined by $\bar{\tau}(\mathfrak{z}) = \|\mathfrak{z}\|^2$ for all $\mathfrak{z} \in \mathbb{C}^2$.

Put $\mathbb{C}^2_* = \mathbb{C}^2 - \{0\}$. The group $\mathbb{C}_*$, operates on $\mathbb{C}^2_*$ by multiplication:

$$a(z_0, z_1) = (az_0, az_1) \qquad (1.14)$$

where a $\in \mathbb{C}$ and $(z_0, z_1) \in \mathbb{C}^2_*$. The quotient space $\mathbb{P}_1 = \mathbb{C}^2_*/\mathbb{C}_*$ is called the complex projective line. It is a connected, simply connected, compact, complex manifold of dimension 1. The residue class map $\mathbb{P} : \mathbb{C}^2_* \to \mathbb{P}_1$ is holomorphic and open.

We have biholomorphic maps $\nu : \mathbb{P}_1 \to S$ and $\mu : \mathbb{P}_1 \to \overline{\mathbb{C}}$ such that $\mu = p \circ \nu$. If $\mathfrak{z} = (z_0, z_1) \in P_1$ and $(\xi, \eta, \zeta) \in S$ and $z \in \mathbb{C}$, then

$$\nu(\mathbb{P}(\mathfrak{z})) = \left(\frac{\bar{z}_0 z_1 + \bar{z}_1 z_0}{2(|z_0|^2 + |z_1|^2}, \frac{\bar{z}_0 z_1 - \bar{z}_1 z_0}{2i(|z_0|^2 + |z_1|^2)}, \frac{|z_1|^2 - |z_0|^2}{2(|z|^2 + |z_0|^2}\right) \qquad (1.15)$$

$$\nu^{-1}(\xi, \eta, \zeta) = \mathbb{P}(1 - 2\zeta, 2(\xi + i\eta)) \qquad (1.15')$$

$$\mu(\mathbb{P}(\mathfrak{z})) = \frac{z_1}{z_0} \text{ if } z_0 \neq 0, \text{ and } \mu(\mathbb{P}(\mathfrak{z})) = \infty \text{ if } z_0 = 0 \qquad (1.16)$$

$$\mu^{-1}(z) = \mathbb{P}((1, z)) \text{ if } z \in \mathbb{C} \text{ also } \mu^{-1}(\infty) = \mathbb{P}(0, 1). \qquad (1.16')$$

With these identifications, we obtain

$$\mathbb{P}^*(\Omega) = dd^c \log\tau \qquad \text{on } \mathbb{C}^2_* \tag{1.17}$$

If $z \in \mathbb{P}_1$ and $w \in \mathbb{P}_1$, then $\mathfrak{z}$ and $\mathfrak{w}$ exist in $\mathbb{C}^2_*$ with $\mathbb{P}(\mathfrak{z}) = z$ and $\mathbb{P}(\mathfrak{w}) = w$. Then

$$\| z \ddot{\wedge} w \| = \frac{|\mathfrak{z} \wedge \mathfrak{w}|}{\|\mathfrak{z}\| \, \|\mathfrak{w}\|}. \tag{1.18}$$

A linear isomorphism $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ induces an automorphism $\alpha : \mathbb{P}_1 \to \mathbb{P}_1$ such that $\alpha \circ \mathbb{P} = \mathbb{P} \circ \alpha$. A linear map $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ is called an isometry if and only if $\|\alpha(\mathfrak{z})\| = \|\mathfrak{z}\|$ for all $\mathfrak{z} \in \mathbb{C}^2$. If α is an isometry, then α is an isomorphism with $(\alpha(\mathfrak{z})|\alpha(\mathfrak{w})) = (\mathfrak{z}|\mathfrak{w})$ and $|\alpha(\mathfrak{z}) \wedge \alpha(\mathfrak{w})| = |\mathfrak{z} \wedge \mathfrak{w}|$ for all $\mathfrak{z}$ and $\mathfrak{w}$ in $\mathbb{C}^2$. Hence

$$\| \alpha(z) \ddot{\wedge} \alpha(w) \| = \| z \ddot{\wedge} w \| \qquad \text{if } z \in \mathbb{P}_1, w \in \mathbb{P}_1. \tag{1.19}$$

The inversion $j(z_0, z_1) = (\bar{z}_1, -\bar{z}_0)$ is an isometry with

$$< \mathfrak{z}, j(\mathfrak{w}) > = \mathfrak{z} \wedge \mathfrak{w}. \tag{1.20}$$

Then $j(z) = 1/z$ if $z \in \mathbb{C}_*$ and $j(0) = \infty$ and $j(\infty) = 0$.

A *divisor* is an integral valued function $\nu : \mathbb{C} \to \mathbb{Z}$ whose support

$$\operatorname{supp} \nu = \{z \in \mathbb{C} | \nu(z) \neq 0\} \tag{1.21}$$

is a closed set of isolated points in $\mathbb{C}$. If $a \in \mathbb{Z}$, then $a\nu$ is also a divisor. If ν_1 and ν_2 are divisors, then $\nu_1 + \nu_2$ is a divisor. Thus the set $\mathfrak{D}$ of divisors on $\mathbb{C}$ is a module. If $\nu \in \mathfrak{D}$, the *counting function* n_ν and the *valence function* N_ν of ν are defined by

$$n_\nu(t) = \sum_{z \in \mathbb{C}[t]} \nu(z) \qquad \text{for } 0 \leq t \in \mathbb{R} \tag{1.22}$$

$$N_\nu(r, s) = \int_s^r n_\nu(t) \frac{dt}{t} \qquad \text{for } 0 < s < r \in \mathbb{R}. \tag{1.23}$$

Here n_ν and N_ν are additive in $\nu \in \mathfrak{D}$. If $\nu \geq 0$, then $n_\nu \geq 0$ and $N_\nu(\cdot, s) \geq 0$ increase.

Let $f \not\equiv 0$ be a holomorphic function on $\mathbb{C}$. Take $z_0 \in \mathbb{C}$. Then there is one and only one holomorphic function g on $\mathbb{C}$, with $g(z_0) \neq 0$ and one and only one non-negative integer p such that

$$f(z) = (z - z_0)^p g(z) \qquad \text{for all } z \in \mathbb{C}. \tag{1.24}$$

then $\mu_f(z_0) = p$ is said to be the *zero multiplicity* of f at z_0 and $\mu_f : \mathbb{C} \to \mathbb{Z}$ is called the *zero divisor* of f. Then supp $\mu_f = f^{-1}(0)$. If $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$ are both holomorphic functions on $\mathbb{C}$ then $\mu_{f_1 \cdot f_2} = \mu_{f_1} + \mu_{f_2}$.

A meromorphic function f on $\mathbb{C}$ is a holomorphic map $f : \mathbb{C} \to \mathbb{P}_1$ with $f \not\equiv \infty$. Also a holomorphic map $f : \mathbb{C} \to \mathbb{P}_1$ with $f \not\equiv \infty$ is a meromorphic function. The concepts coincide. Let $f : \mathbb{C} \to \mathbb{P}_1$ be a holomorphic map. For $0 < t \in \mathbb{R}$, the *spherical image function* A_f of f is defined by

$$A_f(t) = \int_{\mathbb{C}[t]} f^*(\Omega) \geq 0. \tag{1.25}$$

A_f increases. The limit

$$A_f(\infty) = \lim_{t \to \infty} A_f(t) \leq \infty \tag{1.26}$$

exists. $A_f(\infty) = 0$ if and only if f is constant. $A_f(\infty) < \infty$ if and only if f is rational.

For $0 < s < r \in \mathbb{R}$, the (Ahfors-Shimizu) *Characteristic* T_f of f is defined by

$$T_f(r, s) = \int_s^r A_f(t) \frac{dt}{t} \geq 0 \tag{1.27}$$

which increases with r. Then

$$\lim_{r \to \infty} \frac{T_f(r, s)}{\log r} = A_f(\infty). \tag{1.28}$$

A holomorphic vector function $0 \not\equiv \mathfrak{w} = (w_0, w_1) : \mathbb{C} \to \mathbb{C}^2$ is said to be a *representation* of f if $f(z) = \mathbb{P}(\mathfrak{w}(z))$ for all $z \in \mathbb{C}$ with $\mathfrak{w}(z) \neq 0$. A representation $\mathfrak{v} = (v_0, v_1) : \mathbb{C} \to \mathbb{C}^2_*$ is said to be *reduced*. A reduced representation of f exists. An entire function h

exists such that $\mathfrak{w} = h\mathfrak{v}$. The divisor $\mu_{\mathfrak{w}} = \mu_h$ is independent of the choice of $\mathfrak{v}$ and is called the *divisor of* $\mathfrak{w}$. If $0 < s < r \in \mathbb{R}$, then

$$T_f(r,s) = \int_{\mathbb{C}\langle r\rangle} \log \|\mathfrak{w}\|\sigma - \int_{\mathbb{C}\langle s\rangle} \log \|\mathfrak{w}\|\sigma - N_{\mu_{\mathfrak{w}}}(r,s) \quad (1.29)$$

$$T_f(r,s) = \int_{\mathbb{C}\langle r\rangle} \log \|\mathfrak{v}\|\sigma - \int_{\mathbb{C}\langle s\rangle} \log \|\mathfrak{v}\|\sigma. \quad (1.30)$$

Thus if $\alpha : \mathbb{C}^2 \to \mathbb{C}^2$ is an isometry, then $T_{\alpha o f} = T_f$; in particular $T_{1/f} = T_f$. If f_1 and f_2 are meromorphic functions, then there are constants $c_1(s)$ and $c_2(s)$ depending on s such that

$$T_{f_1+f_2}(r,s) \le T_{f_1}(r,s) + T_{f_2}(r,s) + c_1(s) \quad (1.31)$$

$$T_{f_1 f_2}(r,s) \le T_{f_1}(r,s) + T_{f_2}(r,s) + c_2(s) \quad (1.32)$$

for all $r > s$.

Let $f : \mathbb{C} \to \mathbb{P}_1$ and $g : \mathbb{C} \to \mathbb{P}_1$ be different holomorphic maps. Take reduced representations $\mathfrak{v}$ of f and $\mathfrak{w}$ of g. Then $\mathfrak{v} \wedge \mathfrak{w} \not\equiv 0$ is an entire function. Its zero divisor $\mu_{f\wedge g} = \mu_{\mathfrak{v}\wedge\mathfrak{w}}$ does not depend on the choice of $\mathfrak{v}$ and $\mathfrak{w}$. Abbreviate $n_{f\wedge g} = n_{\mu_{f\wedge g}}$ and $N_{f\wedge g} = N_{\mu_{f\wedge g}}$.

For $r > 0$ the compensation function $m_{f\wedge g}$ of the pair f, g is defined by

$$m_{f\wedge g}(r) = \int_{\mathbb{C}\langle r\rangle} \log \frac{1}{\|f \wedge g\|}\sigma > 0. \quad (1.33)$$

If $0 < s < r \in \mathbb{R}$ the *First Main Theorem* states

$$\boxed{\begin{aligned} T_f(r,s) + T_g(r,s) = N_{f\wedge g}(r,s) &+ m_{f\wedge g}(r) \\ &- m_{f\wedge g}(s) \end{aligned}} \quad (1.34)$$

If $g \equiv a$ is constant, then $T_g \equiv 0$ and we obtain *Nevanlinna's First Main Theorem*

$$\boxed{T_f(r,s) = N_{f\wedge a}(r,s) + m_{f\wedge a}(r) - m_{f\wedge a}(s)} \quad (1.35)$$

Easily we calculate

$$\int_{a\in\mathbb{P}_1} \log \frac{1}{\|w\ddot{\wedge}a\|}\ \Omega(a) = \frac{1}{2}. \tag{1.36}$$

An exchange of integral yields

$$\int_{a\in\mathbb{P}_1} m_{f,a}(r)\ \Omega(a) = \frac{1}{2}. \tag{1.37}$$

Thus (1.37) and (1.35) imply

$$\int_{a\in\mathbb{P}_1} N_{f\wedge a}(r,s)\ \Omega(a) = T_f(r,s) \tag{1.38}$$

while (1.35) shows

$$N_{f\wedge a}(r,s) \leq T_f(r,s) + m_{f,a}(s). \tag{1.39}$$

Thus $N_{f\wedge a}$ must grow about as quickly as T_f for almost all a. The Second Main Theorem and the Defect Relation are refinements of this observation.

Let $f : \mathbb{C} \to \mathbb{P}_1$ and $g : \mathbb{C} \to \mathbb{P}_1$ be different holomorphic maps and assume that at least one of them is not constant. Then $T_f(r,s)+T_g(r,s) \to \infty$ for $r \to \infty$; thus, by (1.34) the defect $\delta(f\ddot{\wedge}g)$ can be defined by

$$\begin{aligned} 0 \leq \delta(f\ddot{\wedge}g) &= \liminf_{r\to\infty} \frac{m_{f,g}(r)}{T_f(r,s)+T_g(r,s)} \\ &= 1 - \limsup_{r\to\infty} \frac{N_{f\wedge g}(r,s)}{T_f(r,s)+T_g(r,s)} \leq 1 \end{aligned} \tag{1.40}$$

The function g is said to grow slower than f if and only if

$$\frac{T_g(r,s)}{T_f(r,s)} \to 0 \quad \text{for} \quad r \to \infty. \tag{1.41}$$

The set $\mathfrak{K}(f)$ of a meromorphic functions g on $\mathbb{C}$ which grow slower than f is a field. Let $\mathfrak{G}$ be a finite subset of $\mathfrak{K}(f) \cup \{\infty\}$. In 1986 Steinmetz [77] proved the *Defect Relation*

$$\boxed{\sum_{g\in\mathfrak{G}} \delta(f \dot{\wedge} g) \leq 2} \tag{1.42}$$

If $\mathfrak{G}$ consists of constant maps g only, this defect relation is due to Nevanlinna [45] with collaboration by Collingwood and Littlewood in 1923–24. In this case, the defect relation is the immediate result of the *Second Main Theorem*:

Let $\mathfrak{G}$ be a finite subset of $\mathbb{P}_1$. Let $f : \mathbb{C} \to \mathbb{P}_1$ be a non-constant holomorphic map. Let $\mathfrak{v} : \mathbb{C} \to \mathbb{C}^2$ be a reduced representation of f. Then the divisor ρ of $\mathfrak{v} \wedge \mathfrak{v}' \not\equiv 0$ does not depend on the choice of $\mathfrak{v}$. Take $s \in \mathbb{R}^+ = \{x \in \mathbb{R} | x > O\}$. Then there is a constant $c_s > 0$ and a subset E_s of $\mathbb{R}^+$ of finite measure such that

$$N_\rho(r,s) + \sum_{a\in\mathfrak{G}} m_{f\dot{\wedge}a}(r) \leq 2T_f(r,s) + c_s \log (r\ T_f(r,s)) \tag{1.43}$$

for all $r \in \mathbb{R}^+ - E_s$ with $r > s$.

2. THE GREEN RESIDUE THEOREM ON PARABOLIC MANIFOLDS

A. Parabolic manifolds

Parabolic manifolds are convenient domains for value distribution theory. Let M be a connected complex manifold of dimension m. Let $\tau \geq 0$ be a function on M. For $A \subseteq M$ and $r \in \mathbb{R}^+$ define

$$A[r] = \{x \in A | \tau(x) \leq r^2\} \quad A(r) = \{x \in A | \tau(x) < r^2\} \quad (2.1)$$

$$A\langle r\rangle = \{x \in A | \tau(x) = r^2\} \qquad A_* = \{x \in A | \tau(x) > 0\}. \quad (2.2)$$

An unbounded, continuous, non-negative function τ on M is said to be an *exhaustion* of M, if $M[r]$ is compact for every $r > 0$. If so, then $M[r]$ is called the *closed pseudoball*, $M(r)$ is called the *open pseudoball* and $M\langle r\rangle$ the *pseudosphere*, all of radius r. If $\tau \geq 0$ is a function of class C^∞ on M and if $M_* \neq \emptyset$ for this τ, define

$$\upsilon = dd^c\tau \qquad \omega = dd^c \log \tau \qquad \sigma = d^c \log \tau \wedge \omega^{m-1} \quad (2.3)$$

on M respectively M_*. The function τ is said to be *parabolic* on M if

$$\omega \geq 0 \qquad d\sigma = \omega^m \equiv 0 \not\equiv \upsilon^m \qquad \text{on } M_*. \quad (2.4)$$

Then $\upsilon \geq 0$ on M. If in addition $\upsilon > 0$ on M, then τ is said to be *strictly parabolic*. If τ is a parabolic exhaustion of M, then (M, τ) is called a *parabolic manifold*, *strict* if τ is strict.

There are many examples of parabolic manifolds. A non-compact Riemann surface is parabolic if and only if every subharmonic function bounded above is constant. If $(M_1, \tau_1), \ldots, (M_n, \tau_n)$ are parabolic manifolds, define $M = M_1 x \ldots x M_n$ and define τ by

$$\tau(x_1, \ldots, x_n) = \tau_1(x_1) + \ldots + \tau_n(x_n) \quad (2.5)$$

for all $(x_1, \ldots, x_n) \in M$. Then (M, τ) is a parabolic manifold. In particular $(\mathbb{C}^m, \tau_0)$ is a strictly parabolic manifold with

$$\tau_0(z_1, \ldots, z_m) = |z_1|^2 + \ldots + |z_m|^2 \tag{2.6}$$

for all $(z_1, \ldots, z_m) \in \mathbb{C}^m$.

Let $\tilde{M}$ and M be connected, complex manifolds of dimension m. Let $\pi : \tilde{M} \to M$ be a proper, surjective holomorphic map. Let τ be a parabolic exhaustion of M. Then $\tilde{\tau} = \tau o \pi$ is a parabolic exhaustion of M. Here $(\tilde{M}, \tilde{\tau})$ is called a parabolic covering space of (M, τ). A connected, m-dimensional, affine algebraic manifold M can be spread over $\mathbb{C}^m$ by a surjective, holomorphic, proper map $\pi : M \to \mathbb{C}^m$. Thus connected, affine algebraic manifolds are parabolic but $\mathbb{C}^{m-1} x (\mathbb{C} - \mathbb{Z})$ is parabolic but not affine algebraic. There are not many strictly parabolic manifolds.

Uniformization Theorem (Stoll [97]) If (M, τ) is a strictly parabolic manifold of dimension m, then there is a biholomorphic map $h : M \to \mathbb{C}^m$ which is an isometry of exhaustions $\tau = \tau_0 \circ h$.

Let (M, τ) be a parabolic manifold. Let $\mathfrak{E}_\tau$ be the set of all $r \in \mathbb{R}^+$ such that $M(r)$ is a Stokes domain.[3] Let $\hat{\mathfrak{E}}_\tau$ be the set of all $r \in \mathbb{R}^+$ such that $d\tau(x) \neq 0$ for all $x \in M\langle r \rangle$. Then $\hat{\mathfrak{E}}_\tau \subseteq \mathfrak{E}_\tau$ and $\mathbb{R}^+ - \hat{\mathfrak{E}}_\tau$ has measure zero by Sard's Theorem. If $r \in \mathfrak{E}_\tau$, abbreviate $\int_{M\langle r \rangle} = \int_{dM(r)}$. If $r \in \hat{\mathfrak{E}}_\tau$, then $M\langle r \rangle = \partial M(r) = dM(r)$. If $0 < s < r \in \mathfrak{E}_\tau$ with $s \in \mathfrak{E}_\tau$, Stokes theorem implies

$$\int_{M\langle r \rangle} \sigma - \int_{M\langle s \rangle} \sigma = \int_{M(r) - M[s]} d\sigma = 0. \tag{2.7}$$

Thus

$$\varsigma = \int_{M\langle r \rangle} \sigma \tag{2.8}$$

is constant. We have

$$\tau^2 \omega = \tau \upsilon - d\tau \wedge d^c\tau. \tag{2.9}$$

Since $d\tau \wedge d\tau = 0 = d^c\tau \wedge d^c\tau$, the binomial theorem implies

$$\tau^{p+1} \omega^p = \tau \upsilon^p - p d\tau \wedge d^c\tau \wedge \upsilon^{p-1}. \tag{2.10}$$