數學科技叢書18(上)

APOSTOL

微積分題解(上)

亞帕斯特　著　　張中堯等編寫

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序

信步於杜鵑花城附近的書舖，大多數「新鮮人」都會因琳瑯滿目的微積分題解而佇足迷惘。的確，際此微積分解答充斥之時，本書是否有刋行的必要？每位讀過 Apostol 初微或高微的同學，對於這個問題的看法，許是和作者相去不遠。

Apostol 的微積分以其深入淺出的內容，一向在數學界享有令譽，特別是其精彩絕倫的習題，更令人歎爲觀止，這是我們爲何會懷著肅然起敬的心情出版本書，也是水牛書局繼出版 Johnson 微積分解答之後仍然推出本書的原因。

由於原書的習題過多，因此我們刪去了部分較基本的練習題，在編排方面，爲了使讀者易於參閱起見，完全依著原書的章節，順序及題號，所以讀者將很清楚地發現我們所刪除的，並非是我們解不出來的難題，而是些相當 trivial 的題目。

作者解題時，力求詳盡，因此具有中等程度的學生使用本書時將不會感到太吃力。

有位名教授的補考方式是在假期內作完五百題微積分，的確，要想學好微積分、打好數學的根基，唯有多作練習，而本書對於加強讀者的演練技巧將有極大的助益。

因爲印刷技術上的關係，我們用 O 來表示 little O ,（絕大部分在第七章），而爲免混淆起見，我們也儘量避免使用 big O 解題。

爲了便利讀者起見，我們特別附上原書題目，以便重溫微積分或參加各種考試的讀者。

Apostol 的原書既是微積分中的佼佼者，而本書又擷取了其中的精華，因此本書之刊行，顯非作者個人之力，在此，要向同學程君及周君，深致謝意。

作者於

醉月湖畔

目　　次(上册)

第　一　章

第　二　章

第　三　章

第　四　章

第　五　章

第 一 章

2. If x is an arbitrary real number, prove that there are integers m and n such that $m < x < n$.

[Sol]

For every real x there exists a positive integer n such that

· $n > x$, So

(1) If $x \geqq 0$ then $x > -1 = m$

(2) If $x < 0$ then $-x > 0$

and there exists a positive integer $m \in N$ such that

$-m > -x$, so $x > m$ and $m \in I$

From(1) and (2), we know there exists m, $n \in I$ such that

$m < x < n$

4. If x is an arbitrary real number, prove that there is exactly one integer n which satisfies the inequalities $n \leq x < n + 1$. This n is called the greatest integer in x and is denoted by $[x]$. For example, $[5] = 5$, $[\frac{5}{2}] = 2$, $[-\frac{8}{3}] = -3$.

[Sol]

由上題知 For any $x \in R$, there exists $p, q \in I$ such that

$p < x < q$, 令 $P = \{ p_1 | p_1 \leqq x < q, \ p_1 \in I \}$

因爲 $p \in P$ 所以 $P \neq \phi$ 又對任一 $p_1 < q$ 所以 q 爲 P 之一上界，而

sup P 存在令 sup $P=n$ 則 $n\leqq x$

若 $x\geqq n+1$ 則因 $n+1\in I$ 而 $n+1\leqq x<q$ 所以 $n+1\in P$

所以 $n+1\leqq \sup P=n$ 此爲不合理 故 $n\leqq x<n+1$ 因此 $n=\sup P$

爲惟一之値，所以恰存在 $-n$ 使得任一 $x\in R$ 皆滿足 $n\leqq x<n+1$

5. If x is an arbitrary real number, prove that there is exactly one integer n which satisfies $x\leq n<x+1$.

[Sol]

由第1題知有二整數 p,q 存在，對任一實數 x 滿足 $p<x<q$

令 $Q=\{q_1 \mid x\leqq q_1,\ q_1\in I\}$ 故 $\inf Q$ 存在，令 $\inf Q=n$

由整序性知 $n\in I$

若 $n\geqq x+1$ 則 $n-1\geqq x$，$n-1\in I$ 故 $n-1\in Q$

而 $n-1>\inf Q=n$ 此爲不合理，故 $n<x+1$

所以恰有一個 n 存在使得 $x\leqq n<x+1$

6. If x and y are arbitrary real numbers, $x<y$, prove that there exists at least one rational number r satisfying $x<r<y$, and hence infinitely many. This property is often described by saying that the rational numbers are *dense* in the real-number system

[Sol]

令 $0<a<b$ 則存在 $n_1a>1$ 又 $b-a>0$ 則存在 $n_2(b-a)>1$

令 $n=\max\{n_1,n_2\}$ 則 $na>1$ $n(b-a)>1$

取 $m\in N$ 使得 $m>na(\geqq 1)$ 若 $m-1>na$ 仍成立則取 $m-1$

若 $m-1>na$ 仍成立則改取 $m-2$，餘此類推.

在有限步驟後必有 $-k\in N$ 使 $k>na(\geqq 1)$ 而 $k-1\leqq na$

所以 $a<\dfrac{k}{n}$ 又 $nb-na>1$，$na\geqq k-1$ 所以 $nb>k$

故 $b>\dfrac{k}{n}$ 所以有 k,b 存在，使 $a<\dfrac{k}{n}<b$

x, y 表二實數且 $x<y$ 由阿基米德性質知必有一 $n\epsilon N$ 存在，

使 $n\cdot 1>-x$ 即使 $n+x>0$ 而 $n+y>n+x>0$

所以 $n+y$， $n+x$ 均爲正實數，由上題知必有一有理數 r

使 $n+x<r<n+y$ 即 $x<r-n<y$ 而 $r-n$ 爲有理數故得證

12. The Archimedean property of the real-number system was deduced as a consequence of the least-upper-bound axiom. Prove that the set of rational numbers satisfies the Archimedean property but not the least-upper-bound property. This shows that the Archimedean property does not imply the least-upper-bound axiom.

對任一 p ，$q\epsilon Q$ 則 p ，$q\epsilon \mathbf{R}$ 故存在一 n 使得 $np>q$ 證明有理數集合不滿足 l.u.b Axiom用 I3·15 節的 *decimal expansion*

例如 $\sqrt{2}$ 令 $b_n=a_0+\frac{a_1}{10}+\cdots+\frac{a^n}{10^n}<\sqrt{2}$

則 $\{b_n\}(n\epsilon N)$ 是 Bounded above 的有理數，但它的 l.u.b 是 $\sqrt{2}$， $\sqrt{2}\neq Q$ 故得證

10 Let b denote a fixed positive integer. Prove the following statement by induction: For every integer $n\geq 0$, there exist nonnegative integers q and r such that

$$n=qb+r, \quad 0\leq r<b.$$

$b\epsilon N$， $b\geq 1$

$n=0$ 時 $0=0\cdot b+0$

$n=k$ 時爲眞 即存在 q ，r 使得 $k=qb+r$ $(b>r\geq 0)$

$n=k+1$ $\quad k+1=qb+r+1$ $\qquad (b>r\geq 0$ 故 $b\geq r+1>0)$

若 $b=r+1$ 則 $k+1=(q+1)b$

若 $b>r+1$ 則 $k+1=qb+(r+1)$ $(b>r+1>0)$ 故得證

11 Let n and d denote integers. We say that d is a *divisor* of n if $n=cd$ for some integer c. An integer n is called a *prime* if $n>1$ and if the only positive divisors of n are 1 and n. Prove by induction, that every integer $n>1$ is either a prime or a product of primes.

$n, d\epsilon I$ $\quad n=2$ $\quad$ 2 爲一 prime

令 $n=2, 3, \cdots k-1$ 時爲眞

欲證 $n=k$ 時亦爲眞，若 k 是 prime number 則得證

若 k 不是 prime number 則 $k=m\cdot n$, $m\neq 1$, $n\neq 1$,

且 $2\leq m<k$, $2\leqq n<k$ ，由假設知 m , n 或爲 prime number

或爲 product of prime number 故 k is a product of prime

number 得證

12 Describe the fallacy in the following "proof" by induction

Statement Given any collection of n blonde girls If at least one of the girls has blue eyes, then all n of them have blue eyes

"*Proof*" The statement is obviously true when $n = 1$. The step from k to $k + 1$ can be illustrated by going from $n = 3$ to $n = 4$ Assume, therefore, that the statement is true when $n = 3$ and let G_1, G_2, G_3, G_4 be four blonde girls, at least one of which, say G_1, has blue eyes. Taking G_1, G_2, and G_3 together and using the fact that the statement is true when $n = 3$, we find that G_2 and G_3 also have blue eyes Repeating the process with G_1, G_2, and G_4, we find that G_4 has blue eyes. Thus all four have blue eyes. A similar argument allows us to make the step from k to $k + 1$ in general

Corollary. All blonde girls have blue eyes

Proof Since there exists at least one blonde girl with blue eyes, we can apply the foregoing result to the collection consisting of all blonde girls

Note· This example is from G Polya, who suggests that the reader may want to test the validity of the statement by experiment

當 $n=k$ 爲眞欲證 $n=k+1$ 亦眞時，令 G_1 是 blue eyes，給 G_1 配上 $(k-1)$ 個 G_i ($i=2$, ‥, $k+1$)，然後運用 $n=k$ 成立之假設，證明 G_i 亦爲 blue eyes。

但此證明需要一個先決條件卽 $k+1>2$ 但 $k+1=2$ 時，顯然此論證法不能用 $k=1$ 爲眞之假設來證。

卽這個論證法在 $k=1$ 到 $k=2$ 時不能成立，此爲其證明謬誤之處。

9 Prove, by induction, that the sum $\sum_{k=1}^{2n}(-1)^k(2k+1)$ is proportional to n, and find the constant of proportionality

[Sol] $\sum_{k=1}^{2n}(-1)^k(2k+1)=-3+5-7+\cdots+(4n+1)=2n$

當 $n=1$ 時 $\sum_{k=1}^{2}(-1)^k(2k+1)=-3+5=2\cdot 1=2n$

令 $n=p$ 時成立卽 $\sum_{k=1}^{2n}(-1)^k(2k+1)=2p$

當 $n=p+1$ 時

$$\sum_{k=1}^{2(p+1)}(-1)^k(2k+1)$$

$$=\sum_{k=1}^{2p}(-1)^k(2k+1)+(-1)^{2p+1}(4p+3)+(-1)^{2p+2}(4p+5)$$

$$=2p-(4p+3)+(4p+5)=2p+2=2(p+1)$$

由歸納法知 $\sum_{k=1}^{2n}(-1)^k(2k+1)=2n$

13. Prove that $2(\sqrt{n+1}-\sqrt{n})<\frac{1}{\sqrt{n}}<2(\sqrt{n}-\sqrt{n-1})$ if $n\geq 1$. Then use this to prove that

$$2\sqrt{m}-2<\sum_{n=1}^{m}\frac{1}{\sqrt{n}}<2\sqrt{m}-1$$

if $m\geq 2$. In particular, when $m=10^6$, the sum lies between 1998 and 1999.

[Sol] $2(\sqrt{n+1}-\sqrt{n})<\frac{1}{\sqrt{n}}<2(\sqrt{n}-\sqrt{n-1})$

(1) $\frac{1}{2(\sqrt{n+1}-\sqrt{n})}=\frac{\sqrt{n+1}+\sqrt{n}}{2}>\frac{\sqrt{n}+\sqrt{n}}{2}=\sqrt{n}$

所以 $2(\sqrt{n+1}-\sqrt{n})<\frac{1}{\sqrt{n}}$

(2) $$\frac{1}{2(\sqrt{n}-\sqrt{n-1})}=\frac{\sqrt{n}+\sqrt{n-1}}{2}<\frac{\sqrt{n}+\sqrt{n}}{2}=\sqrt{n}$$

故 $2(\sqrt{n}-\sqrt{n-1})>\frac{1}{\sqrt{n}}$ 得證

$$2(\sqrt{n+1}-\sqrt{n})<\frac{1}{\sqrt{n}}<2(\sqrt{n}-\sqrt{n-1})$$

$$\sum_{n=1}^{m} 2(\sqrt{n+1}-\sqrt{n})<\sum_{n=1}^{m}\frac{1}{\sqrt{n}}$$

而 $\sum_{n=1}^{m} 2(\sqrt{n+1}-\sqrt{n})=2\sqrt{m+1}-2>2\sqrt{m}-2$

then $\sum_{n=1}^{m}\frac{1}{\sqrt{n}}=1+\sum_{n=2}^{m}\frac{1}{\sqrt{n}}<1+\sum_{n=2}^{m}2(\sqrt{n}-\sqrt{n-1})$

$$=1+(2\sqrt{m}-2)=2\sqrt{m}-1$$

12. (a) Use the binomial theorem to prove that for n a positive integer we have

$$\left(1+\frac{1}{n}\right)^n=1+\sum_{k=1}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}.$$

(b) If $n>1$, use part (a) and Exercise 11 to deduce the inequalities

$$2<\left(1+\frac{1}{n}\right)^n<1+\sum_{k=1}^{n}\frac{1}{k!}<3$$

[Sol]

(a) $$1+\sum_{k=1}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}=1+\sum_{k=1}^{n}\frac{n\cdots(n-k+1)}{k!\ n^k}$$

$$=1+\sum_{k=1}^{n}\left(\frac{1}{n}\right)^k\binom{n}{k}$$

$$=\sum_{k=0}^{n}\left(\frac{1}{n}\right)^k\binom{n}{k}=\left(1+\frac{1}{n}\right)^n$$

(b) $2^k<k!$ 所以 $\frac{1}{2^k}>\frac{1}{k!}$

$$\sum_{k=1}^{n}\frac{1}{k!}<\sum_{k=1}^{n}\frac{1}{2^k}<2$$

$$\therefore\quad 1+\sum_{k=1}^{n}\frac{1}{k!}<3$$

Binomial theorem $\quad (a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$

(a) $\quad (1+\frac{1}{n})^n=1+\sum_{k=1}^{n}\{\frac{1}{k!}\prod_{r=0}^{k-1}(1-\frac{r}{n})\}$

利用(a) exercise 11. $\quad n\geqq 4$ 時 $2^n<n!$

$$(1+\frac{1}{n})^n=1+\sum_{k=1}^{n}\frac{1}{k!}\prod_{r=0}^{k-1}(1-\frac{r}{n})<1+\sum_{k=1}^{n}\frac{1}{k!}$$

因爲 $\prod_{r=0}^{k-1}(1-\frac{r}{n})<1$ 　　當 $k\geq 2$ 時

$$\sum_{k=1}^{n}\frac{1}{k!}=1+\frac{1}{2}+\frac{1}{6}+\sum_{k=4}^{n}\frac{1}{k!}<\frac{5}{3}+\sum_{k=4}^{n}\frac{1}{2^k}$$

$$=\frac{5}{3}+\frac{\frac{1}{2^4}(1-(\frac{1}{2})^n)}{(1-\frac{1}{2})}<\frac{5}{3}+\frac{1}{2^3}$$

$$=\frac{40+3}{24}=\frac{43}{24}<2$$

故 $1+\sum_{k=1}^{n}\frac{1}{k!}<3$

$$\sum_{k=1}^{n}\{\frac{1}{k!}\prod_{r=0}^{k-1}(1-\frac{r}{n})\}=1+\sum_{k=2}^{n}\{\frac{1}{k!}\prod_{r=0}^{k-1}(1-\frac{r}{n})\}>1$$

所以 $(1+\frac{1}{n})^n > 1+1=2$

13. (a) Let p be a positive integer. Prove that

$$b^p - a^p = (b-a)(b^{p-1} + b^{p-2}a + b^{p-3}a^2 + \cdots + ba^{p-2} + a^{p-1}).$$

[*Hint:* Use the telescoping property for sums.]

(b) Let p and n denote positive integers. Use part (a) to show that

$$n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p.$$

[Sol]

(a) $(b-a)(b^{p-1}+b^{p-2}a+b^{p-3}a^2+\cdots+ba^{p-2}+a^{p-1})$

$= b(b^{p-1}+b^{p-2}a+\cdots+a^{p-1}) - a(b^{p-1}+b^{p-2}a+\cdots+a^{p-1})$

$= \sum_{k=1}^{p} b^k a^{p-k} - \sum_{k=1}^{p} a^k b^{p-k}$

$= \sum_{k=1}^{p} (b^k a^{p-k} - a^k b^{p-k})$ 但 $\sum_{k=1}^{p} a^k b^{p-k} = \sum_{k=1}^{p} a^{p-k+1} b^{k-1}$

$= \sum_{k=1}^{p} (a^{p-k} b^k - a^{p-k+1} b^{k-1}) = b^p - a^p$ #

(b) 由(a)

$(n+1)^{p+1} - n^{p+1}$

$= [(n+1)-n][(n+1)^p + (n+1)^{p-1}n + \cdots + (n+1)n^{p-1} + n^p]$

$= (n+1)^p + (n+1)^{p-1}n + \cdots + (n+1)n^{p-1} + n^p$

$< (n+1)^p + (n+1)^{p-1}(n+1) + \cdots + (n+1)(n+1)^{p-1} + (n+1)^p$

$= (p+1)(n+1)^p$

又 $(n+1)^p + (n+1)^{p-1}n + (n+1)^{p-2}n^2 + \cdots + (n+1)n^{p-1} + n^p$

$$> n^p + n^{p-1} n + n^{p-2} n^2 + \cdots + n \cdot n^{p-1} + n^p$$

$$= (p+1) n^p$$

所以 $(p+1) n^p < (n+1)^{p+1} - n^{p+1} < (p+1)(n+1)^p$

即 $$n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$$

(c) Use induction to prove that

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p .$$

Part (b) will assist in making the inductive step from n to $n+1$

(c)(1) $n = 2$ 時 $\sum_{k=1}^{2-1} k^p = 1^p = 1$

$$\frac{2^{p+1}}{p+1} = \frac{2^{p+1}}{p+1}$$

$$\sum_{k=1}^{2} k^p = 1^p + 2^p = 1 + 2^p$$

$$\frac{2^{p+1}}{p+1} - 1 = \frac{2^{p+1} - (p+1)}{p+1} = \frac{2^{p+1} - 1 - p}{p+1} > 0$$

（因為由(b) $\frac{2^{p+1} - 1}{p+1} > 1$

所以 $2^{p+1} - 1 > p+1$ 即 $2^{p+1} - 1 - p > 1 > 0$）

$$1 + 2^p - \frac{2^{p+1}}{p+1} = 2^p - \frac{2^{p+1}}{p+1} + 1 > 0$$

（因為由(b) $2^p > \frac{2^{p+1} - 1}{p+1}$