点筛法

Point Sieve Method

包那 著

内蒙古大学出版社

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序

“有人成功地发现了他人一直未能发现的事物；在上一世纪尚混沌一片的东西，在这一世纪却豁然开朗了；科学和艺术是在反反复复的加工提炼中逐渐成型的。我没有放弃用各种方法去探索、去尝试我的力量还不足以发现的东西，我为后人创造方便，提供更灵活便得的方法。我的后继者也会作出同样的努力。这就是困难不能使我们失望，也不能让我们无所作为的原因。”(蒙田（Montaigne)：《随笔》)。论作者的条件，写一本书是很困难的，正是由于蒙田所说的原因，这本书还是写下来了。并且这个原因还是本书的主要目的之一。本书的几乎全部内容，都是作者本人所作的工作。同一个方面的工作归为一章，同一章各节的内容各自形成完整，各有各的参考文献。

在本书的写作过程中得到了内蒙古师范大学数学系吴·嘎日迪同志的大力协助，内蒙古师范大学计算机系王强同志为本书的电脑排版作了大量的工作，对他们所给予的帮助表示诚挚的谢意。

由于时间比较仓促，加上作者的水平有限，书中考虑不周，不妥和遗漏之处一定会有，甚至缺点错误也在所难免。敬请读者指正。

包　那

一九九五年五月

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POINT SIEVE METHOD

§ 1　Introduction

Sieve method is one kind important method of studing famous difficult problem of number theory in world, i,e. Goldbach's supposition. In the past two hundred years, many mathematician having studied the Goldbach's supposition from various not alike point of view, and obtained plentiful methods and results. In which some important results have been obtained by using sieve method. But it is on very difficult problem to carry forward the studing progress of Goldbach's supposition, and many people expect to have new studing methods. In this paper, we produce a new method of studing Goldbach's supposition which called point sieve method.

Let N is a large even number, whole point set

$$A = \{ (n,\ N-n) | n = O,\ 1,\ 2,\ \cdots,\ N-1 \} \tag{1}$$

Suppose addition

$$\boldsymbol{P} = \{ P | P \text{ be prime number, and } P \leqslant Ne^{-\gamma} \} \tag{2}$$

where γ be Euler constant. In view of $e^{-\gamma} > e^{-\ln 2} = \frac{1}{e^{\ln 2}} = \frac{1}{2}$, hence $N^{e^{-\gamma}} > \sqrt{N}$, and

$$\boldsymbol{P}_0 = \{ P | P \text{ be prime number, and } P \leqslant \sqrt{N} \} \subset \boldsymbol{P} \tag{3}$$

Denote $N_0 = Ne^{-\gamma}$, $P_1, P_2, \cdots, P_{\pi(N_0)}$ denote the front $\pi(N_0)$ amount of prime numbers.

In the following recount, the first coordinate of whole point $(n, N-n)$ is called x coordinate, the second coordinate is called y coordinate. We abandon every points from A that x coordinate of which whole divided by P_1, denote the set that is consist of all remained points and the number of its points by $A(P_1(x))$ and $N\Big(A(P_1(x))\Big)$ respectively. We call this process that the point set A be sieved by using $P_1 x$; we abandon every points from $A(P_1(x))$ that y coordinate of which whole divided by P_1, denote the set that is consist of all remained points and the number of its points by $A(P_1(x,y))$ and $N\Big(A(P_1(x, y))\Big)$ respectively. We call this process that the point set $A(P_1(x))$ be sieved by using $P_1 y$; We call above two process that the set A be sieved by P_1.

We sieve again $A(P_1(x, y))$ by using $P_2 x$, and denote the set that is consist of all remained points and the number of its points by $A(P_1(x, y), P_2(x))$ and $N\Big(A(P_1(x, y), P_2(x))\Big)$ respectively. Sieve $A(P_1(x, y), P_2(x))$ by using $P_2 y$, denote the set that is consist of all remained points and the number of its points by $A(P_1(x, y), P_2(x, y))$ and $N\Big(A(P_1(x, y), P_2(x, y))\Big)$ respectively. And so on and so forth, untill sieve point set $A(P_1(x, y), P_2(x, y), \cdots, P_{\pi(N_0)-1}(x, y))$ by using $P_{\pi(N_0)} x$ denote the set that is consist of all remained points and the number of its points by $A(P_1(x, y), P_2(x, y), \cdots, P_{\pi(N_0)-1}(x, y), P_{\pi(N_0)}(x))$ and $N\Big(A(P_1(x, y), P_2(x, y), \cdots, P_{\pi(N_0)-1}(x, y),$

$P_{\pi(N_0)}(x))\Big)$. Sieve $A(P_1(x,y),P_2(x,y),\cdots,P_{\pi(N_0)-1}(x,y),P_{\pi(N_0)}(x))$ by using $P_{\pi(N_0)}y$,denote the set that is consist of all remained points and the number of its points by $A(P_1(x,y),\cdots,P_{\pi(N_0)-1}(x,y),P_{\pi(N_0)}(x,y))$ and $N\Big(A(P_1(x,y),\cdots,P_{\pi(N_0)-1}(x,y),P_{\pi(N_0)}(x,y))\Big)$. We call above whole process that the P point sieve A.

Let F is a any subset of A. P_j is a prime number, and $P_1 <P_j \leqslant P_{\pi(N_0)}$. We abandon every points from F that y coordinate of which whole divided by P_j, denote the set that is consist of all remained points and the number of its points by $F(P_j(y))$ and $N\Big(F(P_j(y))\Big)$, that is, P_jy sieve F in the first place, and after that P_jx sieve $F(P_j(y))$, denote the set that is consist of all remained points and the number of its points by $F(P_j(y,x))$ and $N\Big(F(P_j(y,x))\Big)$ respectively. Then the order about x sieve and y sieve can be exchanged when prime number P_j point sieve F.

Proposition 1 We have

1° $F(P_j(x,\ y))=F(P_j(y,\ x))$;

2° $N\Big(F(P_j(x,\ y))\Big)=N\Big(F(P_j(y,\ x))\Big)$.

It is easy to prove 1° by using the method of proving two sets be equal, from 1°, it is obvious to see that 2° is true.

Let $R=\{P_1,P_2,\cdots,P_k\}\subset P$, but $R'=\{P'_1,P'_2,\cdots,P'_k\}$, where $P'_1,P'_2,\cdots,P'_k$ is a new arrange of $P_1,P_2,\cdots,P_k$. From meaning of point sieve method, it is easy to see that the following proposition is true:

Proposition 2 We have

1° $F(P_1(x, y), \cdots, P_k(x, y))$

$= F(P'_1(x, y), \cdots, P'_k(x, y));$

2° $N\Big(F(P_1(x, y), \cdots, P_k(x, y))\Big)$

$= N\Big(F(P'_1(x, y), \cdots, P'_k(x, y))\Big)$

In above 1°, 2°, every $P_j(x, y)$ can be exchanged by $P_j(y,x)$, or every $P'_j(x,y)$ can be exchanged by $P'_j(y,x)$, 1°, 2° are also true.

This means when some prime numbers point sieve F, the order of these prime numbers can be arbitrary exchanged.

Proposition 3 We have

1° $F(P_1(x, y), \cdots, P_k(x, y))$

$= F(P_1(x), \cdots, (P_k(x), P_1(y), \cdots, P_k(y))$

2° $N\Big(F(P_1(x, y), \cdots, P_k(x, y))\Big)$

$= N\Big(F(P_1(x), \cdots, P_k(x), P_1(y), \cdots, P_k(y))\Big)$

Proposition 4 Let point set $F \subset A$, $E \subset A$, and $F \bigcap E = \emptyset$, denote $C = F + E$, then

1° $C(P_1(x, y), \cdots, P_k(x, y))$

$= F(P_1(x, y), \cdots, P_k(x, y))$

$+ E(P_1(x, y), \cdots, P_k(x, y))$

2° $N\Big(C(P_1(x, y), \cdots, P_k(x, y))\Big)$

$= N\Big(F(P_1(x, y), \cdots, P_k(x, y))\Big)$

$$+N\Big(E(P_1(x, y), \cdots, P_k(x, y))\Big)$$

Above four propositions can be easy proved by meaning of point sieve method, in here they are omitted.

After using P point sieve A, suppose the set is consist of all remained points is not empty, that is $A(P_1(x,y),\cdots,P_{\pi(N_0)}(x,y))\neq$

Might as well suppose point $(P, q)\in A(P_1(x, y), \cdots P_{\pi(N_0)}(x,y))$, then p and $\bigcap_{j=1}^{\pi(N_0)} P_j$ are coprime, from the definition of N_0, it is easy to see that p be sure a prime number, for the same reason, q be sure a prime number too, and $p+q=N$, hence we have following

Theorem 1 To prove Goldbach's supposition as long as to prove the following

$$N\Big(A(P_1(x,y),\cdots,P_{\pi(N_0)}(x,y))\Big)>0.$$

§ 2 About sieve function

$N\Big(A(q_1(x, y), \cdots, q_l(x, y))\Big)$

In the following sections, we will count a series of sieve functions given in above section. In order to count these sieve functions, we will separate a few sections, in this section, our main work is to count the sieve function $N\Big(A(q_1(x, y),\cdots, q_k(x, y))\Big)$.

Set

$$P_1 = \{ q | q \in P, \ q | N \} \tag{4}$$

$$P_2 = \{ p | p \in P, \ p \nmid N \} \tag{5}$$

Suppose the number of elements in P_1 is l, the number of elements in P_2 is s, then

$$P_1 = \{q_1, q_2, \cdots, q_l\}, \quad P_2 = \{P_1, P_2, \cdots, P_s\}$$

and $P_1 \cup P_2 = P$, $l + s = \pi(N_0)$.

The following two lemmas are obvious.

Lemma 1 Suppose point set $F \subset A$, $q \in P_1$, then

(Ⅰ) $$F(q(x)) = F(q(y)) = F(q(x, y)) = F(q(y, x)) \tag{6}$$

(Ⅱ) $$N\Big(F(q(x))\Big) = N\Big(F(q(y))\Big) = N\Big(F(q(x, y))\Big) = N\Big(F(q(y, x))\Big) \tag{7}$$

Lemma 2 Let P'_1 is any not empty subset of P_1, that is $P'_1 = \{q'_1, \cdots, q'_j\}$, $q'_i \in P_1$, $i = 1, 2, \cdots, j$, $1 \leqslant j \leqslant l$. Let $F \subset A$. Then

$$F(q'_1(x), q'_2(x), \cdots, q'_j(x)) = F(q'_1(y), q'_2(y), \cdots, q'_j(y))$$
$$= F(q'_1(x, y), \cdots, q'_j(x, y)) = F(q'_1(y, x), \cdots, q'_j(y, x)) \tag{8}$$

Theorem 2 We have

$$N\Big(A(q_1(x, y), q_2(x, y), \cdots, q_l(x, y))\Big) = N \bigcap_{i=1}^{l} (1 - \frac{1}{q_i}) \tag{9}$$

Proof In lemma 1, we take $F = A$, $q = q_1 \in P_1$ from (Ⅱ) in lemma 1, we have

$$N\Big(A(q_1(x))\Big)=N\Big(A(q_1(y))\Big)=N\Big(A(q_1(x,\ y))\Big)$$
$$=N\Big(A(q_1(y,\ x))\Big)$$

The point set A in all have N points, the first coordinate of these N points are 0, 1, 2, ···, $N-1$ respectively. In these N whole number, have $\frac{N}{q_1}$ whole number be whole divided by q_1. Hence we have

$$N\Big(A(q_1(x,y))\Big)=N\Big(A(q_1(x))\Big)=N-\frac{N}{q_1}=N\Big(1-\frac{1}{q_1}\Big)$$

For set $A(q_1(x,\ y))$, using lemma 1, again, take $F=A(q_1(x,\ y))$, $q=q_2$, for the same reason, we get

$$N\Big(A(q_1(x,y),q_2(x,y))\Big)=N\Big(1-\frac{1}{q_1}\Big)-\frac{1}{q_2}N\Big(1-\frac{1}{q_1}\Big)$$
$$=N\Big(1-\frac{1}{q_1}\Big)\Big(1-\frac{1}{q_2}\Big)$$

And so on and so forth, we obtain

$$N\Big(A(q_1(x,y),\cdots,q_l(x,y))\Big)=N(1-\frac{1}{q_1})(1-\frac{1}{q_2})\cdots(1-\frac{1}{q_l})$$

The proof of theorem 2 is completed.

§ 3 About sieve function

$N\Big(A(q_1(x,\ y),\cdots,q_l(x,\ y),P_1(x,\ y))\Big)$

In this section, we will give the expression and the calculating formula of sieve function $N\Big(A(q_1(x,\ y),\cdots.q_l(x,\ y),P_1(x,y))\Big)$.

For this purpose, prove some lemmas first.

Let $1<a<N$, a is a whole number and $(a, N)=1$, that is a and N are coprime. Let point set

$$G_a=\{(a, N-a), (2a, N-2a), \cdots, (\left[\frac{N}{a}\right]a, N-\left[\frac{N}{a}\right]a)\}$$

$$G'_a=\{(N-\left[\frac{N}{a}\right]a, \left[\frac{N}{a}\right]a), (N-(\left[\frac{N}{a}\right]-1)a, (\left[\frac{N}{a}\right]-1)a), \cdots, (N-a, a)\}$$

For above two point sets, the following two lemmas are obvious true.

Lemma 3 We have

(Ⅰ) $G_a \bigcap G'_a =$ (10)

(Ⅱ) $N\left(G_a\right)=N\left(G'_a\right)=\left[\frac{N}{a}\right]$ (11)

Suppose $q_j \in \boldsymbol{P}_1$, $j=1, 2, \cdots, l$; $P_1 \in \boldsymbol{P}_2$. Let point set

$$G_{P_1}=\{(P_1, N-P_1), (2P_1, N-2P_1), \cdots, (\left[\frac{N}{P_1}\right]P_1, N-\left[\frac{N}{P_1}\right]P_1)\}$$

$$G'_{P_1}=\{(N-\left[\frac{N}{P_1}\right]P_1, \left[\frac{N}{P_1}\right]P_1), (N-(\left[\frac{N}{P_1}\right]-1)P_1, (\left[\frac{N}{P_1}\right]-1)P_1), \cdots, (N-P_1, P_1)\}$$

About $\boldsymbol{P}_1$ point sieve G_{P_1} and G_{P_2}, we have following

Lemma 4 We have

(Ⅰ) $G_{P_1}(q_1(x), q_2(x), \cdots, q_l(x))=G_{P_1}(q_1(y), \cdots, q_l(y))$

$=G_{P_1}(q_1(x, y), \cdots, q_l(x, y))$

(Ⅱ) $G'_{P_1}(q_1(x), \cdots, q_l(x)) = G'_{P_1}(q_1(y), \cdots, q_l(y))$

$= G'_{P_1}(q_1(x, y), \cdots, q_l(x, y))$

(Ⅲ) $G_{P_1}(q_1(x, y), \cdots, q_l(x, y)) \cap G'_{P}(q_1(x, y), \cdots, q_l(x, y))$

$= \varnothing$

(Ⅳ) $N\Big(G_{P_1}(q_1(x, y), \cdots, q_1(x, y))\Big)$

$= N\Big(G'_{P_1}(q_1(x, y), \cdots, q_l(x, y))\Big)$

$$= \left[\frac{N}{P_1}\right] - \left[\frac{N/P_1}{q_1}\right] - \cdots - \left[\frac{[N/p_1]}{q_l}\right] + \left[\frac{[N/P_1]}{q_1 q_2}\right]$$

$$+ \cdots + \left[\frac{[N/P_1]}{q_{l-1} q_l}\right] - \cdots + (-1)^l \left[\frac{[N/P_1]}{q_1 q_2 \cdots q_l}\right] \qquad (12)$$

Theorem 3 We have

$N\Big(A(q_1(x, y), \cdots, q_l(x, y), P_1(x, y))\Big)$

$= N\Big(A(q_1(x, y), \cdots, q_l(x, y))\Big)$

$- N\Big(G_{P_1}(q_1(x, y), \cdots, q_l(x, y))\Big)$

$- N\Big(G'_{P_1}(q_1(x, y), \cdots, q_l(x, y))\Big)$

$= N\Big(A(q_1(x, y), \cdots, q_l(x, y))\Big)$

$- 2N\Big(G_{P_1}(q_1(x, y), \cdots, q_l(x, y))\Big)$

$$= N\left(1 - \frac{1}{q_1}\right)\cdots\left(1 - \frac{1}{q_l}\right)\left(1 - \frac{2}{P_1}\right) + 2\left(\left\{\frac{N}{P_1}\right\}\right.$$

$$-\left\{\frac{N}{P_1 q_1}\right\}-\cdots-\left\{\frac{N}{P_1 q_l}\right\}+\left\{\frac{N}{P_1 q_1 q_2}\right\}+\cdots$$

$$+\left\{\frac{N}{P_1 q_{l-1} q}\right\}-\cdots+(-1)^l\left\{\frac{N}{p_1 q_1 \cdots q_l}\right\}\Bigg) \qquad (13)$$

Proof The first point (0, N) in A be out $A(q_1(x,y), \cdots, q_l(x, y))$ already, hence, it is independent of (0, N) when the prime numbers in P_2 point sieve $A(q_1(x, y), \cdots, q_l(x, y))$.

Set

$$A_0 = A - \{(0, N)\} = \{(1, N-1), (2, N-2), \cdots, (N-1, 1)\}$$

When $P_1 x$ sieve A_0, the set is consist of all abandoned points is just right G_{P_1}, and $G_{P_1}(q_1(x,y),\cdots,q_l(x,y))$ is the set be consist of all remained points when $q_1,\cdots,q_l$ point sieve G_{P_1}, that is the first coordinate of every points in $G_{P_1}(q_1(x,y),\cdots,q_l(x,y))$ are multiple of P_1, but not whole divided by any q_j ($j=1, 2, \cdots, l$). Hence.

$$G_{P_1}(q_1(x, y), \cdots, q_l(x, y)) \subset A(q_1(x, y), \cdots, q_l(x, y))$$

When $P_1 x$ sieve $A(q_1(x,y),\cdots,q_l(x,y))$, the set is consist of all abandoned points is just right $G_{P_1}(q_1(x,y),\cdots,q_l(x,y))$, hence,

$$\begin{aligned} &N\Big(A(q_1(x,y),\cdots,q_l(x,y),P_1(x))\Big) \\ &= N\Big(A(q_1(x, y), \cdots, q_l(x, y))\Big) \\ &\quad - N\Big(G_{P_1}(q_1(x, y), \cdots, q_l(x, y))\Big) \end{aligned} \qquad (14)$$

From lemma 4 (Ⅲ)

$$G_{P_1}(q_1(x,y), \cdots, q_l(x,y)) \cap G'_{P_1}(q_1(x,y), \cdots, q_l(x,y)) = \varnothing \qquad (15)$$

When $P_1 y$ sieve A_0, the set be consist of all abandoned points is just right G'_{P_1}, and $G_{P_1}(q_1(x,y),\cdots,q_l(x,y))$ is the set be consist of all remained points when $q_1,\cdots,q_l$ point sieve G'_{P_1} that is the second coordinate of every points in $G'_{P_1}(q_1(x,y),\cdots,q_l(x,y))$ are multiple of P_1, but not whole divided by any $q_j (j=1, 2, \cdots, l)$. Hence,

$$G'_{P_1}(q_1(x, y), \cdots, q_l(x, y)) \subset A(q_1(x, y), \cdots, q_l(x, y)) \quad (16)$$

From (15), we know that the points in $G'_{P_1}(q_1(x, y), \cdots, q_l(x,y))$ not be abandoned any one when $P_1 x$ sieve $A(q_1(x,y),\cdots,q_l(x,y))$. Hence,

$$G_{P_1}(q_1(x, y), \cdots, q_l(x, y)) \subset A(q_1(x, y), \cdots, q_l(x, y), P_1(x)) \quad (17)$$

Therefore, when $P_1 y$ sieve $A(q_1(x,y),\cdots,q_l(x,y),P_1(x))$, the set is consist of all abandoned points is just right $G'_{P_1}(q_1(x,y),\cdots,q_l(x,y))$. Hence, notice (14), (12), we get

$$\begin{aligned} &N\Big(A(q_1(x, y), \cdots, q_l(x, y), P_1(x, y))\Big) \\ &= N\Big(A(q_1(x, y), \cdots, q_l(x, y), P_1(x))\Big) \\ &\quad - N\Big(G_{P_1}(q_1(x, y), \cdots, q_l(x, y))\Big) \\ &= N\Big(A(q_1(x, y), \cdots, q_l(x, y))\Big) \\ &\quad - 2N\Big(G_{P_1}(q_1(x, y), \cdots, q_l(x, y))\Big) \end{aligned}$$

$$= N\prod_{i=1}^{l}\left(1-\frac{1}{q_i}\right)-2\left(\left[\frac{N}{P_1}\right]-\left[\frac{[N/P_1]}{q_1}\right]-\cdots\right.$$

$$-\left[\frac{[N/P_1]}{q_l}\right]+\left[\frac{[N/P_1]}{q_1q_2}\right]+\cdots+\left[\frac{[N/P_1]}{q_{l-1}q_l}\right]-\cdots$$

$$\left.+(-1)^l\left[\frac{[N/P_1]}{q_1\cdots q_l}\right]\right)$$

$$= N\left(1-\frac{1}{q_1}\right)\cdots\left(1-\frac{1}{q_l}\right)\left(1-\frac{2}{P_1}\right)+2\left(\left\{\frac{N}{P_1}\right\}-\left\{\frac{N}{P_1q_1}\right\}\right.$$

$$-\cdots-\left\{\frac{N}{P_1q_l}\right\}+\left\{\frac{N}{P_1q_1q_2}\right\}+\cdots+\left\{\frac{N}{P_1q_{l-1}q_l}\right\}$$

$$\left.-\cdots+(-1)^l\left\{\frac{N}{P_1q_1\cdots q_l}\right\}\right)$$

The proof of theorem 3 is completed.

§ 4 About sieve function

$N\Big(A(q_1(x,y),\cdots,q_l(x,y),P_1(x,y),P_2(x,y))\Big)$

In above section, we solved the expression and calculating problem of sieve function $N\Big(A(q_1(x,y),\cdots,q_l(x,y),P_1(x,y))\Big)$. In this section, we study the expression and calculating problem of sieve function $N\Big(A(q_1(x,y),\cdots,q_l(x,y),P_1(x,y),P_2(x,y))\Big)$, in order to solve this problem, we prove some lemmas in the first place.

Lemma 5 Let P_1, $P_2 \in \boldsymbol{P}_2$, and