院士文丛

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Hermite 展开与广义函数

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序　言

自从 1950 年 Schwartz L 的分布论（Théorie des distributions，Tom Ⅰ，Ⅱ）出版之后，广义函数的理论和应用已经有了很大的发展，几经演变，其理论可以说日趋完善，特别是经过了 Miknsinshi J，Temple G，Lighthill M J 等人的工作，把广义函数理论大大地简化，以致可以在大学课程中讲授. 中国华罗庚教授早在 20 世纪 50 年代就注意到广义函数，并提出了自己的想法，他的想法有其独特之处. 他本人只在特殊情形，即一维周期广义函数的情形，详细阐述了他的观点. 由于后来事情繁杂，他未能在此一方向作出更大发展. 这本小册子正是沿着华罗庚教授指出的思路，通过直线上的 Hermite 展式进行工作的一个小结. 我们发现，他的想法的确有其优越之处，例述如下：

首先我们可以引进一较 Schwartz 分布论更加广泛的广义函数类，我们称之为弱函数. 在某种意义下，这应该是最广的. 它还保持 Schwartz 分布的特点，即其 Fourier 变换仍然为弱函数. 次之按照华的定义稍稍延伸，我们就可以引进广义数和广义弱函数的概念. 利用这些概念我们可以很快解决弱函数的乘积，当然也就包括了 Schwartz 分布的乘积. 这个问题长期以来被称为广义函数论的一个困难问题，李邦河运用非标准分析最早给出了它的解答. 我们现在运用华的思想，重新

予以处理.

在中国广义函数论的先驱者中，还应提及冯康教授. 是他最早在中国介绍广义函数论，并作了广义函数的 Mellin 变换. 在本书中我们也沿着华的观点，重新作了处理.

时至今日广义函数理论本身似乎已经很完善了，其实还有许多工作需要做. 特别在应用上，虽然有了许多精美的结果，例如在线性偏微分方程论上的应用，特别是在非线性双曲型守恒律方面的应用等. 但对经典分析方面应用就还很少，进一步的发展还是值得充分注意的.

这本小册子只是这方面工作的一个导引. 还有许多丰富的内容有待进一步完善，这只有待于来日.

作　者

2005 年 5 月

Preface

Ever since the publication of L. Schwartz's theory of distributions (Théorie des distributions, Tom Ⅰ, Ⅱ) in 1950 — 1951, theory of generalized functions and its applications have developed dramatically. After having gone through some changes, it can be said that the theory has grown more and more in perfection. J. Miknsinshi, G. Temple, and M. J. Lighthill have even simplified theory of generalized functions, and made it a college course. Chinese mathematician Luoken Hua had noticed the generalized functions as early as the 1950's and had proposed his own interpretation. However, he only worked on a special case — the one dimensional periodic generalized functions. He did not continue to develop the area because he was occupied with other work. This present book is a collection of research on generalized functions according to Hua's ideas based on Hermite expansions. We have discovered that Hua's interpretation certainly has its advantages, as follows:

First, we can introduce a type of generalized functions more general than Schwartz's distributions, which we call weak functions. In some sense, this type is the most generalized. These generalized functions still keep the main characteristic of the Schwartz's temperate distributions, i.

e., the Fourier transform of each weak function is still a weak function. Secondly, extending Hua's definition a little further, we can introduce concepts of generalized numbers and generalized weak functions. Using these concepts, we can resolve the multiplication of weak functions, which therefore has included the multiplication of Schwartz's distributions. Such multiplication has been regarded as a difficult problem for a long time. Banghe Li was the first to resolve this problem by using non-standard analysis. In this work, we again research on this problem, by Hua's idea.

Among the Chinese pioneer researchers of generalized functions, we should mention Professor Kang Feng. He was the first person to introduce generalized functions theory in China and did research on Mellin transforms. In this book, we also again treat Mellin transforms from Hua's point of view.

Certainly, theory of generalized functions itself seems to have evolved quite far in our time. Indeed, there have been many successful applications that have produced elegant results. Examples are in its applications to linear partial differential equations theory and to nonlinear hyperbolic conservation laws. However, there have been few applications to classical analysis, so it is worth paying more attention to developing further applications to such analysis.

This book is only an introduction to such research results. There is still much rich content to be improved upon and perfected, which is only to be discovered in the future.

the Author

May, 2005

内容简介

本书以 Hermite 多项式为工具，引进了新的广义函数，作者们称之为弱函数。它包括了许多经典的广义函数。继而引进了广义数，广义弱函数，解决了弱函数乘法问题。作者们还讨论了该理论在经典分析例如 Riemann zeta一函数论和非线性双曲型守恒律论上的应用。

本书是华罗庚教授关于广义函数论思想的进一步发展。

Abstract

By Hermite polynomials, this monograph proposes a new theory of generalized functions. The authors introduced a new kind of generalized functions called weak functions, which include many classical ones. Furthermore they introduced the concepts of generalized numbers and generalized weak functions, and then solved the multiplication problem of weak functions. They also discussed the applications of the theory above in some classical analysis such as the theory of Riemann zeta-function and the theory of nonlinear hyperbolic conservation laws.

The present monograph develops Professor Hua Luoken's idea in his theory of generalized functions.

目　　录

第一章　Hermite 多项式的基本性质

Hermite 多项式是一组正交多项式，它在数学物理和数学分析中是很有用处的. 我们现在讨论它的一些基本性质.

1. 定义与生成函数

定义 1.1.1　Hermite 多项式 $H_n(x)$ 的定义如下：

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, n = 0, 1, 2, \cdots \tag{1.1.1}$$

容易看出：$H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12x, \cdots$，它的一般形式为

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k}, \tag{1.1.2}$$

其中 $\left[\frac{n}{2}\right]$ 代表 $\frac{n}{2}$ 的整数部分.

这样的 $H_n(x)$ 还可从下面的展式

$$w(x,t) = e^{2xt-t^2} = e^{x^2} e^{-(t-x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{1.1.3}$$

中得出. 为此只需注意到

$$\left(\frac{\partial^n w}{\partial t^n}\right)_{t=0} = e^{x^2} \frac{d^n}{dt^n} e^{-(x-t)^2} \bigg|_{t=0}$$

$$= e^{x^2}(-1)^n \frac{\partial^n e^{-u^2}}{\partial u^n}\bigg|_{u=x} = H_n(x). \quad (1.1.4)$$

由于 $w(x,t)$ 为 x 与 t 的整函数，故(1.1.3)式在 x,t 的任何有限区域内都是一致收敛的，且逐项可微，因此我们有

$$\frac{\partial w}{\partial t} - 2(x-t)w = 0, \quad (1.1.5)$$

$$\frac{\partial w}{\partial x} - 2tw = 0. \quad (1.1.6)$$

由(1.1.5)式，有

$$\sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!}t^n - 2x\sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^n + 2\sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^{n+1} = 0.$$

比较 t^n 的系数，就得到

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \ (n = 1,2,\cdots). \quad (1.1.7)$$

由(1.1.6)式，我们有

$$\sum_{n=0}^{\infty} \frac{H'_n(x)}{n!}t^n - 2\sum_{n=0}^{\infty} \frac{H_n(x)}{n!}t^{n+1} = 0,$$

故得

$$H'_n(x) = 2nH_{n-1}(x) \quad (n = 1,2,\cdots). \quad (1.1.8)$$

由(1.1.7)式和(1.1.8)式，得

$$H_{n+1}(x) - 2xH_n(x) + H'_n(x) = 0,$$

对上式逐项微分，就有

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0 \quad (n = 1,2,\cdots). \quad (1.1.9)$$

此为 $H_n(x)$ 满足的微分方程.

如果令

$$u = e^{-\frac{x^2}{2}} H_n(x), \tag{1.1.10}$$

则

$$u' = e^{-\frac{x^2}{2}}(-xH_n(x) + H'_n(x)).$$

容易得出

$$u'' + (2n+1)u - x^2 u = 0. \tag{1.1.11}$$

如果在等式(1.1.3)中令 $x=0$,就得到

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \quad H_{2n+1}(0) = 0. \tag{1.1.12}$$

由等式(1.1.3)知

$$w(x,u)w(x,v) = e^{2x(u+v)-u^2-v^2}$$
$$= \sum_{m,n} \frac{H_m(x)}{m!} \frac{H_n(x)}{n!} u^m v^n,$$

得

$$e^{-(x-(u+v))^2+2uv} = \sum_{m,n=0}^{\infty} \frac{H_m(x)H_n(x)e^{-x^2}}{m!\ \ n!} u^m v^n,$$

故

$$\int_{-\infty}^{+\infty} e^{-(x-(u+v))^2} e^{2uv} dx = \sum_{m,n=0}^{\infty} \frac{\int_{-\infty}^{+\infty} H_m(x)H_n(x)e^{-x^2} dx}{m!\ \ n!} u^m v^n$$
$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2uv)^n}{n!}.$$

故得

$$\int_{-\infty}^{+\infty} H_m(x)H_n(x)e^{-x^2} dx = 0, \quad m \neq n,$$
$$\int_{-\infty}^{+\infty} H_n^2(x)e^{-x^2} dx = 2^n n! \sqrt{\pi} = c_n^2.$$

今后我们还令

$$\psi_n(x) = \frac{e^{-\frac{x^2}{2}} H_n(x)}{c_n}. \qquad (1.1.13)$$

2. Mehler 公式,参看[B6]

如果 $|t|<1$,

$$\sum_{n=0}^{\infty} \frac{e^{-\frac{1}{2}(x^2+y^2)}}{2^n n!} t^n H_n(x) H_n(y)$$

$$= \frac{1}{\sqrt{1-t^2}} \exp\left\{\frac{x^2-y^2}{2} - \frac{(x-yt)^2}{1-t^2}\right\}, \qquad (1.2.1)$$

此式称为 Mehler 公式.

证 易知

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2+2ixu} du.$$

因此(1.2.1) 式

$$左边 = \frac{e^{\frac{x^2+y^2}{2}}}{\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(-2tuv)^n}{n!} e^{-u^2-v^2+2ixu+2iyv} du dv$$

$$= \frac{e^{\frac{1}{2}(x^2+y^2)}}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-u^2-v^2+2ixu+2iyv-2tuv} du dv$$

$$= \frac{e^{\frac{1}{2}(x^2-y^2)}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-(1-t^2)u^2+2i(x-yt)u} du$$

$$= \frac{1}{\sqrt{1-t^2}} \exp\left\{\frac{1}{2}(x^2-y^2) - \frac{(x-yt)^2}{1-t^2}\right\}.$$

上式积分与求和之可交换是由于

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \frac{(2\rho uv)^n}{n!} e^{-u^2-v^2+2Au+2Bv} du dv.$$

当 $\rho<1$ 时为收敛.

3. $H_n(x)$的渐近性

在这一节我们将探讨 $H_n(x)$ 和 $\psi_n(x)$ 当 $n\to\infty$ 时的渐近性态. 令 $u=e^{-\frac{x^2}{2}}H_n(x)$,则由(1.1.11)式知 u 满足微分方程

$$u''+(2n+1)u=x^2u.$$

将右端看作已知,$u(x)$ 满足初值 $u(0)=H_n(0)$,$u'(0)=H'_n(0)$,则我们又得出 u 的另一表达式

$$u(x)=H_n(0)\cos\sqrt{2n+1}x+H'_n(0)\frac{\sin\sqrt{2n+1}x}{\sqrt{2n+1}}$$
$$+\frac{1}{\sqrt{2n+1}}\int_0^x y^2u(y)\sin[\sqrt{2n+1}(x-y)]\mathrm{d}y. \tag{1.3.1}$$

由(1.1.12)式知

$$H_{2m}(0)=(-1)^m\frac{\Gamma(2m+1)}{\Gamma(m+1)},\quad H_{2m+1}(0)=0,$$

$$H'_{2m}(0)=0,\quad H'_{2m+1}(0)=2(-1)^m\frac{\Gamma(2m+2)}{\Gamma(m+1)},$$

则(1.3.1)式可写为

$$u(x)=\alpha_n\left\{\cos\left(\sqrt{2n+1}x-\frac{n\pi}{2}\right)+r_n(x)\right\}, \tag{1.3.2}$$

其中

$$\alpha_n=\begin{cases}\dfrac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)}, & n\text{ 为偶数;}\\[2ex] \dfrac{2\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\sqrt{2n+1}}, & n\text{ 为奇数,}\end{cases} \tag{1.3.3}$$

$$r_n = \frac{1}{\alpha_n\sqrt{2n+1}}\int_0^x y^2 u(y)\sin[\sqrt{2n+1}(x-y)]\mathrm{d}y, \tag{1.3.4}$$

当 x 为任意实数时，

$$|r_n(x)| \leqslant \frac{1}{\alpha_n\sqrt{2n+1}}\left\{\int_0^{|x|} y^4\mathrm{d}y\right\}^{\frac{1}{2}}\left\{\int_0^{|x|} u^2(y)\mathrm{d}y\right\}^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\alpha_n\sqrt{2n+1}}\left\{\int_0^{|x|} y^4\mathrm{d}y\right\}^{\frac{1}{2}}\left\{\int_0^{\infty} u^2(y)\mathrm{d}y\right\}^{\frac{1}{2}}$$

$$= \frac{(2^n n!\sqrt{\pi})^{\frac{1}{2}}}{\alpha_n\sqrt{2n+1}}\frac{|x|^{\frac{5}{2}}}{\sqrt{2}\sqrt{5}} = \beta_n|x|^{\frac{5}{2}}.$$

由 Stirling 公式知，当 $n\to\infty$ 时，

$$\alpha_n \sim 2^{\frac{n+1}{2}}n^{\frac{n}{2}}\mathrm{e}^{-\frac{n}{2}},\quad 2^n n!\sqrt{\pi} \sim 2^{n+\frac{1}{2}}\mathrm{e}^{-n}n^{n+\frac{1}{2}}\pi, \tag{1.3.5}$$

故对任何 n，$\beta_n n^{\frac{1}{4}}$ 为有界，因此

$$|r_n(x)| \leqslant c\frac{|x|^{\frac{5}{2}}}{n^{\frac{1}{4}}}. \tag{1.3.6}$$

由此知对任何有限值 x 而言

$$u(x) \sim \alpha_n\cos\left(\sqrt{2n+1}x - \frac{n\pi}{2}\right)\qquad (n\to\infty),$$

此即

$$\mathrm{H}_n(x) \sim 2^{\frac{n+1}{2}}n^{\frac{n}{2}}\mathrm{e}^{-\frac{n}{2}}\mathrm{e}^{\frac{x^2}{2}}\left\{\cos\left(\sqrt{2n+1}x - \frac{n\pi}{2}\right) + O\left(\frac{|x|^{\frac{5}{2}}}{n^{\frac{1}{4}}}\right)\right\}, \tag{1.3.7}$$

$$\psi_n(x) = \frac{\mathrm{H}_n(x)}{(2^n n!\sqrt{\pi})^{\frac{1}{2}}}\mathrm{e}^{-\frac{x^2}{2}}$$

$$\sim\sqrt{\frac{1}{\pi}}\left(\frac{2}{n}\right)^{\frac{1}{4}}\cdot\left\{\cos\left(\sqrt{2n+1}x-\frac{n\pi}{2}\right)+O\left(\frac{|x|^{\frac{5}{2}}}{n^{\frac{1}{4}}}\right)\right\}. \tag{1.3.8}$$

4. $L_2(-\infty,+\infty)$的一组正交基

我们令

$$\psi_n(x)=\frac{\varphi_n(x)}{c_n},\ \varphi_n(x)=\mathrm{e}^{-\frac{x^2}{2}}\mathrm{H}_n(x),$$

$$c_n^2=2^n n!\sqrt{\pi},\ \mathrm{h}_n(x)=\frac{\mathrm{H}_n(x)}{c_n}, \tag{1.4.1}$$

则由前面所述，易知$\{\psi_n(x)\}$构成一组正交系，$\{\mathrm{h}_n(x)\}$构成以 e^{-x^2} 为权的一组正交系，即

$$\int_{-\infty}^{+\infty}\psi_m(x)\psi_n(x)\mathrm{d}x=\delta_{m,n},\quad \delta_{m,n}=\begin{cases}1,\ m=n;\\0,\ m\neq n,\end{cases}$$

此即

$$\int_{-\infty}^{+\infty}\mathrm{h}_m(x)\mathrm{h}_n(x)\mathrm{e}^{-x^2}\mathrm{d}x=\delta_{m,n}.$$

下面将证明$\psi_m(x)$构成$L_2(-\infty,+\infty)$的一组完全正交基，我们称之为 Hermite 基. 即有

定理 1.4.1　如果 $f(x)\in L_2(-\infty,+\infty)$，令

$$a_m=\int_{-\infty}^{+\infty}f(x)\psi_m(x)\mathrm{d}x, \tag{1.4.2}$$

（积分显然为收敛）则我们有

$$\int_{-\infty}^{+\infty}\left|f(x)-\sum_{n=0}^{N}a_m\psi_m(x)\right|^2\mathrm{d}x\to 0,当\ N\to\infty. \tag{1.4.3}$$

证　由 Mehler 公式，如果令