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D. V. Anosov (Ed.)

Dynamical Systems IX

Dynamical Systems with Hyperbolic Behaviour

动力系统 IX

带有双曲性的动力系统

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Contents

Dynamical Systems with Hyperbolic Behaviour

D.V. Anosov, S.Kh. Aranson, V.Z. Grines, R.V. Plykin,
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Translated from the Russian
by G.G. Gould

Contents

Preface

This volume is devoted to the "hyperbolic theory" of dynamical systems (DS), that is, the theory of smooth DS's with hyperbolic behaviour of the trajectories (generally speaking, not the individual trajectories, but trajectories filling out more or less "significant" subsets in the phase space. Hyperbolicity of a trajectory consists in the property that under a small displacement of any point of it to one side of the trajectory, the change with time of the relative positions of the original and displaced points resulting from the action of the DS is reminiscent of the motion next to a saddle. If there are "sufficiently many" such trajectories and the phase space is compact, then although they "tend to diverge from one another" as it were, they "have nowhere to go" and their behaviour acquires a complicated intricate character. (In the physical literature one often talks about "chaos" in such situations.) This type of behaviour would appear to be the opposite of the more customary and simple type of behaviour characterized by its own kind of stability and regularity of the motions (these words are for the moment not being used as a strict terminology but rather as descriptive informal terms).[1] The ergodic properties of DS's with hyperbolic behaviour of trajectories (Bunimovich et al. 1985) have already been considered in Volume 2 of this series. In this volume we therefore consider mainly the properties of a topological character (see below for further details).[2]

When at the beginning of the 60's the significance of hyperbolicity was recognized, investigations were first made of DS's in which such behaviour of a trajectory was expressed in the sharpest way — hyperbolicity is, so to speak, "total" (the hyperbolic character, noted above, of the change in time of the relative positions of two phase points, namely the original one and the slightly displaced one, holds under any direction of the displacement to one side of the original trajectory) and "uniform" (uniformity of the inequalities expressing the hyperbolicity with respect to the points, the displacements, and time). The precise formulation of this version of hyperbolicity has led to the so-called hyperbolic sets, whose properties (both topological and metric (ergodic)) have aroused great interest. But in ergodic theory a lot of significant work has been done since then, in connection with the weakening of the conditions of hyperbolicity in various directions. In the broadest terms, this progress

[1] The general definition of hyperbolicity is such that exponential stability turns out to be a particular case of it (cf. hyperbolicity of periodic trajectories in (Anosov et al. 1985, Chap. 1, Sect. 2.4)), while Morse-Smale systems (Anosov et al. 1985, Chap. 2, Sect. 3), which can be regarded as the simplest DS's, are the objects of hyperbolic theory. Thus it deals with both types of behaviour of trajectories, but its novelty is in connection with what occurs when there are "many" hyperbolic trajectories.

[2] In connection with our mention of the ergodic properties of DS's with hyperbolic behaviour of the trajectories it is appropriate to add that there has recently appeared a textbook on the ergodic theory of smooth DS's (Mañé 1987), almost half of which relates to hyperbolicity in one way or another.

is in connection with technical improvements rather than with new ideas of principle at the paradigmatic level. But however that may be, there have been considerable achievements in the metric theory occasioned by weakening the requirements of hyperbolicity. This has not been done in the topological theory (again speaking in the broadest terms). However, this does not mean that since the 60's there have been no significant achievements beyond the framework of total uniform hyperbolicity in the theory of smooth DS's, or that none of these achievements relates to DS's displaying some weaker form of hyperbolicity in their behaviour. Nevertheless, in the topological theory of DS's (so far?) no precise and workable notion of weakened hyperbolicity has been concocted. It would appear that here the gateway that might lead beyond the realm of total uniform hyperbolicity requires new ideas of principle. (At the same time, there remain many unsolved problems even within the bounds of these conditions.)

For all that, something now has emerged beyond the limits of the original version of hyperbolicity, although, as it seems to me, in the two most important cases, namely, DS's with a Lorenz attractor and certain cascades (DS's with discrete time) on surfaces, it is a question not so much of going beyond these limits, but rather of stretching them. (In essence, what have been weakened in these cases are not total and uniform hyperbolicity, but certain other "accompanying" conditions, so to speak. These are the conditions of continuity and smoothness (as ordinarily encountered in analysis); weakening them consists in violating them in some sense at certain points or on certain lines, and the condition that if a hyperbolic set of a flow contains equilibrium points, then they are isolated points of this set). In one way or another, these DS's are considered in the current volume. In this connection, certain information is also given on their metric properties, whereas questions of ergodic theory relating to hyperbolic sets are not touched upon in this volume; here we have nothing to add to the earlier results (given in (Bunimovich et al. 1985)).

The last two articles of this volume may outwardly appear to have no relation to hyperbolicity. In fact, the origin of their subject matter is only partially and fairly tenuously related to hyperbolicity. However in the development of this theme such connections have arisen and have turned out to be essential. These connections are a sufficient motivation for the inclusion of the last two articles in this volume although, of course, their contents deal with other matters. It is therefore worth saying a few words about these articles.

The investigation of cascades on closed surfaces is closely related to the classification of homeomorphisms of surfaces, more precisely, with the classification of isotopy classes of such homeomorphisms. The latter was the concern of topologists back in the 20's. One of the important questions here consisted in choosing "good" representatives of these isotopy classes; it must be recalled that successful representatives have usually turned out to be of great utility in many respects. Naturally, the representatives of some classes possess no hyperbolic properties. Until the 70's it was only such representatives that were known (leaving out the torus). Essential progress was achieved when "good" representatives in the remaining isotopy classes had been successfully chosen. Leaving aside "mixed" (in technical jargon, reducible) cases, when the corresponding homeomorphisms behave differently in different parts of the surface, it can be said that the new representatives generate cascades with "typically hyperbolic" properties. Their difference from objects of ordinary hyperbolic theory is related not to the weakening of hyperbolicity conditions, but to the slight breakdown of smoothness at certain points. In this manner, progress in the theory of homeomorphisms of surfaces in the 70's was in no small degree stimulated by the development of the hyperbolic theory of DS's in the 60's.

What has just been said is only one aspect of the theory of cascades on surfaces, this being the best known and most important. In the article included here, it is considered in the general context of this theory.

The article on certain DS's of algebraic origin, namely, homogeneous flows, occupies a special position in this volume. It is, to some extent, devoted to their ergodic properties. It might appear that this is close to Volume 2, but there is no such article there. The latter is partly accounted for by the fact that there was simply no room, due to the abundance of material in Volume 2. But the real reason is more one of principle. The investigation of the ergodic properties of homogeneous flows has its own specific character. Here an important role is played by the theory of Lie groups and their representations, while those ideas and methods forming the subject matter of Volume 2 recede into the background (although, of course, some of this is essentially used). In a number of cases homogeneous flows possess a certain amount of hyperbolicity (partial but uniform) and related to this are the corresponding geometric properties, as is the case throughout this volume. In essence, the algebra is then required for sorting out this geometry (without always mentioning this explicitly).

In connection with the discussion of the contents of this volume it is appropriate to mention three related sections of the theory of DS's that are not touched upon or only partly dealt with here. The first is certain questions of bifurcation theory where one has to deal with hyperbolicity. Information on this is contained in one of the earlier volumes of this edition (Arnol'd et al. 1986) (it is worth mentioning another new book (Wiggins 1988)); we have given only a few mentions of this. The second is "one-dimensional dynamics", that is, the study of iterations of maps (in general, non-invertible) in a one-dimensional real or complex domain[3] (in the latter case one talks about conformal dynamics whenever the iterated map is conformal). It arose independently of hyperbolic theory (and, if one is talking about conformal dynamics, somewhat earlier) but received an appreciable stimulus from the

[3] The reader should be warned that this terminology is used in another sense, namely, the dynamics of a one-dimensional "chain" (or some other system of particles, etc., on the line). The dimension of the phase space of such a DS is greater than 1 (and increases without bound as the number of particles increases, while if the system has distributed parameters, then it is infinite).

latter when it was realized (around 1970) that there was a "similarity" of behaviour of the trajectories between these two. There is also an influence the other way round: irreversible one-dimensional maps play an important auxiliary or heuristic role in the investigation of certain invertible DS's in higher dimensions that are peculiar to hyperbolic phenomena. It is only in this respect that these maps are referred to in the articles of the present volume. Even if such references had been discussed in greater detail, they would have reflected only part of one-dimensional dynamics and, of course, this arouses interest for other reasons. From a utilitarian point of view, it is simply a question that non-invertible one-dimensional maps arise in various questions of science and technology. Of course, invertible transformations relate more to the original ideas on DS's (as described at the beginning of (Anosov et al. 1985)), but there are nevertheless such problems that directly, or via some indirect route, lead to non-invertible one-dimensional maps.[4] In its conceptual aspects, one-dimensional dynamics admits the almost unique possibility of a fairly full investigation of the complicated behaviour of DS's; that is, the behaviour *both in the sense of the qualitative picture in phase space and in* the sense of dependence on the parameters.[5] In this connection it is worth mentioning the appropriate literature. In (Bunimovich et al. 1985) there is a small chapter on one-dimensional dynamics (primary attention being given to ergodic questions). In addition to the books and surveys referred to there, one can also mention (Eremenko and Lyubich 1989), (Lyubich 1986), (Sharkovskij et al. 1989), (Sharkovskij, Maistrenko and Romanenko 1986), (de Melo 1989), (Milnor 1990), (Nitecki 1982).

Another contiguous area left completely untouched in this book is hyperbolicity and bifurcations connected with it for infinite-dimensional systems. As is well known, reversion to such systems gives a satisfactory treatment of a number of problems for partial differential equations and for ordinary differential equations with a delay. It is natural that in this connection local questions (or questions of a similar character) should be developed in the first instance; but now hyperbolicity is also brought into play. Apparently, such

[4] With regard to "direct" examples it is customary first of all to refer to the "propagandist" article (May 1976). As for "indirect routes", there are apparently various examples; cf. the use of one-dimensional dynamics in the present volume and at the beginning of (Sharkovskij, Maistrenko and Romanenko 1986). Several bibliographical references to applications (both "direct" and "indirect") can be found in (Nitecki 1982).

[5] Thus, relatively amenable to investigation are those one-dimensional DS's for which opposite types of behaviour of the trajectories are realized in different parts of phase space, see the very beginning of this preface. Even if these actual types are known (albeit partially) for "ordinary" DS's (otherwise this book would not have been written!), what is less understood is the complicated way in which they can combine, or what else can occur. (The theory, of course, describes certain versions of such a combination, but unfortunately, it does not exhaust all the "typical" situations. Even for numerical experiments which, it might appear, ought clearly to demonstrate what is occurring in some DS or other, one does not always succeed in finding a convincing theoretical interpretation.)

non-local entities have not been reflected at the textbook or survey-article level.

Finally, I should like to draw attention to four books of a relatively popular character that bear a relation to our theme. The books (Devaney 1989) and (Ruelle 1989) are introductions to a wide range of questions relating, in particular, to the complex behaviour of trajectories. Further, in such a situation there are so-called "fractal" sets which are quite unlike the usual geometric figures, being so violently "jagged" that it is reasonable to ascribe a fractional dimension to them (whence their name). The book (Falconer 1990) serves as an introduction to this topic. There is an album with coloured diagrams representing the results of numerical experiment giving rise to such objects (three quarters of these diagrams relate to conformal dynamics) and supplied with a certain amount of explanatory text (Pitgen and Richter 1986). The figures are very beautiful — in this respect they produce the same impression both for the specialist and for the complete outsider, mathematician or not.

To read this volume one needs general familiarity with the elements of the theory of DS's, not so much with regard to any advanced theorems, but rather with the general system of concepts, terminology, and so on. All this is contained in the article "Smooth dynamical systems", published in Volume 1 of the present series (Anosov et al. 1985) and to some extent precedes all the articles on DS's included in the other volumes. The requisite material from other branches of mathematics is in the main summarized in the preface to (Anosov et al. 1985), where one can also find the notation of a general mathematical character that is used (which is fairly standard). In certain parts additional material is required which is either recalled or (as the authors hope) is clear from the context. The chapter on homogeneous flows naturally takes up a special position in this respect. The reading of it requires not episodical, but constant (and sufficiently systematic) acquaintance with a number of questions that go beyond the above framework. All this is mentioned at the beginning of that chapter.

D.V. Anosov

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* For the convenience of the reader, references to reviews in Zentralblatt für Mathematik (Zbl.), compiled using the MATH database, have, as far as possible, been included in this bibliography.

Chapter 1
Hyperbolic Sets

D.V. Anosov, V.V. Solodov[1]

§1. Preliminary Notions

1.1. Definition of a Hyperbolic Set. Hyperbolicity of a compact (two-sided) invariant set A (that is, a set consisting of entire trajectories) of a flow or cascade $\{g^t\}$ ($t \in \mathbb{R}$ or $t \in \mathbb{Z}$) given on a phase manifold M is defined in terms of the restriction over A of the tangent linear extension $\{Tg^t\}$ (see Anosov et al. 1985, Chap. 1, Sect. 2.2), that is, the properties of the DS $\{Tg^t|p^{-1}A\}$, where $p : TM \to M$ is the natural projection. In other words, we are dealing with the behaviour of the solutions of the variational equations along the trajectories of our DS. (Speaking somewhat freely, we also have in mind not only "true" variational equations (and so on) for a flow, but also their analogues for a cascade $\{g^k\}$. In the latter case, the role of the system of variational equations along the trajectory $\{g^k x\}$ is played by the representation of $Tg^k(x)$ in the form of the composite

$$Tg(g^{k-1}x) \circ \ldots \circ Tg(gx) \circ Tg(x),$$

while the role of the solutions of this system (which is more important) is played by the function of discrete time $k \mapsto Tg^k(x)\zeta$ with $\zeta \in T_xM$. The map $Tg^k(x)$ plays the role of the Cauchy matrix of the system of variational equations along the trajectory $\{g^k x\}$.) It is worth recalling, in connection with the term "tangent linear extension", that in (Anosov et al. 1985, Chap. 3, Sect. 5.2) the more general notion of a linear extension of a DS was introduced. In this terminology the tangent linear extension $\{Tg^t\}$ is, in fact, a linear extension of the DS $\{g^t\}$, while its restriction to the closed invariant subset $TM|A = p^{-1}A$, which is a vector bundle over A, is the linear extension of the DS $\{g^t|A\}$.[2]

First we consider the simpler case of discrete time. By a *hyperbolic set* of a cascade $\{g^k\}$ (or diffeomorphism g) we mean a compact invariant set $A \subset M$

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[1] Sect. 4.2 was written by V.V. Solodov, the rest by D.V. Anosov.

[2] We take the opportunity to correct a somewhat careless formulation in (Anosov et al. 1985). The phase velocity $\mathbf{V}$ of the flow $\{Tg^t\}$ is $J T\mathbf{v}$, where $\mathbf{v} : M \to TM$ is the phase velocity of $\{g^t\}$ and J is the standard involution in TTM. We take local coordinates x^i in M and use the local coordinates $(x^i, \delta x^i)$ in TM and $(x^i, \delta x^i, v^i, \delta v^i)$ in TTM associated with them. (The coordinates of a point $X \in TM$ are written in the following order: first the coordinates of the point $x = pX$, then the coordinates of X as a vector in T_xM; the latter form the "vector part" of the coordinates of X. Similarly for $Y \in TTM$ first we write the coordinates of $X \in TM$ in the fibre over which Y lies, then the "vector part" of the coordinates of Y, that is, the coordinates of Y as a vector in $T_X TM$.) In this terminology,

such that for each point $x \in A$ the tangent space T_xM decomposes into a direct sum

$$T_xM = E_x^s \oplus E_x^u \tag{1}$$

of two subspaces, namely, a *stable* (*shrinking*) *space* E and an *unstable* (*expanding*) *space* E_x^u, with the following properties: for $\xi \in E_x^s$, $\eta \in E_x^u$, $k \geq 0$

a) $$|Tg^k(x)\xi| \leq a|\xi|e^{-ck}, \quad |Tg^{-k}(x)\xi| \geq b|\xi|e^{ck},$$

b) $$|Tg^k(x)\eta| \geq b|\eta|e^{ck}, \quad |Tg^{-k}(x)\eta| \leq a|\eta|e^{-ck},$$

where a, b, c are positive constants that are independent of x, ξ, η, k. The norm of the tangent vector is taken with respect to a fixed Riemannian metric on M; if the latter is changed, one merely has to alter the constants a, b, c. (This is because A is compact. It is immaterial whether the phase manifold M is compact, since for the moment we are only interested in what is happening near A.[3]) Clearly the subspaces E_x^s, E_x^u are uniquely defined by their properties a), b) (if $\zeta \in T_xM \backslash (E_x^s \cup E_x^u)$, then $|Tg^k(x)\zeta| \to \infty$ as $|k| \to \infty$). It is also easy to prove that their dimensions are locally constant (as functions of $x \in A$), while the subspaces themselves depend continuously on x and

$$Tg(x)E_x^s = E_{gx}^s, \; Tg(x)E_x^u = E_{gx}^u. \tag{2}$$

The unions $E^s = \bigcup\{E_x^s;\; x \in A\}$, $E^u = \bigcup\{E_x^u;\; x \in A\}$ are vector subbundles of the restriction $TM|A$ of the tangent bundle of M to A, and

---

$$(x^i, \delta x^i, v^i, \delta v^i) \overset{J}{\leftrightarrow} (x^i, v^i, \delta x^i, \delta v^i).$$

We verify that $\mathbf{V} = JT\mathbf{v}$. In fact,

$$\mathbf{v} : (x^i) \mapsto (x^i, v^i(x)), \quad T\mathbf{v} : (x^i, \delta x^i) \mapsto (x^i, v^i(x), \delta x^i, \delta v^i(x, \delta x)),$$

where

$$\delta v^i(x, \delta x) = \sum_j \frac{\partial v^i(x)}{\partial x^j} \delta x^j,$$

$\mathbf{V}(x, \delta x) = (x^i, \delta x^i$, vector part of coordinates of $\mathbf{V}) = (x^i, \delta x^i, (x^i)^{\cdot}, (\delta x^i)^{\cdot}) = (x^i, \delta x^i, v^i(x), \delta v^i(x, \delta x)) = JT\mathbf{v}(x, \delta x)$.

It might be thought that the description of $\mathbf{V}$ should play a significant role. However, this is not the case. Meanwhile, in the hyperbolic theory the explicit form of the variational equations is used principally in the examination of geodesic flows (and flows related to them), while there is another description for these equations that is given in Riemannian geometry.

[3] For the same reason the map g need not be defined outside some neighbourhood of A; similarly, in the case of a flow $\{g^t\}$, to be considered later, it is permissible for $g^t x$ not to be defined for all t if $x \neq A$; it is also possible that the phase velocity field is defined only near A. The branches of the theory of DS's in which one is concerned with the behaviour of a trajectory not in the entire phase space but in a neighbourhood of some invariant set A, which is generally more "extensive" than in the traditional local theory (where one considers a neighbourhood of a periodic trajectory) are sometimes referred to as semilocal. In such theories the DS, properly speaking, may not be a DS in the strict sense of the word (cf. Anosov et al. 1985, Chap. 1, Sect. 1.6).

$TM|A = E^s \oplus E^u$ (Whitney sum). (Here we allow the slight but fairly obvious generalization of the notion of a vector bundle in which the fibres can have different dimensions over different parts of the base A which, of course, are at a positive distance from each other.) These subbundles are invariant with respect to Tg and are called the *stable* and *unstable bundles*, respectively (for A, g and $\{g^k\}$).

A closed invariant subset of a hyperbolic set and unions of a finite number of hyperbolic sets are hyperbolic sets. A hyperbolic set A decomposes into the disjoint closed invariant sets

$$A_j = \{x \in A;\ \dim E_x^s = j\}.$$

When studying the properties of the hyperbolic sets themselves (including the behaviour of trajectories near to them), we can consider the A_j individually, that is, we may suppose at the outset that $\dim E_x^s$ takes the same value throughout A. The situation is different, however, when one says that some set A is hyperbolic in some DS. (This is particularly the case if the definition of A by itself bears no relation to hyperbolicity and A plays a significant role in the global qualitative picture. Thus in what follows we shall give great attention to DS's in which the set of nonwandering points is hyperbolic.) In this case the constancy of E_x^s on A can certainly not be taken for granted; A may quite well consist of several A_j.

The simplest example of a hyperbolic set (in fact one already known to us) is a hyperbolic periodic trajectory l (see Anosov et al. 1985, Chap. 1, Sect. 2.4)[4] or a finite system of such trajectories. If $x \in l$ and the (minimal) period of l is equal to τ, then in accordance with the decomposition (1) we need to take as E_x^s, E_x^u the invariant subspaces of the linear transformation $Tg^\tau(x) : T_xM \to T_xM$ corresponding to the eigenvalues λ with $|\lambda| < 1$ or $|\lambda| > 1$ respectively. It is then clear that inequalities a) and b) hold for some a, b, c. The constants a, b, c can be chosen to be independent of the point x for the simple reason that there are now only a finite number of such points.

In this example the motion is highly regular (as in the Preface, regularity is not a precise term, but rather an appeal to intuition; what could be more regular than a periodic sequence?). Furthermore, they could well be (although they do not have to be) stable, in which case the stability is then exponential; this is the strongest version of stability (in the sense of the nature of the dependence of the trajectory on the starting point). But clearly, the notion of a hyperbolic set has not been introduced for the sake of this example. More complicated examples will be given in Sect. 2. The behaviour of the trajectories in them is quite different in that it is complicated, irregular, unstable and, to some degree, imitating a random process. It is remarkable that such disparate objects fall under the same common definition based on the essentially simple notion of uniform exponential behaviour of the solutions

[4] Hyperbolicity of l means that there are no eigenvalues (that is, eigenvalues of the transformation $Tg^\tau(x)$, where $x \in l$ and τ is the period of l) of modulus 1.

of the variational equations along all the trajectories in the set in question. (Here uniformity means that the constants a, b, c in a) and b) can be chosen the same for all x, ξ, η.)

We now consider the case of a flow (that is, a DS with continuous time) $\{g^t\}$ defined on a manifold M by means of a smooth vector field $\mathbf{v}$. By a *hyperbolic set* of a flow $\{g^t\}$ we mean a compact invariant set $A \subset M$ such that: 1) if a point of A is an equilibrium point, then it is hyperbolic[5] (consequently, there are only a finite number of such points); 2) the set

$$B = A \backslash \{x : \mathbf{v}(x) = 0\}$$

is closed and for each point $x \in B$ the tangent space $T_x M$ decomposes into a direct sum

$$T_x M = E_x^{\mathrm{s}} \oplus E_x^{\mathrm{u}} \oplus \mathbb{R}\mathbf{v}(x) \tag{3}$$

of subspaces, the third of which is spanned by the phase velocity vector, while the properties of the first two are similar to those of $E_x^{\mathrm{s}}, E_x^{\mathrm{u}}$ in the discrete-time case; namely, for $\xi \in E_x^{\mathrm{s}}, \eta \in E_x^{\mathrm{u}},\ t \geq 0$

a) $\qquad |Tg^t(x)\xi| \leq a|\xi|e^{-ct}, \quad |Tg^{-t}(x)\xi| \geq b|\xi|e^{ct},$

b) $\qquad |Tg^t(x)\eta| \geq b|\eta|e^{ct}, \quad |Tg^{-t}(x)\eta| \leq a|\eta|e^{-ct},$

where a, b, c are positive constants independent of x, ξ, η, t. Everything that was said earlier (for discrete-time), namely, about the Riemannian metric, the nomenclature and properties of the subspaces $E_x^{\mathrm{s}}, E_x^{\mathrm{u}}$, the bundles formed by them, the finite unions of hyperbolic sets, the closed invariant subsets of hyperbolic sets and the parts of them with constant $\dim E_x^{\mathrm{s}}, \dim E_x^{\mathrm{u}}$, carries over word for word. Only this time there are points ζ in $T_x M$ for which $|Tg^t(x)\zeta|$ stays within certain positive limits for all t; these are the non-zero points of $\mathbb{R}\mathbf{v}(x)$ (we recall that

$$Tg^t(x)\mathbf{v}(x) = \mathbf{v}(g^t x), \tag{4}$$

(see Anosov et al. 1985, Chap. 1, Sect. 2.2) and only these. The subspace $\mathbb{R}\mathbf{v}(x)$ is called the *neutral* or *centre* subspace. The second name is explained by a certain analogy between it (and to an even greater extent the trajectory $\{g^t x\}$) and the centre manifolds in the local qualitative theory (see Il'yashenko 1985, Chap. 3, Sect. 4.2 and Chap. 6, Sect. 2.3), which also is distinguished by the "small" (non-exponential) speed of the motions occurring in it. Accordingly, this subspace can also be denoted by E_x^{n} or E_x^{c}, and sometimes the notation

[5] We recall (see Anosov et al. 1985, Chap. 1, Sect. 2.4) that in the theory of smooth DS's an equilibrium point is said to be hyperbolic if none of its eigenvalues (that is, the eigenvalues of the coefficient matrix corresponding to the linearized system) lies on the imaginary axis. (We do not exclude the possibility that their real parts are all of the same sign; in this regard the terminology differs from that which arose earlier in the qualitative theory of differential equations, where a hyperbolic point refers to a saddle, but not a node or a focus.)