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Yu. D. Burago　V. A. Zalgaller (Eds.)

Geometry III

Theory of Surfaces

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曲面理论

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I. The Geometry of Surfaces in Euclidean Spaces

Yu.D. Burago, S.Z. Shefel'

Translated from the Russian
by E. Primrose

Contents

Preface

The original version of this article was written more than five years ago with S.Z. Shefel', a profound and original mathematician who died in 1984. Since then the geometry of surfaces has continued to be enriched with ideas and results. This has required changes and additions, but has not influenced the character of the article, the design of which originated with Shefel'. Without knowing to what extent Shefel' would have approved the changes, I should nevertheless like to dedicate this article to his memory. (Yu.D. Burago)

We are trying to state the qualitative questions of the theory of surfaces in Euclidean spaces in the form in which they appear to the authors at present. This description does not entirely correspond to the historical development of the subject. The theory of surfaces was developed in the first place mainly as the theory of surfaces in three-dimensional Euclidean space E^3; however, it makes sense to begin by considering surfaces F in Euclidean spaces of any dimension $n \geqslant 3$. This approach enables us, in particular, to put in a new light some unsolved problems of this developed (and in the case of surfaces in E^3 fairly complete) theory, and in many cases to refer to the connections with the present stage of development of the theory of multidimensional submanifolds.

The leading question of the article is the problem of the connection between classes of metrics and classes of surfaces in E^n. The first chapter is a brief survey of general questions in the theory of surfaces from this point of view. Chapters 2 and 3 are devoted to a more detailed consideration of convex and saddle surfaces respectively. The subject of Chapter 4 consists of classes of metrics not associated directly with the condition that the Gaussian curvature has a definite sign, and G-stable immersions of them.

A whole series of important questions in the theory of surfaces remain outside the framework of the article. We only touch on questions of the purely extrinsic geometry of surfaces. This applies above all to the most developed and complete theory of convex surfaces. Thus, the geometric theory of equations (basically of Monge-Ampère type) is only recalled, and there is no description of existence and uniqueness theorems for surfaces with given conditional curvatures. The reader can become acquainted with these questions from the monographs Bakel'man, Verner and Kantor (1973), Pogorelov (1969), Pogorelov (1975). We do not consider boundary-value problems of the theory of bending of convex surfaces, infinitesimal bendings of high orders, or subtle questions of the bending of surfaces in a neighbourhood of an isolated zero of the curvature. For these questions see Part III of the present book.

Chapter 1
The Geometry of Two-Dimensional Manifolds and Surfaces in E^n

§ 1. Statement of the Problem

As the title itself emphasizes, in our article we consider only questions in the theory of surfaces in E^n, although many of the results recalled carry over automatically to surfaces in spaces of constant curvature, and sometimes in Riemannian manifolds. Of course, there are aspects that are specific for such spaces; we shall not dwell on them, see Pogorelov (1969), Milka (1980), for example.

1.1. Classes of Metrics and Classes of Surfaces. Geometric Groups and Geometric Properties. It is well known that every (for simplicity, sufficiently smooth) surface in E^n, considered from the viewpoint of its intrinsic metric, uniquely determines a Riemannian manifold. On the other hand, an abstractly defined Riemannian manifold can always be isometrically immersed in some E^n, but such an immersion is not unique, and generally speaking the properties of the Riemannian metric do not have an appreciable influence on the geometry of the immersed surface. In the natural problem of the connection between properties of a surface and properties of its intrinsic metric we shall be mainly interested in the following two aspects.

Firstly, we have the question of which of the intrinsic properties of a surface can be guaranteed by some completely determined extrinsic geometrical properties of it. (Of course, the answer to this question depends on what one understands by a "geometric" property of a surface.) Secondly, there is the question of the restriction of the class of admissible immersions to "regular" ones, that is, immersions for which the properties of the metric have an appreciable influence on the extrinsic properties of the surface. The following definition of a geometric property of a surface is basic for our later arguments.

A property of a surface is said to be *geometric* if it is preserved by transformations of E^n that belong to some group G. We always assume that G contains the group of similarities and is distinct from it. Such groups are called *geometric*. A classification of geometric groups was obtained in G.S. Shefel' (1984), G.S. Shefel' (1985). Leaving a detailed discussion of this question to 2.2 of Ch. 4, we note that it is meaningful to consider only the group of affine transformations[1], the pseudogroup of Möbius transformations (generated by similarities and inver-

[1] Since the dimension n of the ambient space is not fixed, it is a question, strictly speaking, of an infinite choice of groups A_n of affine transformations of E^n for all $n > 2$ and similarly in the other cases.

sions when $n > 2$) and, to rather different ends, the group of all diffeomorphisms of fixed smoothness.

The given definition of a geometric property makes more precise the first of the questions posed above and suggests an answer to the second. Namely by the "regularity" of an immersion we shall understand its G-stability.

Definition. A surface F in E^n is called a *G-stable* immersion of the metric of some class $\mathscr{K}$ if any transformation belonging to the group G takes F into a surface whose intrinsic metric also belongs to the class $\mathscr{K}$.

Here it is assumed that G is a geometric group (or pseudogroup) of transformations in E^n. Since the identity transformation id belongs to G, it is obvious that the intrinsic metric of the surface F itself belongs to $\mathscr{K}$. In this definition the class of metrics $\mathscr{K}$ is not necessarily exhausted by Riemannian metrics. Correspondingly, by an immersion of a metric here we understand a C^0-smooth (topological) immersion which is an isometry.

It is essential that the requirement of G-stability of a surface does not impose any a priori restrictions on the dimension n of the ambient space. We note that G-stable immersions of metrics of some class $\mathscr{K}$ (not exhausting all admissible metrics) always have a certain general geometric property. Transition from any immersions to G-stable ones enables us to establish a dual connection between extrinsic and intrinsic properties of surfaces.

The naturalness of the concept of G-stability is illustrated by the following assertions, proved in the most general form in S.Z. Shefel' (1969), S.Z. Shefel' (1970), Sabitov and S.Z. Shefel' (1976). The only affine-stable immersions in E^n, $n \geqslant 3$, for the class of two-dimensional Riemannian metrics of positive curvature are locally convex surfaces in some $E^3 \subset E^n$. The class of affine stable immersions for two-dimensional Riemannian metrics of negative curvature is by no means exhausted by surfaces in E^3, but all such immersions belong to the class of so-called saddle surfaces, that is, surfaces that locally do not admit strictly supporting hyperplanes; for the details see 3.1 of Ch. 3. Now suppose that G is the group of diffeomorphisms in E^n of smoothness C^∞. Then the only G-stable immersions for the class of Riemannian metrics of smoothness $C^{l,\alpha}$, $l \geqslant 2$, $0 < \alpha < 1$, are surfaces of the same smoothness.

The most attractive situation is that in which the class of metrics $\mathscr{K}$, the group G and the class of surfaces $\mathscr{M}$ have the following relations.

1°. The class of surfaces $\mathscr{M}$ coincides with the class of all G-stable immersions of metrics of the corresponding class of metrics $\mathscr{K}$.

2°. Every metric of the class $\mathscr{K}$ admits an immersion in the form of a surface of class $\mathscr{M}$.

In this case the class of surfaces $\mathscr{M}$ and the class of metrics $\mathscr{K}$ are said to be *G-connected.*

Later we shall also use the concept of G-connectedness "in the small" and G-connectedness "in the large"; for details see the next section.

The given definition admits gradations depending on how we understand the terms surface, metric, and immersion of a metric. For example, affine-stable immersions in E^n of a one-element class of plane metrics on E^2 contain all cylinders (with rectifiable directrix) or consists only of smooth cylinders, depending on whether we understand by a surface any C^0-immersion or only a smooth one. We must take into account that the fact that a surface and all its images under affine transformations have a smooth intrinsic metric does not imply, generally speaking, that the surface itself is smooth[2]. Therefore in § 2 all metrics, surfaces and immersions are a priori assumed to be smooth. In the examination of non-regular surfaces and metrics, by isometric immersions we understand topological (of smoothness C^0) immersions that are isometries.

Otherwise it is a question of immersions that are stable with respect to the group of diffeomorphisms; see § 5 of Ch. 4.

§ 2. Smooth Surfaces

2.1. Types of Points. We assume that F is a smooth surface, that is, an immersion of smoothness C^l, $l \geqslant 3$, of a two-dimensional manifold M in E^n, $n \geqslant 3$. In differential geometry it is usual to describe surfaces by means of the first and second fundamental forms. The first fundamental form specifies the intrinsic (induced) metric of the surface – a metric where the distance between points is equal to the greatest lower bound of lengths of curves joining these points on the surface. The second fundamental form determines at each point of the surface a family of osculating paraboloids. Let us explain this.

Let B be the second fundamental form of a surface F at a fixed point p. If F is specified by a vector-valued function $r(u^1, u^2)$, then

$$B(X, Y) = \sum_{i,j=1}^{2} X^i Y^j (r_{ij})^N.$$

Here X^i and Y^j are the coordinates of vectors X and Y tangent to F in the basis (r_1, r_2), where $r_i = \partial r/\partial u^i$, $r_{ij} = \partial^2 r/\partial u^i \partial u^j$, and the index N denotes projection into the normal (that is, orthogonal to T_pF) subspace.

Every projection of the graph Γ of the map $X \mapsto B(X, X)$ onto the three-dimensional space spanned by T_pF and some normal v is a paraboloid (or degenerates into a cylinder) and is called the *osculating paraboloid*. In the case of degeneracy to a cylinder we shall call the latter a parabolic paraboloid by analogy with elliptic and hyperbolic paraboloids.

We note that the subspace spanned by Γ is said to *osculate* F at the point p. Its dimension is at most five. For it is spanned in E^n by the two-dimensional subspace T_pF and the vectors $(r_{11})^N$, $(r_{12})^N$, $(r_{22})^N$.

In the case of a surface in E^3 the family of osculating paraboloids consists of one paraboloid. According to the type of osculating paraboloid the points of a

[2] A remarkable exception consists of smooth metrics of positive curvature under locally convex immersions; see § 3 of Ch. 2.

surface in E^3 are traditionally divided into elliptic, hyperbolic and parabolic (in particular, flat points), which forms the only possible affine classification of points of a surface in E^3 up to infinitesimals of the second order. The affine classification of points coincides with the classification according to the sign of the Gaussian curvature.

When $n > 3$ the affine classification of points of a smooth surface in E^n is also determined by the affine-invariant properties of the family of osculating paraboloids at a point p and leads to eight different types of points (S.Z. Shefel' (1985)). Without giving the classification itself here, we note that for two of these types the Gaussian curvature[3] of the surface at p is zero. Points of these two types are called *parabolic*. For another type of point the Gaussian curvature is positive (*elliptic point*). For three other types of point the Gaussian curvature is negative (*hyperbolic point*), and in two cases the sign of the Gaussian curvature is not determined by the type of point (such points are said to be *movable*). The first two of these types – parabolic points – have a common property: among the osculating paraboloids there are no elliptic or hyperbolic ones. One type – elliptic point – is characterized by the fact that among the osculating paraboloids there are elliptic but no hyperbolic or non-degenerate parabolic ones. Three more types are characterized by the fact that there are hyperbolic paraboloids but no elliptic ones (hyperbolic point). Finally, the two remaining types are characterized by the fact that at a point there are elliptic, hyperbolic and parabolic paraboloids.

2.2. Classes of Surfaces. The classification of points enables us to distinguish six classes of smooth surfaces. Surfaces of the first three classes M^+, M^-, M_0 consist, respectively, of only elliptic, hyperbolic or parabolic points. Surfaces of class M_0^+ consist only of elliptic and parabolic points, and surfaces of class M_0^- consist only of hyperbolic and parabolic points. Finally, the class M is formed by all smooth surfaces.

Surfaces of the class M_0^+ are called *normal* surfaces of non-negative curvature, and surfaces of the classes M_0^- and M^- are called *saddle* surfaces and *strictly saddle* surfaces respectively.

Theorem 2.2.1 (S.Z. Shefel' (1970)). *The class M^+ in E^n consists of locally convex sufaces each lying in some $E^3 \subset E^n$. A complete surface of class M^+ is a complete convex surface (the boundary of a convex body in E^3). Normal surfaces of non-negative curvature (of class M_0^+) are characterized by the fact that either every point of such a surface has a neighbourhood in the form of a convex surface or through this point there passes a rectilinear generator with its ends on the boundary of the surface, and the tangent plane along this rectilinear generator is stationary. A complete surface of class M_0^+ is either a convex surface in E^3 or a cylinder in E^n.*

[3] By the Gaussian curvature K of a smooth surface in E^n we always have in mind the Gaussian (that is, sectional) curvature of its intrinsic metric. By the generalized Gauss theorem $K = B(X, X)B(Y, Y) - B(X, Y)^2$ when $\|X \wedge Y\| = X^2Y^2 - \langle X, Y\rangle^2 = 1$.

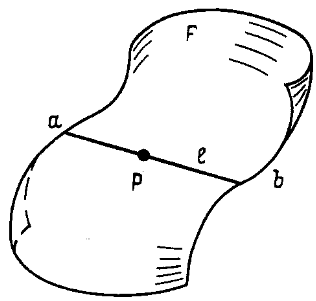

Fig. 1

The class M_0 consists of developable surfaces. The complete surfaces of this class are cylinders.

Saddle surfaces F (the class M_0^-) can be characterized by the property that no hyperplane cuts out from F a *crust*, that is, a region whose closure is compact and does not go out to the boundary of F.

Fig. 1 shows the case when a surface of class M_0^+ in a neighbourhood of a point p is neither locally convex nor developable (ab is a rectlinear generator). We should emphasize that, in contrast to the class M^+, surfaces of the class M_0^+, like all the subsequent classes, can be essentially n-dimensional for any $n > 3$, that is, they do not lie in any proper subspace of E^n.

Thus, the theory of convex surfaces is, by necessity, the theory of surfaces in E^3, while surfaces of all the remaining classes are naturally regarded as surfaces in E^n for all $n \geqslant 3$.

The reason for such an exceptional position of convex surfaces has a simple algebraic nature. Let B be the second fundamental form of a surface F at some point p. Consider a linear map L of the normal space to F at p into $\mathbb{R}^3$ according to the following rule: we fix a basis in T_pF and associate with each normal ν an ordered triple of numbers (a, b, c), the coefficients of the quadratic form $B^\nu(X, X) := \langle B(X, X), \nu \rangle$, where $\langle\ ,\ \rangle$ is the scalar product. The type of osculating paraboloid corresponding to the normal ν (and vectors parallel to it) is determined by the sign of the discriminant $ac - b^2$. In particular, every direction for which the osculating paraboloid is elliptic is mapped inside the cone $ac - b^2 > 0$, Fig. 2. Therefore all osculating paraboloids can be elliptic or degenerate only if $q = \dim \text{image } L \leqslant 1$. Similarly at a hyperbolic point, where there are no elliptic paraboloids, we certainly have $q = \dim \text{image } L \leqslant 2$.

If $q = 3$ at all points, then the immersion (surface) is said to be *free*. Surfaces consisting only of variable points form the closure of the set of free immersions in the corresponding topology. In the class of saddle surfaces it is natural to regard the situation of general position as that in which $q = 2$ everywhere (the osculating space is four-dimensional), and in the class of convex surfaces $q = 1$ (the osculating space is three-dimensional). For convex surfaces the condition $q = 1$ (that is, $q \neq 0$) means that the Gaussian curvature does not vanish.

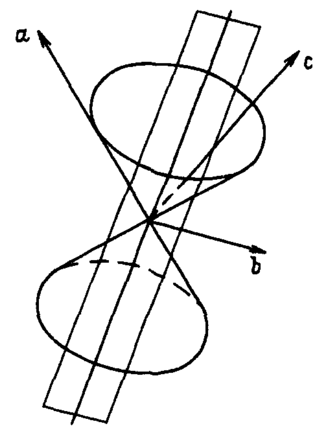

Fig. 2

2.3. Classes of Metrics. According to the sign of the Gaussian curvature it is natural to distinguish the following classes of two-dimensional Riemannian metrics: the classes K^+, K^-, K_0 of Riemannian metrics of positive, negative and zero curvature, the classes K_0^+, K_0^- of metrics of non-negative and non-positive curvature, and the class K of all Riemannian metrics. The classes of surfaces and metrics marked with the same indices will be called corresponding.

2.4. *G*-Connectedness. Local properties of smooth surfaces and metrics usually reduce to conditions on the surface (or metric) at each point of it. As a rule, these conditions describe the behaviour of the surface (metric) in a neighbourhood of a point up to the second order of smallness. Henceforth a geometric property of a surface will be called *local* if it is a property of a point of the surface, and its fulfilment at some point p of the surface F implies its fulfilment at p for any other surface that coincides with F in a neighbourhood of p up to infinitesimals of the second order.

For classes of surfaces and metrics distinguished on the basis of their local properties we shall distinguish G-connectedness in the small and G-connectedness in the large and correspondingly formulate two problems: in the small and in the large.

The class of surfaces $\mathcal{M}$ and the class of metrics $\mathcal{K}$ are said to be *G-connected in the small* if 1) the class of surfaces $\mathcal{M}$ coincides with the class of G-stable immersions of metrics of $\mathcal{K}$, 2) every metric of $\mathcal{K}$ admits a local immersion in the form of a surface of $\mathcal{M}$. The problem in the small consists in looking for classes of surfaces and metrics that are G-connected in the small.

The class $\tilde{\mathcal{M}}$ of complete surfaces and the class $\tilde{\mathcal{K}}$ of complete metrics are said to be *G-connected in the large* if 1) the class of surfaces $\tilde{\mathcal{M}}$ coincides with the class of G-stable immersions of metrics of $\tilde{\mathcal{K}}$, 2) every metric of $\tilde{\mathcal{K}}$ admits an immersion (in the large) in the form of a surface of $\tilde{\mathcal{M}}$. The problem in the large consists in looking for classes of surfaces and metrics that are G-connected in the large.

In contrast to the problem in the small, here even in those cases when local properties are fundamental for the distinction of classes, we need to impose a priori conditions of non-local character on classes of complete surfaces and metrics that are G-connected in the large.

This is because the local conditions that distinguish classes of metrics and surfaces that are G-connected in the small may lead to topological restrictions that are different for surfaces and metrics. For example, on the projective plane there are metrics of positive curvature, but none of them admits affine stable immersions in E^n. Moreover, in the case of classes of surfaces and metrics defined by local conditions that are G-connected in the small there may exist non-local obstructions for G-stable isometric immersions that have not only topological but also mixed topological-metric character. Thus, on a sphere with three punctures there are complete Riemannian metrics of non-positive curvature that are immersible in E^3 and non-immersible as a saddle surface in any E^n; see 1.4 of Ch. 3.

In the case of complete metrics of positive curvature, and correspondingly complete convex surfaces, the only (purely topological) obstruction is non-connectedness. The matter is simple in the case of zero curvature. However, finding all obstructions to immersibility of complete metrics of non-positive (negative) curvature in the form of complete saddle (strictly saddle) surfaces in at least one E^n is a difficult problem. (The case of simply-connected surfaces is simpler; for them it may be that all obstructions are trivial; see 1.3 of Ch. 3 and 4.3 of Ch. 4.)

2.5. Results and Conjectures. In this chapter a fundamental question is that of the correspondence of surfaces and metrics in the case of smooth surfaces[4] and for the affine transformation group, as in the general case, it consists of the problem in the small and the problem in the large. The problem in the small for the classes K^+, K^-, K_0, K has been solved completely; we have the following two theorems.

Theorem 2.5.1. *The classes M^+, M^-, M_0, M of smooth surfaces and the corresponding classes of metrics are pairwise affine connected in the small.*

Theorem 2.5.2. *If we restrict ourselves to those classes of smooth surfaces, each of which is defined by a local geometric property, then there are no pairs that are affine connected in the small other than those listed in Theorem* 2.5.1 *and possibly the pairs K_0^-, M_0^-.*

Theorem 2.5.1 combines the following assertions.

1°. Each of the classes of surfaces mentioned above is affine-invariant.

---

[4] We recall that a smooth surface is always understood to be an immersion of class C^l, $l \geqslant 3$. Special cases, such as C^0-smoothness (topological immersion) or $C^{l,\alpha}$-smoothness, will be treated specially.

2°. The intrinsic metric of a surface of any of these classes belongs to the corresponding class of metrics.

3°. An affine-stable immersion in E^n of a metric of any class belongs to the corresponding class of surfaces.

4°. Every metric of any of the classes admits a local immersion in the form of a surface of the corresponding class in E^n.

Assertions 1°–3° hold for all six classes. The first of them is obvious. The second follows from the many-dimensional generalization of Gauss's theorem. The third assertion is proved in S.Z. Shefel' (1970). The fourth assertion has been proved (see Pogorelov (1969), Poznyak and Shikin (1974)) only for the classes listed in the theorem.

Let us proceed to complete metrics and surfaces.

Theorem 2.5.3. *The classes $\tilde{M}^+$, $\tilde{M}_0$, $\tilde{M}$ of smooth complete simply-connected surfaces and the corresponding classes of Riemannian metrics are affine connected in the large*[5].

Like Theorem 2.5.1, this theorem combines four assertions. The first three of them are the same as in Theorem 2.5.1, and are therefore proved. The fourth assertion is as follows: every complete simply-connected Riemannian manifold of any of the classes $\tilde{K}^+$, $\tilde{K}_0$, $\tilde{K}$ admits an immersion in the form of a (complete) surface of the corresponding class. In the case of $\tilde{K}_0$ this is obvious. Also, every complete Riemannian metric of positive curvature, defined on a sphere or plane, admits an immersion in E^3 in the form of a smooth complete convex surface. This is the solution of Weyl's famous problem and its analogue for non-compact surfaces; for details see Ch. 2. Therefore in the case of the classes $\tilde{K}^+$ and $\tilde{K}_0$ Theorem 2.5.3 is true. It is also true for the class $\tilde{K}$ (even without the requirement of simply-connectedness) by a general theorem of Nash on isometric immersions (Nash (1956)).

Let us state the proposition that the classes $\tilde{K}^-$ and $\tilde{M}^-$ of smooth simply-connected surfaces and metrics are affine-connected in the large. This proposition combines four parts, of which the first three are the same as in Theorems 2.5.1 and 2.5.3, and are automatically true. The fourth part can be stated as follows.

Conjecture A[6]. *A complete simply-connected Riemannian metric of negative curvature admits an isometric immersion in some E^n in the form of a saddle surface.*

Together with Theorem 2.5.3, Conjecture A, when it is true, can be regarded as a generalization of Weyl's problem. In any case all the results about non-immersibility, in the first place Hilbert's classical theorem and the well-known

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[5] Here and later a tilde over a letter implies the completeness of the metric or surface.

[6] This conjecture was made in S.Z. Shefel' (1978), S.Z. Shefel' (1979), but with superfluous generality, without the assumption of simply-connectedness; as we mentioned above, such a generalized conjecture is false.

more general theorem of Efimov (see § 1.1 of Ch. 3), do not contradict our conjecture, since here the class of immersions is restricted not by the dimension of the space but by a geometric property, the saddle form.

The class $\tilde{K}^+$ *of metrics and the corresponding class of surfaces do not form* an affine connected pair. It is true that a complete simply-connected Riemannian manifold of non-negative curvature admits an immersion in E^3 in the form of a convex surface, but the smoothness of this surface may turn out to be substantially lower than the smoothness of the metric at the zeros of the curvature; see the example in 1.1 of Ch. 2. Such a lowering of the smoothness also takes place when considered locally; it is easy to verify this on the basis of an example from Pogorelov (1971). The authors do not have corresponding examples for the classes K_0^- and $\tilde{K}_0^-$. We observe that in the case of analytic metrics and surfaces the classes ${}^aK_0^-$ and ${}^a\tilde{K}_0^-$ of analytic metrics are affine connected in the small with the corresponding classes of surfaces; see Poznyak (1973).

The fact that not all the classes of surfaces under consideration are affine connected with the corresponding classes of metrics is probably stipulated by the eclectic character of these classes: they are distinguished simultaneously by geometric properties (convexity, saddle form, and so on) and the a priori requirement of smoothness. However, as we mentioned at the end of § 1, smoothness is not an affine stable property in general; for details see § 5 of Ch. 4. We can therefore hope that in the case of not necessarily smooth surfaces distinguished on the basis of just geometric properties there arise only classes that are affine connected with the corresponding classes of metrics; see § 3 below.

2.6. The Conformal Group. Let us now dwell on the conformal group of transformations. At each point of any smooth surface either 1) all the osculating paraboloids are paraboloids of rotation or degenerate, or 2) by a conformal transformation we can arrange that the Gaussian curvature of the surface at this point takes any value. Hence it follows easily that apart from the class of all surfaces and the class of all metrics the only ones that are conformally connected in the small are the class of surfaces in E^3 locally congruent to a sphere or a plane, and the class of metrics of constant curvature.

If a group of diffeomorphisms that preserves the subgroup of similarities is not affine or conformal, then by the action of this group we can achieve any value of the Gaussian curvature at some point of the surface (G.S. Shefel' (1985)). Therefore all other groups distinguish only the class of all metrics and the class of all surfaces, and consideration of them from these positions is not meaningful.

The principle of correspondence between classes of surfaces and metrics distinguishes classes of surfaces and metrics that play a central role in the theory of surfaces and in Riemannian geometry, and this is one of the basic forms of connection between intrinsic and extrinsic geometry. Only metrics of constant negative curvature have not found their natural place in this scheme. It is possible that a similar approach in the case of a pseudo-Euclidean space could distinguish such metrics instead of metrics of constant positive curvature.

§ 3. Convex, Saddle and Developable Surfaces with No Smoothness Requirement

3.1. Classes of Non-Smooth Surfaces and Metrics. The classes of surfaces considered above, apart from the general class M, admit synthetic definitions (that is, purely geometric, not requiring any analytic apparatus). These definitions, but without any a priori assumption of smoothness, distinguish the wider classes $\mathscr{M}_0^+$, $\mathscr{M}_0^-$, $\mathscr{M}_0$, $\mathscr{M}^+$, $\mathscr{M}^-$ of generally speaking non-regular surfaces. Complete surfaces of the first three classes are complete convex surfaces, complete saddle surfaces, and cylinders.

These classes, apart from possibly non-simply-connected saddle surfaces, have the compactness property: if compact surfaces F_i, lying in E^n, of one of the classes have the same topology and their boundaries form a compact family, then we can pick out from them a convergent subsequence (it is a question of Fréchet convergence); see Aleksandrov (1939), G.S. Shefel' (1984). The classes $\mathscr{M}_0^+$, $\mathscr{M}_0^-$, $\mathscr{M}_0$ and the corresponding classes $\tilde{\mathscr{M}}_0^+$, $\tilde{\mathscr{M}}_0^-$, $\tilde{\mathscr{M}}_0$ are closed[7] in the sense that a convergent subsequence of surfaces of one class converges to a surface of the same class. The classes $\mathscr{M}^+$ and $\mathscr{M}^-$ are not closed and in this connection they play a minor role.

What we have said here about surfaces can largely be repeated for metrics. The classes of Riemannian metrics considered above admit a simple synthetic description. The classes K_0^+, K_0^-, K_0 are characterized by the fact that the excess (that is, the difference between the sum of the angles and π) of any simply-connected triangle of shortest curves is respectively non-negative, non-positive and equal to zero. (For the classes K^+ and K^- we need to compare the excess with the area of the triangle.) Let us now give up the fact that the metric is Riemannian, that is, we shall consider a two-dimensional manifold with an intrinsic metric (given directly by distances, and not by means of a quadratic form). For precise definitions of a triangle, an angle, and other concepts in such a space, we refer the reader to Aleksandrov and Zalgaller (1962). Then, depending on the sign of the excesses of triangles, we distinguish five classes of generally speaking non-Riemannian metrics. These are the classes $\mathscr{K}_0^+$, $\mathscr{K}_0^-$, $\mathscr{K}_0$ of metrics of non-negative, non-positive and zero curvature (the last class consists merely of flat Riemannian metrics), and two more classes $\mathscr{K}^+$, $\mathscr{K}^-$ of metrics of strictly positive and strictly negative curvature. The classes $\mathscr{K}_0^+$, $\mathscr{K}_0^-$, $\mathscr{K}_0$ are closed, but $\mathscr{K}^+$, $\mathscr{K}^-$ are not closed. Criteria for compactness of these classes are apparently not known.

3.2. Questions of Approximation. Another approach, which leads to non-regular surfaces and metrics, is as follows. We complete the classes M_0^+, M_0^-, M_0

[7] Surfaces in a Euclidean space of fixed dimension form a metric space T with a Fréchet metric. This space is complete. The fact that a class Ψ is closed means that the set $T \cap \Psi$ is closed in T. If we regard $T \cap \Psi$ as a metric space, it is a question of its completeness.