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E. B. Vinberg (Ed.)

Geometry II
Spaces of Constant Curvature

几 何 II
常曲率空间

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I. Geometry of Spaces of Constant Curvature

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Translated from the Russian
by V. Minachin

Contents

Preface

Spaces of constant curvature, i.e. Euclidean space, the sphere, and Lobachevskij space, occupy a special place in geometry. They are most accessible to our geometric intuition, making it possible to develop elementary geometry in a way very similar to that used to create the geometry we learned at school. However, since its basic notions can be interpreted in different ways, this geometry can be applied to objects other than the conventional physical space, the original source of our geometric intuition.

Euclidean geometry has for a long time been deeply rooted in the human mind. The same is true of spherical geometry, since a sphere can naturally be embedded into a Euclidean space. Lobachevskij geometry, which in the first fifty years after its discovery had been regarded only as a logically feasible by-product appearing in the investigation of the foundations of geometry, has even now, despite the fact that it has found its use in numerous applications, preserved a kind of exotic and even romantic element. This may probably be explained by the permanent cultural and historical impact which the proof of the independence of the Fifth Postulate had on human thought.

Nowadays modern research trends call for much more businesslike use of Lobachevskij geometry. The traditional way of introducing Lobachevskij geometry, based on a kind of Euclid-Hilbert axiomatics, is ill suited for this purpose because it does not enable one to introduce the necessary analytical tools from the very beginning. On the other hand, introducing Lobachevskij geometry starting with some specific model also leads to inconveniences since different problems require different models. The most reasonable approach should, in our view, start with an axiomatic definition, but it should be based on a well-advanced system of notions and make it possible either to refer to any model or do without any model at all.

Their name itself provides the description of the property by which spaces of constant curvature are singled out among Riemannian manifolds. However, another characteristic property is more important and natural for them — the property of maximum mobility. This is the property on which our exposition is based.

The reader should realize that our use of the term "space of constant curvature" does not quite coincide with the conventional one. Usually one understands it as describing any Riemannian manifold of constant curvature. Under our definition (see Chap. 1, Sect. 1) any space of constant curvature turns out to be one of the three spaces listed at the beginning of the Preface.

Although Euclidean space is, of course, included in our exposition as a special case, we have no intention of introducing the reader to Euclidean geometry. On the contrary, we make free use of its basic facts and theorems. We also assume that the reader is familiar with the basics of linear algebra and affine geometry, the notion of a smooth manifold and Lie group, and the elements of Riemannian geometry.

For the history of non-Euclidean geometry and the development of its ideas the reader is referred to relevant chapters in the books of Klein [1928], Kagan [1949, 1956], Coxeter [1957], and Efimov [1978].

Chapter 1
Basic Structures

§ 1. Definition of Spaces of Constant Curvature

This chapter provides the definition of spaces of constant curvature and of their basic structures, and describes their place among homogeneous spaces on the one hand and Riemannian manifolds on the other. If the reader's main aim is just to study Lobachevskij geometry, no great damage will be done if he skips Theorems 1.2 and 1.3 and the proof of Theorem 2.1.

1.1. Lie Groups of Transformations. We assume that the reader is familiar with the notions of a (real) smooth manifold and of a (real) Lie group. The word "smooth" (manifold, function, map etc.) always means that the corresponding structure is C^∞. All smooth manifolds are assumed to have a countable base of open subsets. By $T_x(X)$ we denote the tangent space to a manifold X at a point x, and by $d_x g$ the differential of the map g at a point x. If no indication of the point is necessary the subscript is omitted.

We now recall some basic definitions of Lie group theory. (For more details see, e.g. Vinberg and Onishchik [1988].)

A group G of transformations[1] of a smooth manifold X endowed with a Lie group structure is said to be a *Lie group of transformations* of the manifold X if the map

$$G \times X \to X, \qquad (g, x) \mapsto gx,$$

is smooth, which means that the (local) coordinates of the point gx are smooth functions of the coordinates of the element g and the point x. Then the stabilizer

$$G_x = \{g \in G : gx = x\}$$

of any point $x \in X$ is a (closed) Lie subgroup of the group G. Its linear representation $g \mapsto d_x g$ in the space $T_x(X)$ is called the *isotropy representation* and the linear group $d_x G_x$ is called the *isotropy group* at the point x.

The stabilizers of equivalent points x and $y = gx \quad (g \in G)$ are conjugate in G, i.e.

$$G_y = gG_xg^{-1}.$$

The corresponding isotropy groups are related in the following way:

$$d_xG_y = (d_xg)(d_xG_x)(d_xg)^{-1}.$$

In other words, if tangent spaces $T_x(X)$ and $T_y(Y)$ are identified by the isomorphism d_xg, then the group d_xG_x coincides with the group d_yG_y.

[1] By a group of transformations we understand an effective group of transformations, i.e. we assume that different transformations correspond to different elements of the group.

If G is a transitive Lie group of transformations of a manifold X, then for each point $x \in X$ the map

$$G/G_x \to X, \qquad gG_x \mapsto gx$$

is a diffeomorphism commuting with the action of the group G. (The group G acts on the manifold G/G_x of left cosets by left shifts.) In this case the manifold X together with the action of G on it can be reconstructed from the pair (G, G_x).

Definition 1.1. A smooth manifold X together with a given transitive Lie group G of its transformations is said to be a *homogeneous space.*

We denote a homogeneous space by (X, G), or simply X.

A homogeneous space (X, G) is said to be connected or simply-connected[2] if the manifold X has this property.

1.2. Group of Motions of a Riemannian Manifold. A Riemannian metric is said to be defined on a smooth manifold X if a Euclidean metric is defined in each tangent space $T_x(X)$, and if the coefficients of this metric are smooth functions in the coordinates of x. A diffeomorphism g of a Riemannian manifold X is called a *motion* (or an *isometry*) if for each point $x \in X$ the linear map

$$d_x g : T_x(X) \to T_{gx}(X)$$

is an isometry. The set of all motions is evidently a group.

Each motion g takes a geodesic into a geodesic, and therefore commutes with the exponential map, i.e.

$$g(\exp \xi) = \exp dg(\xi)$$

for all $\xi \in T(X)$. Hence each motion g of a connected manifold X is uniquely defined by the image gx of some point $x \in X$ and the differential $d_x g$ at that point. This enables us to introduce coordinates into the group of motions, turning it into a Lie group. To be more precise, the following theorem holds.

Theorem 1.2 (Kobayashi and Nomizu [1981]). *The group of motions of a Riemannian manifold X is uniquely endowed with a differentiable structure, which turns it into a Lie group of transformations of the manifold X.*

If the group of motions of a Riemannian manifold X is transitive, then X is complete. Indeed, in this case there exists $\varepsilon > 0$, which does not depend on x, such that for any point $x \in X$ and for any direction at that point there exists a geodesic segment of length ε issuing from x in that direction. This implies that each geodesic can be continued indefinitely in any direction.

A Riemannian manifold X is said to have *constant curvature* c if at each point its sectional curvature along any plane section equals c.

[2] We assume that any simply-connected space is, by definition, connected.

Simply-connected complete Riemannian manifolds of constant curvature admit a convenient characterization in terms of the group of motions.

Theorem 1.3 (Wolf [1972]). *A simply-connected complete Riemannian manifold is of constant curvature if and only if for any pair of points $x, y \in X$ and for any isometry $\varphi : T_x(X) \to T_y(X)$ there exists a (unique) motion g such that $gx = y$ and $d_x g = \varphi$.*

The first part of the statement follows immediately from the fact that motions preserve curvature and that any given two-dimensional subspace of the space $T_x(X)$ can, by an appropriate isometry, be taken into any given two-dimensional subspace of the space $T_y(X)$. For the proof of the converse statement see Chap. 8, Sect. 1.3.

1.3. Invariant Riemannian Metrics on Homogeneous Spaces. Let (X, G) be a homogeneous space. A Riemannian metric on X is said to be *invariant* (with respect to G) if all transformations in G are motions with respect to that metric. An invariant Riemannian metric can be reconstructed from the Euclidean metric it defines on any tangent space $T_x(X)$. This Euclidean metric is invariant under the isotropy group $d_x G_x$. Conversely, if a Euclidean metric is defined in the space $T_x(X)$ and is invariant under the isotropy group, then it can be moved around by the action of the group G thus yielding an invariant Riemannian metric on X. Thus, an invariant Riemannian metric on X exists if and only if there is a Euclidean metric in the tangent space invariant under the isotropy group.

We now consider the question of when such a metric is unique.

A linear group H acting in a vector space V is said to be irreducible if there is no non-trivial subspace $U \subset V$ invariant under H.

Lemma 1.4. *Let H be a linear group acting in a real vector space V. If H is irreducible, then up to a (positive) scalar multiple there is at most one Euclidean metric in the space V invariant under H.*

Proof. Consider any invariant Euclidean metric (if such a metric exists) turning V into a Euclidean space. Then each invariant Euclidean metric q on V is of the form $q(x) = (Ax, x)$, where A is a positive definite symmetric operator commuting with all operators in H. Let c be any eigenvalue of A. The corresponding eigenspace is invariant under H, and consequently coincides with V. This implies that $A = cE$, i.e. $q(x) = c(x, x)$. □

The Lemma implies that if the isotropy group of a homogeneous space is irreducible, then there exists, up to a (positive) scalar multiple, at most one invariant Riemannian metric.

If a homogeneous space X is connected and admits an invariant Riemannian metric, then the isotropy representation is faithful at each point $x \in X$, since each element of the stabilizer of x, being a motion, is uniquely defined by its differential at that point.

1.4. Spaces of Constant Curvature

Definition 1.5. A simply-connected homogeneous space is said to be a *space of constant curvature* if its isotropy group (at each point) is the group of all orthogonal transformations with respect to some Euclidean metric.

The last condition is called the *maximum mobility axiom*. For the possibility of giving up the condition that the space is simply connected see Sect. 2.5.

Let (X, G) be a space of constant curvature. The maximum mobility axiom immediately implies that there is a unique (up to a scalar multiple) invariant Riemannian metric on X. With respect to this metric X is a Riemannian manifold of constant curvature (the trivial part of Theorem 1.3). The fact that G is a transitive group implies that X is a complete Riemannian manifold. Note also that G is the group of *all* its motions. Indeed, for each motion g and for each point $x \in X$, there exist an element $g_1 \in G$ such that $gx = g_1x$, i.e. $(g_1^{-1}g)x = x$, and an element $g_2 \in G_x$ such that $d_x(g_1^{-1}g) = d_xg_2$. But then $g_1^{-1}g = g_2$ and $g = g_1g_2 \in G$.

Conversely, by Theorem 1.3, any simply-connected complete Riemannian manifold X of constant curvature satisfies the conditions of Definition 1.5 if one takes for G the group of all its motions.

Thus, spaces of constant curvature (in the sense of the above definition) are simply-connected complete Riemannian manifolds of constant curvature considered up to change of scale, which explains their name. However, below in presenting the geometry of these spaces the fact that they are of constant Riemannian curvature is never used directly, and it is quite sufficient for the reader to be familiar with the simplest facts of Riemannian geometry (including the notion of a geodesic but excluding that of parallel translation or curvature).

Let (X, G) be a space of constant curvature. Since the manifold X is simply-connected, firstly it is orientable, and secondly each connected component of the group G contains exactly one connected component of the stabilizer (of each point). Since the isotropy representation is faithful, the stabilizer is isomorphic to the orthogonal group. The orthogonal group consists of two connected components, one including all orthogonal transformations with determinant 1, the other including all orthogonal transformations with determinant -1. Hence the group of motions of a space of constant curvature consists of two connected components, one of which includes all the motions preserving orientation (*proper motions*) and the other includes all motions reversing it (*improper motions*).

1.5. Three Spaces. For each $n \geq 2$ there are at least three n-dimensional spaces of constant curvature.

1. *Euclidean Space* E^n. Denoting the coordinates in the space $\mathbb{R}^n$ by $x_1, \ldots, x_n$, we define the scalar product by the formula

$$(x, y) = x_1 y_1 + \ldots + x_n y_n,$$

thus turning $\mathbb{R}^n$ into a Euclidean vector space.

Let

$$X = \mathbb{R}^n, \quad G = T_n \rtimes O_n \qquad \text{(semidirect product)},$$

where T_n is the group of parallel translations (isomorphic to $\mathbb{R}^n$) and O_n is the group of orthogonal transformations of $\mathbb{R}^n$.

For any parallel translation t_a along a vector $a \in \mathbb{R}^n$ and any orthogonal transformation $\varphi \in O_n$ one has

$$\varphi t_a \varphi^{-1} = t_{\varphi(a)},$$

which shows that G is indeed a group. Evidently, G acts transitively on X.

For each $x \in X$ the tangent space $T_x(X)$ is naturally identified with the space $\mathbb{R}^n$. The isotropy group then coincides with the group O_n.

Thus (X, G) is a space of constant curvature. It is called the n-dimensional *Euclidean space*, and denoted by E^n.

The Riemannian metric on the space E^n is induced by the Euclidean metric on the space $\mathbb{R}^n$, i.e. it is of the form

$$ds^2 = dx_1^2 + \ldots + dx_n^2.$$

Its curvature is 0.

2. *Sphere* S^n. Denoting the coordinates in the space $\mathbb{R}^{n+1}$ by $x_0, x_1, \ldots, x_n$, we introduce a scalar product in $\mathbb{R}^{n+1}$ by the formula

$$(x, y) = x_0 y_0 + x_1 y_1 + \ldots + x_n y_n,$$

which turns $\mathbb{R}^{n+1}$ into a Euclidean vector space.

Let

$$X = \{x \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}, \quad G = O_{n+1}.$$

For each $x \in X$ the tangent space $T_x(X)$ is naturally identified with the orthogonal complement to the vector x in the space $\mathbb{R}^{n+1}$. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis in the space $T_x(X)$, then $\{x, e_1, \ldots, e_n\}$ is an orthonormal basis in the space $\mathbb{R}^{n+1}$.

Since each orthonormal basis in the space $\mathbb{R}^{n+1}$ can, by an appropriate orthogonal transformation, be taken into any other orthonormal basis, the group G acts transitively on X, and the isotropy group at each point x coincides with the group of all orthogonal transformations of the space $T_x(X)$.

For $n \geq 2$ the manifold X is simply-connected and hence (X, G) is a space of constant curvature. It is called the n-dimensional *sphere* and denoted by S^n.

The Riemannian metric on the space S^n is induced by the Euclidean metric on $\mathbb{R}^{n+1}$, i.e. it is of the form

$$ds^2 = dx_0^2 + dx_1^2 + \ldots + dx_n^2.$$

(Remember that the coordinate functions $x_0, x_1, \ldots, x_n$ are not independent on S^n.) The curvature of this metric is 1.

3. *Lobachevskij Space* $Л^n$. Denoting the coordinates in the space $\mathbb{R}^{n+1}$ by $x_0, x_1, \ldots, x_n$, we introduce a scalar product in $\mathbb{R}^{n+1}$ by the formula

$$(x, y) = -x_0y_0 + x_1y_1 + \ldots + x_ny_n,$$

which turns $\mathbb{R}^{n+1}$ into a pseudo-Euclidean vector space, denoted by $\mathbb{R}^{n,1}$.

Each pseudo-orthogonal (i.e. preserving the above scalar product) transformation of $\mathbb{R}^{n,1}$ takes an open cone of time-like vectors

$$C = \{x \in \mathbb{R}^{n,1} : (x, x) < 0\}$$

consisting of two connected components

$$C^+ = \{x \in C : x_0 > 0\}, \qquad C^- = \{x \in C : x_0 < 0\}$$

onto itself.

Denote by $O_{n,1}$ the group of all pseudo-orthogonal transformations of the space $\mathbb{R}^{n,1}$, and by $O'_{n,1}$ its subgroup of index 2 consisting of those pseudo-orthogonal transformations which map each connected component of the cone C onto itself. Let

$$X = \{x \in \mathbb{R}^{n,1} : -x_0^2 + x_1^2 + \ldots + x_n^2 = -1, x_0 > 0\}, \quad G = O'_{n,1}.$$

A basis $\{e_0, e_1, \ldots, e_n\}$ is said to be orthonormal if $(e_0, e_0) = -1, (e_i, e_i) = 1$ for $i \neq 0$ and $(e_i, e_j) = 0$ for $i \neq j$. For example, the standard basis is orthonormal.

For any $x \in X$ the tangent space $T_x(X)$ is naturally identified with the orthogonal complement of the vector x in the space $\mathbb{R}^{n,1}$, which is an n-dimensional Euclidean space (with respect to the same scalar product). If $\{e_1, \ldots, e_n\}$ is an orthonormal basis in it, then $\{x, e_1, \ldots, e_n\}$ is an orthonormal basis in the space $\mathbb{R}^{n,1}$. It follows then (in the same way as for the sphere) that the group G acts transitively on X and the isotropy group coincides with the group of all orthogonal transformations of the tangent space.

The manifold X has a diffeomorphic projection onto the subspace $x_0 = 0$, and is therefore simply-connected. Hence (X, G) is a space of constant curvature. It is called the n-dimensional *Lobachevskij space* (or hyperbolic space) and denoted by $Л^n$.

The Riemannian metric on the space $Л^n$ is induced by the pseudo-Euclidean metric on the space $\mathbb{R}^{n,1}$, i.e. it is of the form

$$ds^2 = -dx_0^2 + dx_1^2 + \ldots + dx_n^2.$$

Its curvature is -1.

Remark 1. For the sake of uniformity the procedure of embedding in $\mathbb{R}^{n+1}$ can also be applied to the Euclidean space E^n. Its motions are then induced by linear transformations in the following way. Let $x_0, x_1, \dots, x_n$ be coordinates in $\mathbb{R}^{n+1}$, and let $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ be the subspace defined by the equation $x_0 = 0$. The space E^n can then be identified with the hyperplane $x_1 = 1$, its motions being induced by linear transformations of the space $\mathbb{R}^{n+1}$ preserving x_0 and inducing orthogonal transformations in $\mathbb{R}^n$ (with respect to the standard Euclidean metric). Note that under this interpretation the subspace $\mathbb{R}^n$ may naturally be regarded as a tangent space of E^n (at any point).

Remark 2. All the above constructions can also be carried out in a coordinate-free form. For example, the sphere S^n can be defined as the set of vectors of square 1 in the $(n+1)$-dimensional Euclidean vector space, and its group of motions as the group of orthogonal transformations of that space. The space $Л^n$ can be defined as the connected component of the set of vectors of square -1 in the $(n+1)$-dimensional pseudo-Euclidean vector space of signature $(n, 1)$, and its group of motions as the index 2 subgroup of the group of pseudo-orthogonal transformations of that space which consists of transformations preserving each connected component of the cone of time-like vectors. Another approach is to consider a coordinate system in which the scalar product is not written in the standard way (but has the right signature).

Remark 3. For $n = 1$ and 0 the above constructions define the following homogeneous spaces:

$E^1 \simeq Л^1$ (Euclidean line),
S^1 (circle),
$E^0 \simeq Л^0$ (point),
S^0 (double point).

The spaces $E^1 \simeq Л^1$ and $E^0 \simeq Л^0$ are spaces of constant curvature while, under our definition, the spaces S^1 and S^0 do not belong to this class as they are not simply connected.

The models of S^n and $Л^n$ constructed above together with the model of E^n given in Remark 1 will be called *vector models*, while the model of E^n described at the beginning of this section will be referred to as the *affine model*. When there is a reason to indicate that a model under discussion is related to a coordinate system in the above manner we will refer to it as a *standard vector (affine) model*.

Unless otherwise stated we will assume that the Riemannian metric in S^n and $Л^n$ is normalized as in this section. If the Riemannian metric is divided by $k > 0$, the curvature is multiplied by k^2.

1.6. Subspaces of the Space $\mathbb{R}^{n,1}$. In view of the extensive use below of the vector model of the Lobachevskij space we now present the classification of subspaces of the pseudo-Euclidean vector space $\mathbb{R}^{n,1}$. A subspace $U \subset \mathbb{R}^{n,1}$ is said to be *elliptic* (respectively, *parabolic, hyperbolic*) if the restriction of the scalar product in $\mathbb{R}^{n,1}$ to U is positive definite (respectively, positive semi-definite and degenerate, indefinite). Subspaces of each type can