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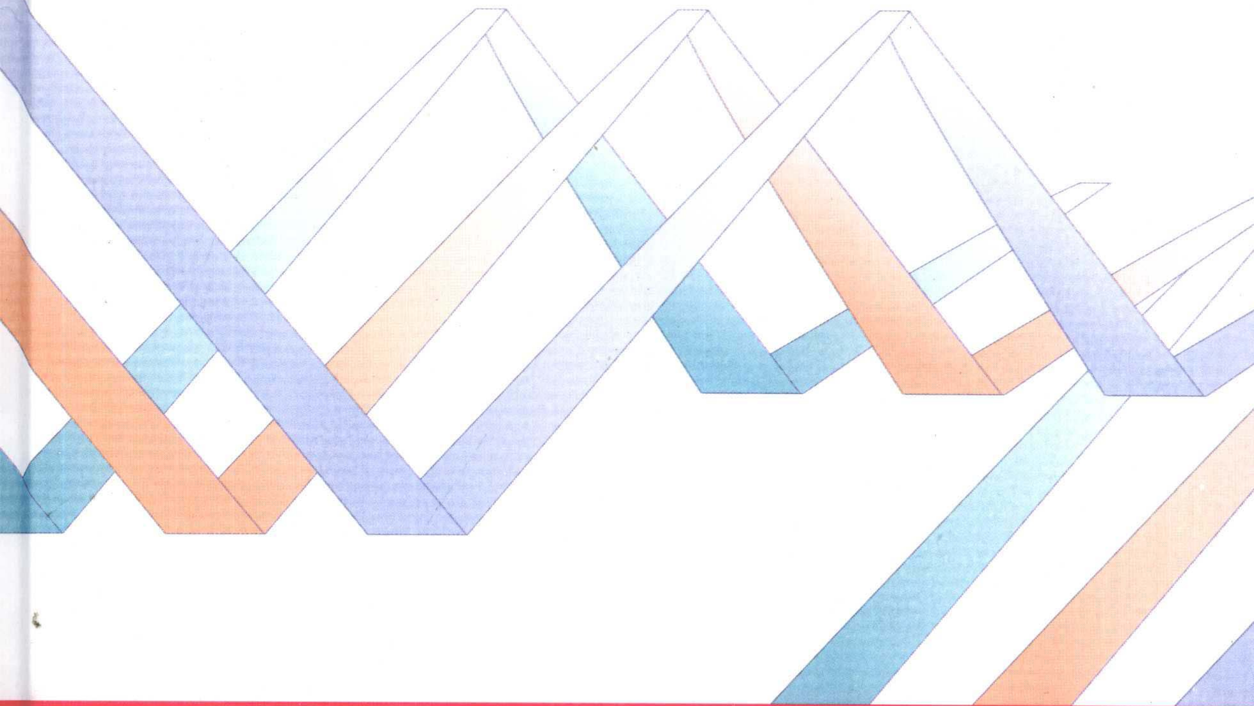

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Jean-Pierre Demailly

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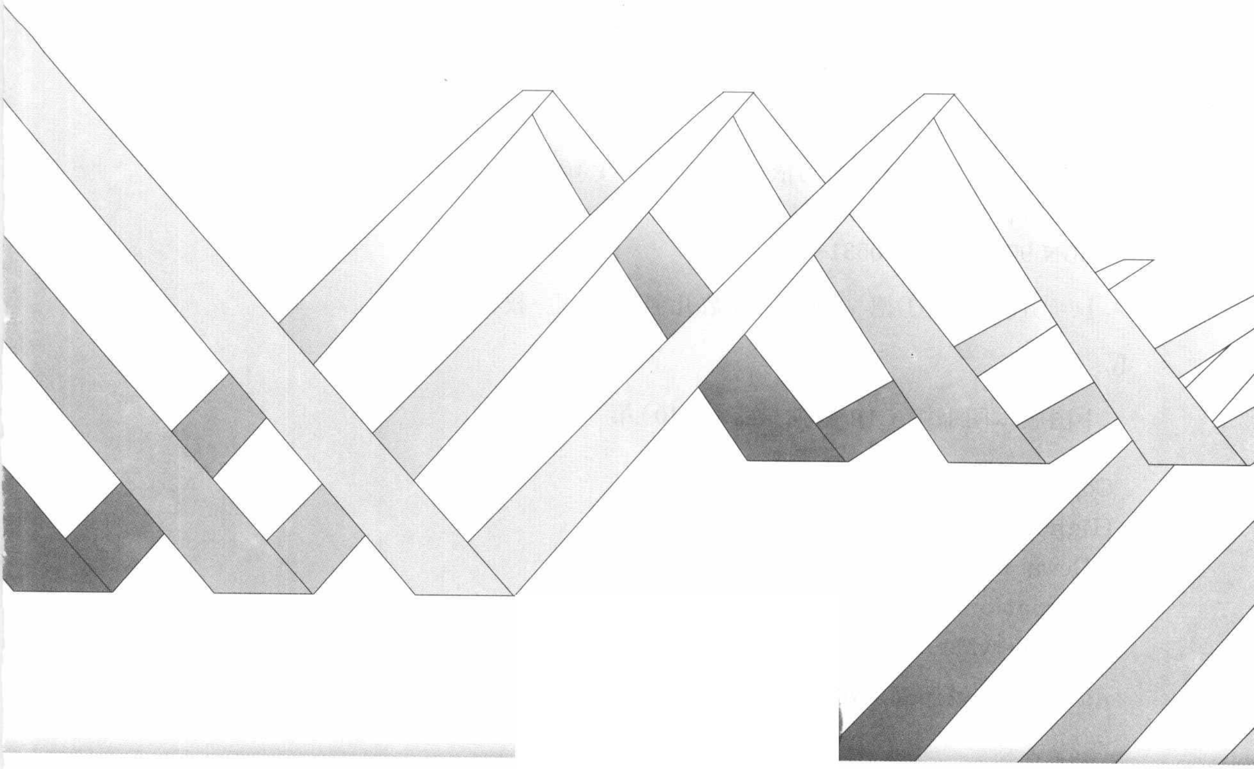

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Jean-Pierre Demailly

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Preface

The main purpose of this book is to describe analytic techniques which are useful to study questions such as linear series, multiplier ideals and vanishing theorems for algebraic vector bundles. One century after the ground-breaking work of Riemann on geometric aspects of function theory, the general progress achieved in differential geometry and global analysis on manifolds resulted into major advances in the theory of algebraic and analytic varieties of arbitrary dimension. One central unifying concept is positivity, which can be viewed either in algebraic terms (positivity of divisors and algebraic cycles), or in more analytic terms (plurisubharmonicity, Hermitian connections with positive curvature). In this direction, one of the most basic results is Kodaira's vanishing theorem for positive vector bundles (1953—1954), which is a deep consequence of the Bochner technique and the theory of harmonic forms initiated by Hodge during the 1940's. This method quickly led Kodaira to the well-known embedding theorem for projective varieties, a far reaching extension of Riemann's characterization of abelian varieties. Further refinements of the Bochner technique led ten years later to the theory of L^2 estimates for the Cauchy-Riemann operator, in the hands of Kohn, Andreotti-Vesentini and Hörmander among others. Not only can vanishing theorems be proved or reproved in that manner, but perhaps more importantly, extremely precise information of a quantitative nature can be obtained about solutions of $\overline{\partial}$-equations, their zeroes, poles and growth at infinity.

We try to present here a condensed exposition of these techniques, assuming that the reader is already somewhat acquainted with the basic concepts pertaining to sheaf theory, cohomology and complex differential geometry. In the final chapter, we address very recent questions and open problems, e.g. results related to the finiteness of the canonical ring and the abundance conjecture, as well as results describing the geometric structure of Kähler varieties and their positive cones.

This book is an expansion of lectures given by the author at the Park City Mathematics Institute in 2008 and was published partly in Analytic and Algebraic Geometry, edited by Jeff McNeal and Mircea Mustaţă, It is a volume in the Park City Mathematics Series, a co-publication of the Park City Mathematics Institute and the American Mathematical Society.

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April 21, 2010

Contents

Introduction

This introduction will serve as a general guide for reading the various parts of this text. The first three chapters briefly introduce basic materials concerning complex differential geometry, Dolbeault cohomology, plurisubharmonic functions, positive currents and holomorphic vector bundles. They are mainly intended to fix notation. Although the most important concepts are redefined, readers will probably need to already possess some related background in complex analysis and complex differential geometry — whereas the expert readers should be able to quickly proceed further.

The heart of the subject starts with the Bochner technique in Chapter 4, leading to fundamental L^2 existence theorems for solutions of $\overline{\partial}$-equations in Chapter 5. What makes the theory extremely flexible is the possibility to formulate existence theorems with a wide assortment of different L^2 norms, namely norms of the form $\int_X |f|^2 e^{-2\varphi}$ where φ is a plurisubharmonic or strictly plurisubharmonic function on the given manifold or variety X. Here, the weight φ need not be smooth, and on the contrary, it is extremely important to allow weights which have logarithmic poles of the form $\varphi(z) = c \log \sum |g_j|^2$, where $c > 0$ and (g_j) is a collection of holomorphic functions possessing a common zero set $Z \subset X$. Following Nadel [Nad89], one defines the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally summable. Then $\mathcal{I}(\varphi)$ is a coherent algebraic sheaf over X and $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ for all $q \geqslant 1$ if the curvature of L is positive as a current. This important result can be seen as a generalization of the Kawamata-Viehweg vanishing theorem [Kaw82, Vie82], which is one of the cornerstones of higher dimensional algebraic geometry, especially in relation with Mori's minimal model program.

In the dictionary between analytic geometry and algebraic geometry, the ideal $\mathcal{I}(\varphi)$ plays a very important role, since it directly converts an analytic object into an algebraic one, and, simultaneously, takes care of the singularities in a very efficient way. Another analytic tool used to deal with singularities is the theory of positive currents introduced by Lelong [Lel57]. Currents can be seen as generalizations of algebraic cycles, and many classical results of intersection theory still apply to currents. The concept of Lelong number of a current is the analytic analogue of the concept of multiplicity of a germ of algebraic variety. Intersections of cycles correspond to wedge products of currents (whenever these products are defined).

Besides the Kodaira-Nakano vanishing theorem, one of the most basic "effective result" expected to hold in algebraic geometry is expressed in the following conjecture of Fujita [Fuj87]: if L is an ample (i.e. positive) line bundle on a projective n-dimensional algebraic variety X, then $K_X + (n+1)L$ is generated by sections and $K_X + (n+2)L$ is very ample. In the last two decades, a lot of efforts have been brought for the solution of this conjecture — but reaching the expected optimal bounds will probably require new ideas. The first major results are the proof of the Fujita conjecture in the case of surfaces by Reider [Rei88] (the case of curves is easy and has been known since a very long time), and the numerical criterion for the very ampleness of $2K_X + L$ given in [Dem93b], obtained by means of analytic techniques and Monge-Ampère equations with isolated singularities. Alternative algebraic techniques were developed slightly later by Kollár [Kol92], Ein-Lazarsfeld [EL93], Fujita [Fuj93], Siu [Siu95, 96], Kawamata [Kaw97] and Helmke [Hel97]. We will explain here Siu's method because it is technically the simplest method; one of the results obtained by this method is the following effective result: $2K_X + mL$ is very ample for $m \geqslant 2 + \binom{3n+1}{n}$. The basic idea is to apply the Kawamata-Viehweg vanishing theorem, and to combine this with the Riemann-Roch formula in order to produce sections through a clever induction procedure on the dimension of the base loci of the linear systems involved.

Although Siu's result is certainly not optimal, it is sufficient to obtain a nice constructive proof of *Matsusaka's big theorem* [Siu93, Dem96]. The result states that there is an effective value m_0 depending only on the intersection numbers L^n and $L^{n-1} \cdot K_X$, such that mL is very ample for $m \geqslant m_0$. The basic idea is to combine results on the very ampleness of $2K_X + mL$ together with the theory of holomorphic Morse inequalities [Dem85b]. The Morse inequalities are used to construct sections of $m'L - K_X$ for m' large. Again this step can be made algebraic (following suggestions by F. Catanese and R. Lazarsfeld), but the analytic formulation apparently has a wider range of applicability.

In the subsequent chapters, we pursue the study of L^2 estimates, in relation with the Nullstellenstatz and with the extension problem. Skoda [Sko72b, 78] showed that the division problem $f = \sum g_j h_j$ can be solved holomorphically with very precise L^2 estimates, provided that the L^2 norm of $|f|\,|g|^{-p}$ is finite for some sufficiently large exponent p ($p > n = \dim X$ is enough). Skoda's estimates have a nice interpretation in terms of local algebra, and they lead to precise qualitative and quantitative estimates in connection with the Bézout problem. Another very important result is the L^2 extension theorem by Ohsawa-Takegoshi [OT87, Ohs88], which has also been generalized later by Manivel [Man93]. The main statement is that every L^2 section f of a suitably positive line bundle defined on a subavariety $Y \subset X$ can be extended to a L^2 section $\tilde{f}$ defined over the whole of X. The positivity condition can be understood in terms of the canonical sheaf and normal bundle to the subvariety. The extension theorem turns out to have an incredible amount of important consequences: among them, let us mention for instance Siu's theorem [Siu74] on the analyticity of Lelong numbers, the basic approximation theorem of closed positive $(1,1)$-currents by divisors, the subadditivity property $\mathcal{I}(\varphi+\psi) \subset \mathcal{I}(\varphi)\mathcal{I}(\psi)$ of multiplier ideals [DEL00], the restriction formula $\mathcal{I}(\varphi_{|Y}) \subset \mathcal{I}(\varphi)_{|Y}$, ... A suitable combination of these results yields another important result of Fujita [Fuj94] on approximate Zariski decomposition, as we show in Chapter 14.

In Chapter 15, we show how subadditivity can be used to derive an "equi-singular" approximation theorem for (almost) plurisubharmonic functions: any such function can be approximated by a sequence of (almost) plurisubharmonic functions which are smooth outside an analytic set, and which define the same multiplier ideal sheaves. From this, we derive a generalized version of the *hard Lefschetz theorem* for cohomology with values in a pseudo-effective line bundle; namely, the Lefschetz map is surjective when the cohomology groups are twisted by the relevant multiplier ideal sheaves.

Chapter 16 explains the proof of Siu's theorem on the invariance of plurigenera, according to a beautiful approach developped by Mihai Păun [Pău07]. The proofs consists of an iterative process based on the Ohsawa-Takegoshi theorem, and a very clever limiting argument for currents.

Chapters 17 and 18 are devoted to the study of positive cones in Kähler or projective geometry. Recent "algebro-analytic" characterizations of the Kähler cone [DP04] and the pseudo-effective cone of divisors [BDPP04] are explained in detail. This leads to a discussion of the important concepts of volume and mobile intersections, following S. Boucksom's PhD work [Bou02]. As a consequence, we show that a projective algebraic manifold has a pseudo-effective canonical line bundle if and only if it is not uniruled.

Chapter 19 presents further important ideas of H. Tsuji, later refined by Berndtsson and Păun, concerning the so-called "super-canonical metrics", and their interpretation in terms of the invariance of plurigenera and of the abundance conjecture. In the concluding Chapter 20, we state Păun's version of the Shokurov-Hacon-McKernan-Siu non vanishing theorem and give an account of the very recent approach of the proof of the finiteness of the canonical ring by Birkar-Păun [BiP09], based on the ideas of Hacon-McKernan and Siu.

Chapter 1

Preliminary Material: Cohomology, Currents

1.A. Dolbeault Cohomology and Sheaf Cohomology

Let X be a $\mathbb{C}$-analytic manifold of dimension n. We denote by $\Lambda^{p,q}T^*_X$ the bundle of differential forms of bidegree (p,q) on X, i.e., differential forms which can be written as

$$u = \sum_{|I|=p,\,|J|=q} u_{I,J} dz_I \wedge d\overline{z}_J.$$

Here $(z_1, \ldots, z_n)$ denote arbitrary local holomorphic coordinates on X, $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ are multi-indices (increasing sequences of integers in the range $[1, \ldots, n]$, of lengths $|I| = p$, $|J| = q$), and

$$dz_I := dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \qquad d\overline{z}_J := d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

Let $\mathscr{E}^{p,q}$ be the sheaf of germs of complex valued differential (p,q)-forms with $\mathscr{C}^\infty$ coefficients. Recall that the exterior derivative d splits as $d = d' + d''$ where

$$d'u = \sum_{|I|=p,\,|J|=q,1\leqslant k\leqslant n} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J,$$
$$d''u = \sum_{|I|=p,\,|J|=q,1\leqslant k\leqslant n} \frac{\partial u_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J$$

are of type $(p+1,q)$, $(p,q+1)$ respectively. The well-known Dolbeault-Grothendieck lemma asserts that any d''-closed form of type (p,q) with $q > 0$ is locally d''-exact (this is the analogue for d'' of the usual Poincaré lemma for d, see e.g. [Hör66]). In other words, the complex of sheaves $(\mathscr{E}^{p,\bullet}, d'')$ is exact in degree $q > 0$; in degree $q = 0$, $\operatorname{Ker} d''$ is the sheaf Ω^p_X of germs of holomorphic forms of degree p on X.

More generally, if F is a holomorphic vector bundle of rank r over X, there is a natural d'' operator acting on the space $\mathscr{C}^\infty(X, \Lambda^{p,q}T^*_X \otimes F)$ of smooth (p,q)-forms with values in F; if $s = \sum_{1\leqslant\lambda\leqslant r} s_\lambda e_\lambda$ is a (p,q)-form expressed in terms of a local holomorphic frame of F, we simply define $d''s := \sum d''s_\lambda \otimes e_\lambda$, observing that the holomorphic transition matrices involved in changes of holomorphic frames do not affect the computation of d''. It is then clear that the Dolbeault-Grothendieck lemma still holds for F-valued forms. For every integer $p = 0, 1, \ldots, n$, the *Dolbeault Cohomology* groups $H^{p,q}(X, F)$ are defined to be the cohomology groups of the complex of global (p,q) forms (graded by q):

$$H^{p,q}(X,F) = H^q\big(\mathscr{C}^\infty(X, \Lambda^{p,\bullet}T^*_X \otimes F)\big). \tag{1.1}$$

Now, let us recall the following fundamental result from sheaf theory (De Rham-Weil isomorphism theorem): let $(\mathscr{L}^\bullet, d)$ be a resolution of a sheaf $\mathscr{A}$ by acyclic sheaves, i.e. a complex of sheaves $(\mathscr{L}^\bullet, \delta)$ such that there is an exact sequence of sheaves

$$0 \longrightarrow \mathscr{A} \xrightarrow{\ j\ } \mathscr{L}^0 \xrightarrow{\ \delta^0\ } \mathscr{L}^1 \longrightarrow \cdots \longrightarrow \mathscr{L}^q \xrightarrow{\ \delta^q\ } \mathscr{L}^{q+1} \longrightarrow \cdots,$$

and $H^s(X, \mathscr{L}^q) = 0$ for all $q \geqslant 0$ and $s \geqslant 1$. Then there is a functorial isomorphism

$$H^q\big(\Gamma(X, \mathscr{L}^\bullet)\big) \longrightarrow H^q(X, \mathscr{A}). \tag{1.2}$$

We apply this to the following situation: let $\mathscr{E}(F)^{p,q}$ be the sheaf of germs of $\mathscr{C}^\infty$ sections of $\Lambda^{p,q}T^*_X \otimes F$. Then $(\mathscr{E}(F)^{p,\bullet}, d'')$ is a resolution of the locally free $\mathscr{O}_X$-module $\Omega^p_X \otimes \mathscr{O}(F)$ (Dolbeault-Grothendieck lemma), and the sheaves $\mathscr{E}(F)^{p,q}$ are acyclic as modules over the soft sheaf of rings $\mathscr{C}^\infty$. Hence by (1.2) we get

(1.3) Dolbeault Isomorphism Theorem (1953). *For every holomorphic vector bundle F on X, there is a canonical isomorphism:*

$$H^{p,q}(X,F) \simeq H^q(X, \Omega^p_X \otimes \mathscr{O}(F)).$$

If X is projective algebraic and F is an algebraic vector bundle, Serre's GAGA theorem [Ser56] shows that the algebraic sheaf cohomology group $H^q(X,\, \Omega^p_X \otimes \mathscr{O}(F))$ computed with algebraic sections over Zariski open sets is actually isomorphic to the analytic cohomology group. These results are the most basic tools to attack algebraic problems via analytic methods. Another important tool is the theory of plurisubharmonic functions and positive currents originated by K. Oka and P. Lelong in the decades 1940–1960.

1.B. Plurisubharmonic Functions

Plurisubharmonic functions have been introduced independently by Lelong and Oka in the study of holomorphic convexity. We refer to [Lel67, 69] for more details.

(1.4) Definition. *A function $u : \Omega \longrightarrow [-\infty, +\infty[$ defined on an open subset $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic (psh for short) if*

(a) *u is upper semicontinuous ;*

(b) *for every complex line* $L \subset \mathbb{C}^n$, $u_{\restriction \Omega \cap L}$ *is subharmonic on* $\Omega \cap L$, *that is, for all* $a \in \Omega$ *and* $\xi \in \mathbb{C}^n$ *with* $|\xi| < d(a, \complement\Omega)$, *the function* u *satisfies the mean value inequality:*

$$u(a) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\,\xi)\,d\theta.$$

The set of psh functions on Ω *is denoted by* $\mathrm{Psh}(\Omega)$.

We list below the most basic properties of psh functions. They all follow easily from the definition.

(1.5) Basic Properties.

(a) Every function $u \in \mathrm{Psh}(\Omega)$ is subharmonic, namely it satisfies the mean value inequality on Euclidean balls or spheres:

$$u(a) \leqslant \frac{1}{\pi^n r^{2n}/n!} \int_{B(a,r)} u(z)\,d\lambda(z)$$

for every $a \in \Omega$ and $r < d(a, \complement\Omega)$. Either $u \equiv -\infty$ or $u \in L^1_{\mathrm{loc}}$ on every connected component of Ω.

(b) For any decreasing sequence of psh functions $u_k \in \mathrm{Psh}(\Omega)$, the limit $u = \lim u_k$ is psh on Ω.

(c) Let $u \in \mathrm{Psh}(\Omega)$ be such that $u \not\equiv -\infty$ on every connected component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u * \rho_\varepsilon$ is $\mathscr{C}^\infty$ and psh on

$$\Omega_\varepsilon = \big\{x \in \Omega\,;\ d(x, \complement\Omega) > \varepsilon\big\},$$

the family $(u * \rho_\varepsilon)$ is increasing in ε and $\lim_{\varepsilon\to 0} u * \rho_\varepsilon = u$.

(d) Let $u_1, \ldots, u_p \in \mathrm{Psh}(\Omega)$ and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function such that $\chi(t_1, \ldots, t_p)$ is increasing in each t_j. Then $\chi(u_1, \ldots, u_p)$ is psh on Ω. In particular $u_1 + \cdots + u_p$, $\max\{u_1, \ldots, u_p\}$, $\log(e^{u_1} + \cdots + e^{u_p})$ are psh on Ω. □

(1.6) Lemma. *A function* $u \in C^2(\Omega, \mathbb{R})$ *is psh on* Ω *if and only if the Hermitian form:*

$$Hu(a)(\xi) = \sum_{1 \leqslant j,k \leqslant n} \partial^2 u/\partial z_j \partial \overline{z}_k(a)\,\xi_j \overline{\xi}_k$$

is semi-positive at every point $a \in \Omega$.

Proof. This is an easy consequence of the following standard formula:

$$\frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\,\xi)\,d\theta - u(a) = \frac{2}{\pi} \int_0^1 \frac{dt}{t} \int_{|\zeta|<t} Hu(a + \zeta\xi)(\xi)\,d\lambda(\zeta),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$. Lemma 1.6 is a strong evidence that plurisubharmonicity is the natural complex analogue of linear convexity. □

For non smooth functions, a similar characterization of plurisubharmonicity can be obtained by means of a regularization process.

(1.7) Theorem. *If $u \in \mathrm{Psh}(\Omega)$, $u \not\equiv -\infty$ on every connected component of Ω, then for all $\xi \in \mathbb{C}^n$*

$$Hu(\xi) = \sum_{1 \leqslant j,k \leqslant n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \xi_j \overline{\xi}_k \in \mathscr{D}'(\Omega)$$

is a positive measure. Conversely, if $v \in \mathscr{D}'(\Omega)$ is such that $Hv(\xi)$ is a positive measure for every $\xi \in \mathbb{C}^n$, there exists a unique function $u \in \mathrm{Psh}(\Omega)$ which is locally integrable on Ω and such that v is the distribution associated to u.

In order to get a better geometric insight of this notion, we assume more generally that u is a function on a complex n-dimensional manifold X. If $\Phi : X \to Y$ is a holomorphic mapping and if $v \in C^2(Y, \mathbb{R})$, we have $d'd''(v \circ \Phi) = \Phi^* d'd'' v$, hence

$$H(v \circ \Phi)(a, \xi) = Hv\big(\Phi(a), \Phi'(a)\xi\big).$$

In particular Hu, viewed as a Hermitian form on T_X, does not depend on the choice of coordinates $(z_1, \ldots, z_n)$. Therefore, the notion of psh function makes sense on any complex manifold. More generally, we have

(1.8) Proposition. *If $\Phi : X \longrightarrow Y$ is a holomorphic map and $v \in \mathrm{Psh}(Y)$, then $v \circ \Phi \in \mathrm{Psh}(X)$.*

(1.9) Example. It is a standard fact that $\log|z|$ is psh (i.e. subharmonic) on $\mathbb{C}$. Thus $\log|f| \in \mathrm{Psh}(X)$ for every holomorphic function $f \in H^0(X, \mathcal{O}_X)$. More generally

$$\log\big(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\big) \in \mathrm{Psh}(X)$$

for every $f_j \in H^0(X, \mathcal{O}_X)$ and $\alpha_j \geqslant 0$ (apply Property 1.5 (d) with $u_j = \alpha_j \log|f_j|$). We will be especially interested in the singularities obtained at points of the zero variety $f_1 = \cdots = f_q = 0$, when the α_j are rational numbers. □

(1.10) Definition. *A psh function $u \in \mathrm{Psh}(X)$ will be said to have analytic singularities if u can be written locally as*

$$u = \frac{\alpha}{2} \log\big(|f_1|^2 + \cdots + |f_N|^2\big) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a locally bounded function and the f_j are holomorphic functions. If X is algebraic, we say that u has algebraic singularities if u can be written as above on sufficiently small Zariski open sets, with $\alpha \in \mathbb{Q}_+$ and f_j algebraic.

We then introduce the ideal $\mathcal{J} = \mathcal{J}(u/\alpha)$ of germs of holomorphic functions h such that $|h| \leqslant C e^{u/\alpha}$ for some constant C, i.e.

$$|h| \leqslant C\big(|f_1| + \cdots + |f_N|\big).$$

This is a globally defined ideal sheaf on X, locally equal to the integral closure $\overline{\mathcal{J}}$ of the ideal sheaf $\mathcal{J} = (f_1, \ldots, f_N)$, thus $\mathcal{J}$ is coherent on X. If $(g_1, \ldots, g_{N'})$ are local generators of $\mathcal{J}$, we still have

$$u = \frac{\alpha}{2} \log\left(|g_1|^2 + \cdots + |g_{N'}|^2\right) + O(1).$$

If X is projective algebraic and u has analytic singularities with $\alpha \in \mathbb{Q}_+$, then u automatically has algebraic singularities. From an algebraic point of view, the singularities of u are in 1:1 correspondence with the "algebraic data" $(\mathcal{J}, \alpha)$. Later on, we will see another important method for associating an ideal sheaf to a psh function.

(1.11) Exercise. Show that the above definition of the integral closure of an ideal $\mathcal{J}$ is equivalent to the following more algebraic definition: $\overline{\mathcal{J}}$ consists of all germs h satisfying an integral equation:

$$h^d + a_1 h^{d-1} + \cdots + a_{d-1} h + a_d = 0, \qquad a_k \in \mathcal{J}^k.$$

Hint. One inclusion is clear. To prove the other inclusion, consider the normalization of the blow-up of X along the (non necessarily reduced) zero variety $V(\mathcal{J})$. □

1.C. Positive Currents

The reader can consult [Fed69] for a more thorough treatment of current theory. Let us first recall a few basic definitions. A *current* of degree q on an oriented differentiable manifold M is simply a differential q-form Θ with distribution coefficients. The space of currents of degree q over M will be denoted by $\mathcal{D}'^q(M)$. Alternatively, a current of degree q can be seen as an element Θ in the dual space $\mathcal{D}'_p(M) := \left(\mathcal{D}^p(M)\right)'$ of the space $\mathcal{D}^p(M)$ of smooth differential forms of degree $p = \dim M - q$ with compact support; the duality pairing is given by

$$(1.12) \qquad \langle \Theta, \alpha \rangle = \int_M \Theta \wedge \alpha, \quad \alpha \in \mathcal{D}^p(M).$$

A basic example is the *current of integration* $[S]$ over a compact oriented submanifold S of M:

$$(1.13) \qquad \langle [S], \alpha \rangle = \int_S \alpha, \quad \deg\alpha = p = \dim_{\mathbb{R}} S.$$

Then $[S]$ is a current with measure coefficients, and Stokes' formula shows that $d[S] = (-1)^{q-1}[\partial S]$, in particular $d[S] = 0$ if S has no boundary. Because of this example, the integer p is said to be the dimension of Θ when $\Theta \in \mathcal{D}'_p(M)$. The current Θ is said to be *closed* if $d\Theta = 0$.

On a complex manifold X, we have similar notions of bidegree and bidimension; as in the real case, we denote by

$$\mathscr{D}'^{p,q}(X) = \mathscr{D}'_{n-p,n-q}(X), \qquad n = \dim X,$$

the space of currents of bidegree (p,q) and bidimension $(n-p,n-q)$ on X. According to [Lel57], a current Θ of bidimension (p,p) is said to be (*weakly*) *positive* if for every choice of smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_p$ on X the distribution

$$(1.14) \qquad \Theta \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha}_p \quad \text{is a positive measure.}$$

(1.15) Exercise. If Θ is positive, show that the coefficients $\Theta_{I,J}$ of Θ are complex measures, and that, up to constants, they are dominated by the trace measure:

$$\sigma_\Theta = \Theta \wedge \frac{1}{p!}\beta^p = 2^{-p}\sum \Theta_{I,I}, \qquad \beta = \frac{i}{2}d'd''|z|^2 = \frac{i}{2}\sum_{1\leqslant j\leqslant n} dz_j \wedge d\overline{z}_j,$$

which is a positive measure.
Hint. Observe that $\sum \Theta_{I,I}$ is invariant by unitary changes of coordinates and that the (p,p)-forms $i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha}_p$ generate $\Lambda^{p,p}T^\star_{\mathbb{C}^n}$ as a $\mathbb{C}$-vector space. □

A current $\Theta = i\sum_{1\leqslant j,k\leqslant n} \Theta_{jk} dz_j \wedge d\overline{z}_k$ of bidegree $(1,1)$ is easily seen to be positive if and only if the complex measure $\sum \lambda_j \overline{\lambda}_k \Theta_{jk}$ is a positive measure for every n-tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$.

(1.16) Example. If u is a (not identically $-\infty$) psh function on X, we can associate with u a (closed) positive current $\Theta = i\partial\overline{\partial}u$ of bidegree $(1,1)$. Conversely, each closed positive current of bidegree $(1,1)$ can be written under this form on any open subset $\Omega \subset X$ such that $H^2_{DR}(\Omega,\mathbb{R}) = H^1(\Omega,\mathcal{O}) = 0$, e.g. on small coordinate balls (exercise to the reader). □

It is not difficult to show that a product $\Theta_1 \wedge \cdots \wedge \Theta_q$ of positive currents of bidegree $(1,1)$ is positive whenever the product is well defined (this is certainly the case if all Θ_j but one at most are smooth; much finer conditions will be discussed in Chapter 2).

We now discuss another very important example of closed positive current. In fact, with every closed analytic set $A \subset X$ of pure dimension p is associated a current of integration $[A]$ such that:

$$(1.17) \qquad \langle [A], \alpha \rangle = \int_{A_{\text{reg}}} \alpha, \qquad \alpha \in \mathscr{D}^{p,p}(X),$$

obtained by integrating over the regular points of A. In order to show that (1.17) is a correct definition of a current on X, one must show that A_{reg} has locally finite area in a neighborhood of A_{sing}. This result, due to [Lel57] is shown as follows.

Suppose that 0 is a singular point of A. By the local parametrization theorem for analytic sets, there is a linear change of coordinates on $\mathbb{C}^n$ such that all projections

$$\pi_I : (z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_p})$$

define a finite ramified covering of the intersection $A \cap \Delta$ with a small polydisk Δ in $\mathbb{C}^n$ onto a small polydisk Δ_I in $\mathbb{C}^p$. Let n_I be the sheet number. Then the p-dimensional area of $A \cap \Delta$ is bounded above by the sum of the areas of its projections counted with multiplicities, i.e.

$$\mathrm{Area}(A \cap \Delta) \leqslant \sum n_I \mathrm{Vol}(\Delta_I).$$

The fact that $[A]$ is positive is also easy. In fact

$$\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p = |\det(\alpha_{jk})|^2 \, \mathrm{i}w_1 \wedge \overline{w}_1 \wedge \cdots \wedge \mathrm{i}w_p \wedge \overline{w}_p$$

if $\alpha_j = \sum \alpha_{jk} dw_k$ in terms of local coordinates $(w_1, \ldots, w_p)$ on A_{reg}. This shows that all such forms are $\geqslant 0$ in the canonical orientation defined by $\mathrm{i}w_1 \wedge \overline{w}_1 \wedge \cdots \wedge \mathrm{i}w_p \wedge \overline{w}_p$. More importantly, Lelong [Lel57] has shown that $[A]$ is d-closed in X, even at points of A_{sing}. This last result can be seen today as a consequence of the Skoda-El Mir extension theorem. For this we need the following definition: a *complete pluripolar* set is a set E such that there is an open covering (Ω_j) of X and psh functions u_j on Ω_j with $E \cap \Omega_j = u_j^{-1}(-\infty)$. Any (closed) analytic set is of course complete pluripolar (take u_j as in Example 1.9).

(1.18) Theorem (Skoda [Sko82], El Mir [EM84], Sibony [Sib85]). *Let E be a closed complete pluripolar set in X, and let Θ be a closed positive current on $X \smallsetminus E$ such that the coefficients $\Theta_{I,J}$ of Θ are measures with locally finite mass near E. Then the trivial extension $\widetilde{\Theta}$ obtained by extending the measures $\Theta_{I,J}$ by 0 on E is still closed on X.*

Lelong's result $d[A] = 0$ is obtained by applying the Skoda-El Mir theorem to $\Theta = [A_{\mathrm{reg}}]$ on $X \smallsetminus A_{\mathrm{sing}}$.

Proof of Theorem 1.18. The statement is local on X, so we may work on a small open set Ω such that $E \cap \Omega = v^{-1}(-\infty)$, $v \in \mathrm{Psh}(\Omega)$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function such that $\chi(t) = 0$ for $t \leqslant -1$ and $\chi(0) = 1$. By shrinking Ω and putting $v_k = \chi(k^{-1} v * \rho_{\varepsilon_k})$ with $\varepsilon_k \to 0$ fast, we get a sequence of functions $v_k \in \mathrm{Psh}(\Omega) \cap \mathscr{C}^\infty(\Omega)$ such that $0 \leqslant v_k \leqslant 1$, $v_k = 0$ in a neighborhood of $E \cap \Omega$ and $\lim v_k(x) = 1$ at every point of $\Omega \smallsetminus E$. Let $\theta \in \mathscr{C}^\infty([0,1])$ be a function such that $\theta = 0$ on $[0, 1/3]$, $\theta = 1$ on $[2/3, 1]$ and $0 \leqslant \theta \leqslant 1$. Then $\theta \circ v_k = 0$ near $E \cap \Omega$ and $\theta \circ v_k \to 1$ on $\Omega \smallsetminus E$. Therefore $\tilde{\Theta} = \lim_{k \to +\infty} (\theta \circ v_k)\Theta$ and

$$d'\tilde{\Theta} = \lim_{k \to +\infty} \Theta \wedge d'(\theta \circ v_k)$$

in the weak topology of currents. It is therefore sufficient to verify that $\Theta \wedge d'(\theta \circ v_k)$ converges weakly to 0 (note that $d''\tilde{\Theta}$ is conjugate to $d'\tilde{\Theta}$, thus $d''\tilde{\Theta}$ will also vanish).