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郭柏灵　著

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序

今年是恩师郭柏灵院士70寿辰，华南理工大学出版社决定出版《郭柏灵论文集》。郭老师的弟子，也就是我的师兄弟，推举我为文集作序。这使我深感荣幸。我于1985年考入北京应用物理与计算数学研究所，师从郭柏灵院士和周毓麟院士。研究生毕业后我留在研究所工作，继续跟随郭老师学习和研究偏微分方程理论。老师严谨的治学作风和对后学的精心培养与殷切期望，给我留下了深刻的印象，同时老师在科研上的刻苦精神也一直深深地印在我的脑海中。

郭老师1936年生于福建省龙岩市新罗区龙门镇，1953年从福建省龙岩市第一中学考入复旦大学数学系，毕业后留校工作。1963年，郭老师服从祖国的需要，从复旦大学调入北京应用物理与计算数学研究所，从事核武器研制中有关的数学、流体力学问题及其数值方法研究和数值计算工作。他全力以赴地做好了这项工作，为我国核武器的发展做出了积极的贡献。1978年改革开放以后，他又在非线性发展方程数学理论及其数值方法领域开展研究工作，现为该所研究员、博士生导师，中国科学院院士。迄今他共发表学术论文300余篇、专著9部，1987年获国家自然科学三等奖，1994年和1998年两度获得国防科工委科技进步一等奖，为我国的国防建设与人才培养作出了巨大贡献。

郭老师的研究方向涉及数学的多个领域，其中包括非线性发展方程的数学理论及其数值解、孤立子理论、无穷维动力系统等，其研究工作的主要特点是紧密联系数学物理中提出的各种重要问题。他对力学及物理学等应用学科中出现的许多重要的非线性发展方程进行了系统深入的研究，其中对 Landau - Lifshitz 方程和 Benjamin - Ono 方程的大初值的整体可解性、解的唯一性、正则性、渐近行为以及爆破现象等建立了系统而深刻的数学理论。在无穷维动力系统方面，郭老师研究了一批重要的无穷维动力系统，建立了有关整体吸引子、惯性流形和近似惯性流形的存在性和分形维数精细估计等理论，提出了一种证明强紧吸引子的新方法，并利用离散化等方法进行理论分析和数值计算，展示了吸引子的结构和图像。下面我从这几个方面介绍郭老师的一些学术成就。

Landau - Lifshitz 方程(又称铁磁链方程)由于其结构的复杂性，特别是强耦合性和不带阻尼时的强退化性，在20世纪80年代之前国内外几乎没有从数学上进行理论研究的成果出现。最先进行研究的，当属周毓麟院士和郭老师，他们在1982年到1986年间，采用 Leray - Schauder 不动点定理、离散方法、Galerkin 方法证明了从一维到高维的各种边值问题整体弱解的存在性，比国外在1992年才出现的同类结果早了将近10年。

20世纪90年代初期，周毓麟、郭柏灵和谭绍滨，郭柏灵和洪敏纯得到了两个在国内外至今影响很大的经典结果。第一，通过差分方法结合粘性消去法，利用十分巧妙的先验估计，证明了一维 Landau - Lifshitz 方程光滑解的存在唯一

性,对于一维问题给出了完整的答案,解决了长期悬而未决的难题。第二,系统分析了带阻尼的二维 Landau - Lifshitz 方程弱解的奇性,发现了 Landau - Lifshitz 方程与调和映照热流的联系,其弱解具有与调和映照热流完全相同的奇性。现在,国内外这方面的文章基本上都引用这个结果。调和映照的 Landau - Lifshitz 流的概念,即是源于此项结果。

20 世纪 90 年代中期,郭老师对于 Landau - Lifshitz 方程的长时间性态、Landau - Lifshitz 方程耦合 Maxwell 方程的弱解及光滑解的存在性问题进行了深入的研究,得到了一系列的成果。铁磁链方程的退化性以及缺少相应的线性化方程解的表达式,对研究解的长时间性态带来很大困难。郭老师的一系列成果克服了这些困难,证明了近似惯性流形的存在性、吸引子的存在性,给出了其 Hausdorff 和分形维数的上、下界的精细估计。此外,我们知道,与调和映照热流比较,高维铁磁链方程的研究至今还很不完善。其中最重要的是部分正则性问题,其难点在于单调不等式不成立,导致能量衰减估计方面的困难。另外一个是 Blow-up 解的存在性问题,至今没有解决;而对于调和映照热流来说,这样的问题的研究是比较成熟的。

对于高维问题,20 世纪 90 年代后期至今,郭老师和陈韵梅、丁时进、韩永前、杨干山一道,得出了许多成果,大大地推动了该领域的研究。首先,证明了二维问题的能量有限弱解的几乎光滑性及唯一性,这个结果类似于 Freire 关于调和映照热流的结果。第二,得到了高维 Landau - Lifshitz 方程初边值问题的奇点集合的 Hausdorff 维数和测度的估计。第三,得到了三维 Landau - Lifshitz - Maxwell 方程的奇点集合的 Hausdorff 维数和测度的估计。第四,得到了一些高维轴对称问题的整体光滑解和奇性解的精确表达式。郭老师还开创了一些新的研究领域。例如,关于一维非均匀铁磁链方程光滑解的存在唯一性结果后来被其他数学家引用并推广到一般流形上。其次,率先讨论了可压缩铁磁链方程测度值解的存在性。最近,在 Landau - Lifshitz 方程耦合非线性 Maxwell 方程方面,也取得了许多新的进展。

多年来,郭老师还对一大批非线性发展方程解的整体存在唯一性、有限时刻的爆破性、解的渐近性态等开展了广泛而深入的研究,受到国内外同行的广泛关注。研究的模型源于数学物理、水波、流体力学、相变等领域,如含磁场的 Schrödinger 方程组、Zakharov 方程、Schrödinger - Boussinesq 方程组、Schrödinger - KdV 方程组、长短波方程组、Maxwell 方程组、Davey - Stewartson 方程组、Klein - Gordon - Schrödinger 方程组、波动方程、广义 KdV 方程、Kadomtsev - Petviashvili(KP)方程、Benjamin - Ono 方程、Newton - Boussinesq 方程、Cahn - Hilliard 方程、Ginzburg - Landau 方程等。其中不少耦合方程组都是郭老师得到了第一个结果,开创了研究的先河,对国内外同行的研究产生了深远的影响。

郭老师在无穷维动力系统方面也开展了广泛的研究,取得了丰硕的成果。

对耗散非线性发展方程所决定的无穷维动力系统,研究了整体吸引子的存在性、分形维数估计、惯性流形、近似惯性流形、指数吸引子等问题。特别是在研究无界域上耗散非线性发展方程的强紧整体吸引子存在性时所提出的化弱紧吸引子成为强紧吸引子的重要方法和技巧,颇受同行关注并广为利用。对五次非线性 Ginzburg - Landau 方程,郭老师利用空间离散化方法将无限维问题化为有限维问题,证明了该问题离散吸引子的存在性,并考虑五次 Ginzburg - Landau 方程的定态解、慢周期解、异宿轨道等的结构。利用有限维动力系统的理论和方法,结合数值计算得到具体的分形维数(不超过 4)和结构以及走向混沌、湍流的具体过程和图像,这是一种寻求整体吸引子细微结构新的探索和尝试,对其他方程的研究也是富有启发性的。1999 年以来,郭老师集中于近可积耗散的和 Hamilton 无穷维动力系统的结构性研究,利用孤立子理论、奇异摄动理论、Fenichel 纤维理论和无穷维 Melnikov 函数,对于具有小耗散的三次到五次非线性 Schrödinger 方程,证明了同宿轨道的不变性,并在有限维截断下证明了 Smale 马蹄的存在性,目前,正把这一方法应用于具小扰动的 Hamilton 系统的研究上。他对于非牛顿流无穷维动力系统也进行了系统深入的研究,建立了有关的数学理论,并把有关结果写成了专著。以上这些工作得到国际同行们的高度评价,被称为“有重大的国际影响”、“对无穷维动力系统理论有重要持久的贡献”。最近,郭老师及其合作者又证明了具耗散的 KdV 方程 L^2 整体吸引子的存在性,该结果也是引人注目的。

郭老师不仅自己辛勤地搞科研,还尽心尽力培养了大批的研究生(硕士生、博士生、博士后),据不完全统计,有 40 多人。他根据每个人不同的学习基础和特点,给予启发式的具体指导,其中的不少人已成为了该领域的学科带头人,有些人虽然开始时基础较差,经过培养,也得到了很大提高,成为该方向的业务骨干。

《郭柏灵论文集》按照郭老师在不同时期所从事的研究领域,分成多卷出版。文集中所搜集的都是郭老师正式发表过的学术成果。把这些成果整理成集出版,不仅系统地反映了他的科研成就,更重要的是对于从事这方面学习、研究的学者无疑大有裨益。这本文集的出版得到了多方面的帮助与支持,特别要感谢华南理工大学校长李元元教授、华南理工大学出版社范家巧社长和华南理工大学数学科学学院吴敏院长的支持。还要特别感谢华南理工大学的李用声教授、华南师范大学的丁时进教授、北京应用物理与计算数学研究所的苗长兴研究员等人在论文的搜集、选择与校对等工作中付出了辛勤的劳动。感谢华南理工大学出版社的编辑对文集的精心编排工作。

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2005 年 8 月于厦门大学

目　录

2005 年

2006 年

2007 年

Two New Types of Bounded Waves of CH-γ Equation *

Guo Boling(郭柏灵) Liu Zhengrong(刘正荣)

Abstract

In this paper, the bifurcation method of dynamical systems and numerical approach of differential equations are employed to study CH-γ equation. Two new types of bounded waves are found. One of them is called the compacton. The other is called the generalized kink wave. Their planar graphs are simulated and their implicit expressions are given. The identity of theoretical derivation and numerical simulation is displayed.

Keywords CH-γ equation; compactons; generalized kink waves

1 Introduction

Recently, Dullin, Gottwald and Holm[1,2] considered the following integrable shallow water equation:

$$m_t + c_0 u_x + u m_x + 2 m u_x = -\gamma u_{xxx}, \tag{1.1}$$

which is called the CH-γ equation. Here $m = u - \alpha^2 u_{xx}$ is a momentum variable, α, c_0 and γ are constants and $\alpha \neq 0$. In Ref. [1], Dullin et al. identified how the dispersion coefficients for the linearized water waves appear as parameters in the isospectral problem for this inverse scattering transform integrable equation. They also determined how the linear dispersion parameters α, c_0, and γ in (1.1) affect the isospectral content of its soliton solutions and the shape of its traveling waves. In Ref. [2] Dullin et al. showed the asymptotically equivalence of Eq. (1.1) and other shallow water wave equations.

Guo and Liu[3] rewrote (1.1) as

$$u_t + c_0 u_x + 3 u u_x - \alpha^2 (u_{xxt} + u u_{xxx} + 2 u_x u_{xx}) + \gamma u_{xxx} = 0, \tag{1.2}$$

and obtained some peaked wave solutions. The limiting property of the periodic cusp waves was also discussed.

Zhang[4] found a transformation which simplifies Eq. (1.2) to CH equation and obtained general explicit expressions of peaked waves.

Tang and Yang[5] added an integral constant to the traveling wave equation of (1.2) and extended the peaked wave solutions of (1.1).

When $\alpha^2 = 1$, $c_0 = 2k$ and $\gamma = 0$, many authors have given the studies on Eq. (1.2), for

* Science in China Ser. A Mathematics, 2005, 48(12): 1618 - 1630.

instance, see Refs. [6 - 9].

In this paper, we study the bounded waves of (1.1) by using the bifurcation method of dynamical systems and numerical approach of differential equations. Two new types of the bounded waves of (1.1) are found. One of them is called the compacton which has been found in RH equation by Rosenau and Hyman[10]. The compactons mean the solitons with the compact support, or the strict localization of solitary waves. The other is called the general kink wave. To our knowledge, the generalized kink wave is our new discovery. Nobody has ever found them in other models. Since it is defined on the semifinal bounded domain and possesses some properties of the kink wave, we call it the generalized kink wave.

It is well-known that a kink wave solution means a traveling wave solution $f(\xi)$ defined on $(-\infty, \infty)$ and $\lim\limits_{\xi\to-\infty} f(\xi) = A$, $\lim\limits_{\xi\to-\infty} f(\xi) = B$, where $\xi = x - ct$, A, B and c are constants and $A \neq B$. In this paper, we find that Eq. (1.1) has some bounded traveling wave solutions $\varphi_i(\xi)$ $(i = 1, 2, 3, 4)$ defined on the semifinal bounded interval. Their implicit expressions will be given in the following Theorem 2.2. Their planar graphs will be simulated in Fig. 5.

This paper is organized as follows. In Section 2, we will state our main results which are the implicit expressions of the compactons and the generalized kink wave solutions of Eq. (1.1). The preliminaries are in Section 3. Sections 4 and 5 present the proofs of the main results. In Section 6, the numerical simulation is given to verify the theoretical results. A short conclusion is given in Section 7.

2 Main Results

To state conveniently, we make the following preparations. For given parameters $\alpha \neq 0$, c_0 and γ, suppose c and g are arbitrary constants, and let

$$\xi = x - ct, \tag{2.1}$$

$$c^* = -(c_0 + 3\gamma/\alpha^2)/2, \tag{2.2}$$

$$q = c + \gamma/\alpha^2, \tag{2.3}$$

$$g_1 = -(c + \gamma/\alpha^2)(c + 2c_0 + 3\gamma/\alpha^2)/2, \tag{2.4}$$

$$g_2 = (c_0 + \gamma/\alpha^2)(c_0 - 4c - 3\gamma/\alpha^2)/8, \tag{2.5}$$

$$\varphi_\pm^0 = \left[(c - c_0) \pm \sqrt{(c - c_0)^2 - 6g}\,\right] \Big/ 3, \tag{2.6}$$

$$\varphi_\pm^* = \frac{1}{2}\left[-c_0 - \frac{\gamma}{\alpha^2} \pm \sqrt{\left(c_0 + \frac{\gamma}{\alpha^2}\right)^2 - 4\left(c_0 + \frac{\gamma}{\alpha^2}\right)\left(c + \frac{\gamma}{\alpha^2}\right) - 8g}\,\right], \tag{2.7}$$

$$a = \left\{c - c_0 - \varphi_0 + \sqrt{(-c + c_0 + \varphi_0)^2 - 4[2g + \varphi_0(-c + c_0 + \varphi_0)]}\right\} \Big/ 2, \tag{2.8}$$

$$d = \left\{c - c_0 - \varphi_0 - \sqrt{(-c + c_0 + \varphi_0)^2 - 4[2g + \varphi_0(-c + c_0 + \varphi_0)]}\right\} \Big/ 2, \tag{2.9}$$

where φ_0 is a constant and its range will be given later.

Our main results, the existence and implicit expressions of compactons and generalized kink waves, will be stated in the following two theorems.

Theorem 2.1 If let $K(k)$ be the complete elliptic integral of the first kind, $\prod(\cdot,\cdot)$ be the Legendre's incomplete elliptic integral of the third kind, $\mathrm{sn}u = \mathrm{sn}(u, k)$ be the sine amplitude u (Jacobian elliptic function), $\mathrm{sn}^{-1}u = \mathrm{sn}^{-1}(u, k)$ be the inverse function of $\mathrm{sn}u$[11], then Eq. (1.1) has compacton $u = \varphi(\xi)$ which is of the following implicit form:

Case 1 If the constants c, g and φ_0 satisfy the conditions $c > c^*$, $g_1 \leqslant g < g_2$ and $\varphi_-^0 < \varphi_0 < \varphi_+^*$, or for any c, $g < g_1$ and $\varphi_-^0 < \varphi_0 < q$, then $\varphi(\xi)$ is concave and possesses the implicit expression

$$\prod\left(\mathrm{sn}^{-1}\sqrt{\frac{(a-\varphi_0)(q-\varphi)}{(q-\varphi_0)(a-\varphi)}}, \alpha_1^2\right) - \mathrm{sn}^{-1}\left(\sqrt{\frac{(a-\varphi_0)(q-\varphi)}{(q-\varphi_0)(a-\varphi)}}, k_1\right) = e_1(\xi_1^0 - |\xi|),$$
$$|\xi| \leqslant \xi_1^0, \tag{2.10}$$

where

$$\alpha_1^2 = \frac{q-\varphi_0}{a-\varphi_0}, \tag{2.11}$$

$$k_1^2 = \frac{(q-\varphi_0)(a-d)}{(a-\varphi_0)(q-d)}, \tag{2.12}$$

$$e_1 = \frac{\sqrt{(a-\varphi_0)(q-d)}}{2|\alpha|(a-q)}, \tag{2.13}$$

and

$$\xi_1^0 = \frac{\prod(\alpha_1^2, k_1) - K(k_1)}{e_1}. \tag{2.14}$$

Case 2 If the constants c, g and φ_0 satisfy the conditions $c < c^*$, $g_1 \leqslant g < g_2$ and $\varphi_-^* < \varphi_0 < \varphi_+^0$, or for any c, $g < g_1$ and $q < \varphi_0 < \varphi_+^0$, then $\varphi(\xi)$ is convex and has an implicit representation

$$\prod\left(\mathrm{sn}^{-1}\sqrt{\frac{(\varphi_0-d)(\varphi-q)}{(\varphi_0-q)(\varphi-d)}}, \alpha_2^2\right) - \mathrm{sn}^{-1}\left(\sqrt{\frac{(\varphi_0-d)(\varphi-q)}{(\varphi_0-q)(\varphi-d)}}, k_2\right) = e_2(\xi_2^0 - |\xi|),$$
$$|\xi| \leqslant \xi_2^0, \tag{2.15}$$

where

$$\alpha_2^2 = \frac{\varphi_0-q}{\varphi_0-d}, \tag{2.16}$$

$$k_2^2 = \frac{(\varphi_0-q)(a-d)}{(a-q)(\varphi_0-d)}, \tag{2.17}$$

$$e_2 = \frac{\sqrt{(a-q)(\varphi_0-d)}}{2(q-d)|\alpha|} \tag{2.18}$$

and

$$\xi_2^0 = \frac{\prod(\alpha_2^2, k_2) - K(k_2)}{e_2}. \tag{2.19}$$

Theorem 2.2 Eq. (1.1) has the generalized kink wave solutions $\varphi_i(\xi)(i=1,2,3,4)$ which are of the following implicit forms:

Case 3 If $c>c^*$ and $g_1\leqslant g<g_2$ or for any c, $g<g_1$, then $\varphi_1(\xi)$ and $\varphi_2(\xi)$ satisfy respectively

$$\left(\frac{\sqrt{m-\varphi_1}+\sqrt{q-\varphi_1}}{\sqrt{m-\varphi_1}-\sqrt{q-\varphi_1}}\right)\left(\frac{\delta_1\sqrt{m-\varphi_1}-\sqrt{q-\varphi_1}}{\delta_1\sqrt{m-\varphi_1}+\sqrt{q-\varphi_1}}\right)^{\delta_1}=\beta_1 e^{\xi/|\alpha|},\ \xi\in(-\infty,l_1) \tag{2.20}$$

and

$$\left(\frac{\sqrt{m-\varphi_2}+\sqrt{q-\varphi_2}}{\sqrt{m-\varphi_2}-\sqrt{q-\varphi_2}}\right)\left(\frac{\delta_1\sqrt{m-\varphi_2}-\sqrt{q-\varphi_2}}{\delta_1\sqrt{m-\varphi_2}+\sqrt{q-\varphi_2}}\right)^{\delta_1}=\beta_2 e^{-\xi/|\alpha|},\ \xi\in(-l_2,\infty), \tag{2.21}$$

where

$$\beta_i=\left(\frac{\sqrt{m-\varphi_i^0}+\sqrt{q-\varphi_i^0}}{\sqrt{m-\varphi_i^0}-\sqrt{q-\varphi_i^0}}\right)\left(\frac{\delta_1\sqrt{m-\varphi_i^0}-\sqrt{q-\varphi_i^0}}{\delta_1\sqrt{m-\varphi_i^0}+\sqrt{q-\varphi_i^0}}\right)^{\delta_1},\ i=1,2, \tag{2.22}$$

$$\delta_1=\sqrt{\frac{q-\varphi_-^0}{m-\varphi_-^0}}, \tag{2.23}$$

$$m=\left[c-c_0+2\sqrt{(c-c_0)^2-6g}\right]\Big/3, \tag{2.24}$$

φ_1^0 and φ_2^0 are two arbitrary constants and belong to the interval (φ_-^0,q) and

$$l_i=-|\alpha|\ln\beta_i,\ i=1,2. \tag{2.25}$$

Case 4 If $c<c^*$ and $g_1\leqslant g<g_2$ or for any c, $g<g_1$ then $\varphi_3(\xi)$ and $\varphi_4(\xi)$ satisfy respectively

$$\left(\frac{\sqrt{\varphi_3-n}-\sqrt{\varphi_3-q}}{\sqrt{\varphi_3-n}+\sqrt{\varphi_3-q}}\right)\left(\frac{\delta_2\sqrt{\varphi_3-n}+\sqrt{\varphi_3-q}}{\delta_2\sqrt{\varphi_3-n}-\sqrt{\varphi_3-q}}\right)^{\delta_2}=\beta_3 e^{\xi/|\alpha|},\ \xi\in(-l_3,\infty) \tag{2.26}$$

and

$$\left(\frac{\sqrt{\varphi_4-n}-\sqrt{\varphi_4-q}}{\sqrt{\varphi_4-n}+\sqrt{\varphi_4-q}}\right)\left(\frac{\delta_2\sqrt{\varphi_4-n}+\sqrt{\varphi_4-q}}{\delta_2\sqrt{\varphi_4-n}-\sqrt{\varphi_4-q}}\right)^{\delta_2}=\beta_4 e^{-\xi/|\alpha|},\ \xi\in(-\infty,l_4), \tag{2.27}$$

where

$$\beta_j=\left(\frac{\sqrt{\varphi_j^0-n}-\sqrt{\varphi_j^0-q}}{\sqrt{\varphi_j^0-n}+\sqrt{\varphi_j^0-q}}\right)\left(\frac{\delta_2\sqrt{\varphi_j^0-n}+\sqrt{\varphi_j^0-q}}{\delta_2\sqrt{\varphi_j^0-n}-\sqrt{\varphi_j^0-q}}\right)^{\delta_2},\ j=3,4, \tag{2.28}$$

$$\delta_2=\sqrt{\frac{\varphi_+^0-q}{\varphi_+^0-n}}, \tag{2.29}$$

$$n=\left[c-c_0-2\sqrt{(c-c_0)^2-6g}\right]\Big/3, \tag{2.30}$$

φ_3^0 and φ_4^0 are two arbitrary constants which belong to the interval $(q,\ \varphi_+^0)$ and

$$l_j = |\alpha| \ln\beta_j,\ j = 3,4. \tag{2.31}$$

We give the following three examples to display the planar graphs of the above compactons and generalized kink waves.

Example 2.1 (corresponding to Case 1 of Theorem 2.1) If take $\alpha = 2$, $c_0 = 2$ and $\gamma = 3$, then from (2.2) it follows that $c^* = -2.125$. Choosing $c = 2 > c^*$, then from (2.3), (2.4) and (2.5) we have $q = 2.75$, $g_1 \approx -11.3438$ and $g_2 \approx -2.83594$. On the interval (g_1, g_2) taking $g = -6$, from (2.6) and (2.7) we get $\varphi_-^0 = -2$, $\varphi_+^* = 1.140\,58$ and $\varphi_+^0 = 2$. Letting $\varphi_0 = 1 \in (\varphi_-^0,\ \varphi_+^*)$, from (2.8) and (2.11)~(2.14) we obtain

$$a \approx 2.8541,\ d \approx -3.8541,\ \alpha_1^2 \approx 0.9439,$$
$$k_1^2 \approx 0.9587,\ e_1 \approx 8.4034,\ \text{and } \xi_1^0 \approx 2.1298. \tag{2.32}$$

Using Maple to (2.10), the graph of $\varphi(\xi)$ is displayed as (a) of Fig. 1.

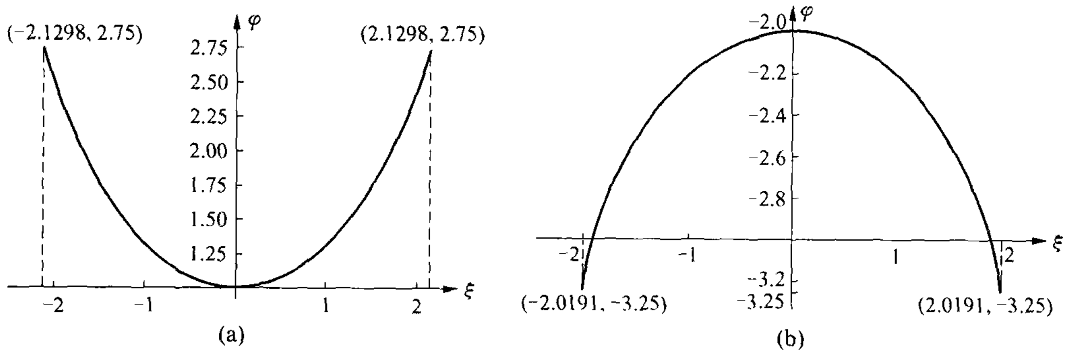

Fig. 1 The graphs of compacton $\varphi(\xi)$ when $\alpha = 2$, $c_0 = 2$ and $\gamma = 3$

(a) (corresponding to (2.10) and Example 1) $c = 2$, $g = -6$ and $\varphi_0 = 1$,

(b) (corresponding to (2.15) and Example 2) $c = -4$, $g = 4$ and $\varphi_0 = -2$

Example 2.2 (corresponding to Case 2 of Theorem 2.1) As Example 2.1, taking $\alpha = 2$, $c_0 = 2$ and $\gamma = 3$, it follows that $c^* = -2.125$. Let $c = -4 < c^*$, then $q = -3.25$, $g_1 = -3.65625$ and $g_2 \approx 5.414\,06$. On the interval (g_1, g_2) take $g = 4$, then $\varphi_-^0 \approx -3.1547$, $\varphi_-^* \approx -3.0567$ and $\varphi_+^0 \approx -0.8453$. Choosing $\varphi_0 = -2 \in (\varphi_-^*,\ \varphi_+^0)$, we have

$$a = 0,\ d = -4,\ \alpha_2^2 = 0.625,\ k_1^2 \approx 0.7692,\ e_2 \approx 0.8498,\ \text{and } \xi_2^0 \approx 2.0191. \tag{2.33}$$

Using Maple to (2.15), we obtain the graph of $\varphi(\xi)$ as (b) of Fig. 1.

Example 2.3 (corresponding to Case 3 of Theorem 2.2) Take the same data as Example 2.1, that is, $\alpha = 2$, $c_0 = 2$, $\gamma = 3$, $c^* = -2.125$, $c = 2$, $q = 2.75$, $g_1 \approx -11.3438$, $g_2 \approx -2.83594$, $g = -6$, $\varphi_-^0 = -2$. Clearly, c and g satisfy the conditions of case 3 of Theorem 2.2. Let $\varphi_1^0 = \varphi_2^0 = 1 \in (\varphi_-^0,\ q)$. From (2.22)~(2.25) we get

$$\beta_1 = \beta_2 \approx 0.7556,\ \delta_1 \approx 0.8898,\ m = 4,\ \text{and } l_1 = l_2 \approx 0.5605. \tag{2.34}$$

Applying Maple to (2.20) and (2.21) we obtain the graphs of $\varphi_1(\xi)$ and $\varphi_2(\xi)$ as Fig. 2.

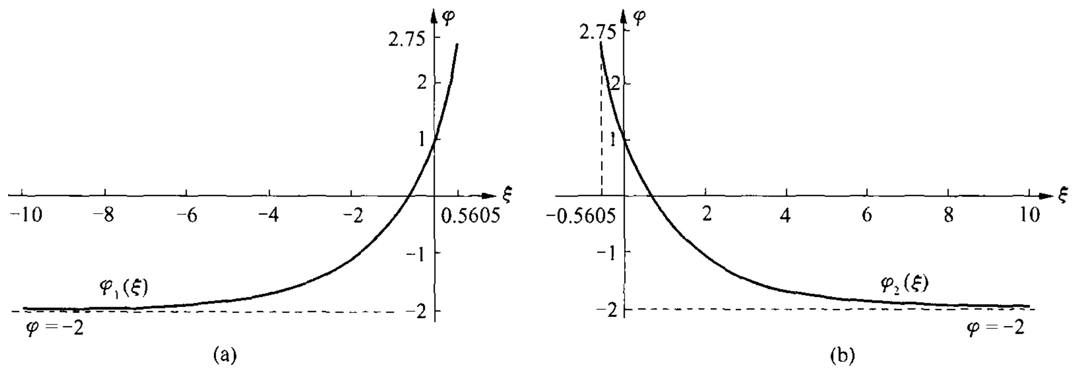

Fig. 2 The graphs of the generalized kink waves solutions $\varphi_1(\xi)$ and $\varphi_2(\xi)$ when $\alpha=2$, $c_0=2$, $\gamma=3$, $c=2$, $g=-6$, and $\varphi_1^0=\varphi_2^0=1$ (see Example 2.3)
(a) corresponding to (2.20), (b) corresponding to (2.21)

3 Preliminary

Letting c be a constant and substituting $u=\varphi(\xi)$ with $\xi=x-ct$ into (1.1), we get

$$(c_0-c)\varphi'+3\varphi\varphi'-\alpha^2(\varphi\varphi'''+2\varphi'\varphi''-c\varphi''')+\gamma\varphi'''=0. \tag{3.1}$$

Integrating (3.1) once we have

$$\varphi''(\alpha^2\varphi-\alpha^2c-\gamma)=g-(c-c_0)\varphi+\frac{3\varphi^2}{2}-\frac{\alpha^2(\varphi')^2}{2}, \tag{3.2}$$

which is called the travelling wave equation.

Letting $y=\varphi'$ we obtain the travelling wave system

$$\frac{d\varphi}{d\xi}=y,\ \frac{dy}{d\xi}=\frac{2g-2(c-c_0)\varphi+3\varphi^2-\alpha^2y^2}{2(\alpha^2\varphi-\alpha^2c-\gamma)}. \tag{3.3}$$

Obviously, (3.3) possesses the following properties[12]:

(1) $$H(\varphi,y)=y^2(\alpha^2\varphi-\alpha^2c-\gamma)-\varphi^3+(c-c_0)\varphi^2-2g\varphi=h \tag{3.4}$$

is a first integral, where h is a constant.

(2) $\psi=q$ is a singular line.

(3) When $c\neq c^*$ and $g<(c-c_0)^2/6$, $(\varphi_-^0,0)$ and $(\varphi_+^0,0)$ are two singular points, and

$$g_1<g_2<(c-c_0)^2/6. \tag{3.5}$$

(4) When $c>c^*$ and $g_1<g<g_2$, $(\varphi_-^0,0)$ is a saddle point. $(\varphi_+^0,0)$ is a center point surrounded by a family of closed orbits. The boundary of the closed orbits passes the point $(\varphi_+^*,0)$ (see (a) of Fig. 3). If φ_0 satisfies $\varphi_-^0<\varphi_0<\varphi_+^*$, then it follows that

$$d<\varphi_-^0<\varphi_0<\varphi_+^*<\varphi_+^0<q<a. \tag{3.6}$$

(5) When $c>c^*$ and $g=g_1$, $(\varphi_-^0,0)$ is a saddle point and $(\varphi_+^0,0)$ is a degenerate center point. If $\varphi_0\in(\varphi_-^0,\varphi_+^*)$, then it follows that

$$d<\varphi_-^0<\varphi_0<\varphi_+^*=\varphi_+^0=q<a \text{ (see (b) of Fig. 3)}. \tag{3.7}$$

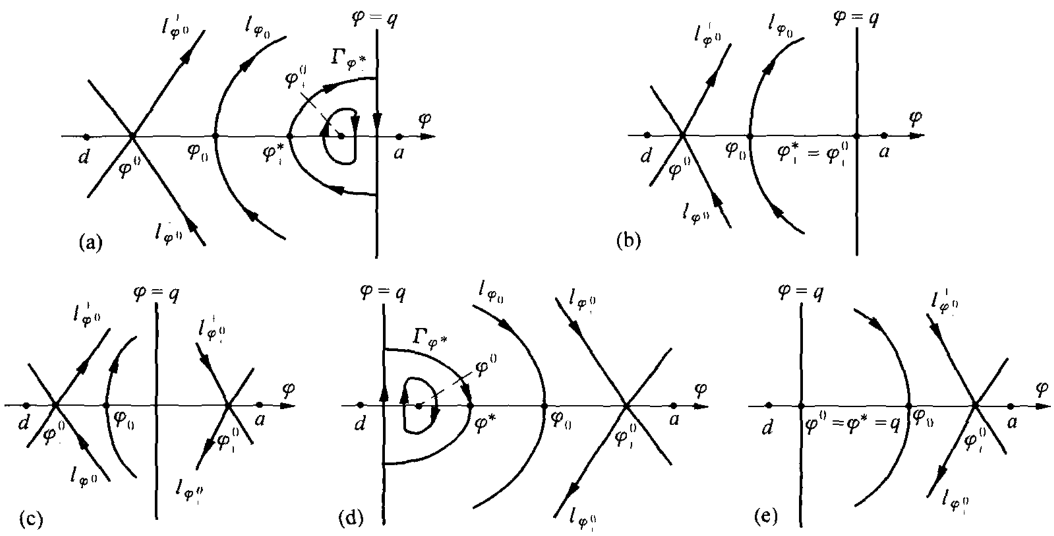

Fig.3 Some orbits and the locations of a, d, q, φ_-^0, φ_+^0, φ_+^* and φ_0 for System (3.3) when $g<g_2$
(a) $c>c^*$ and $g_1<g<g_2$, (b) $c>c^*$ and $g=g_1$, (c) any c and $g<g_1$,
(d) $c<c^*$ and $g_1<g<g_2$, (e) $c<c^*$ and $g=g_1$

(6) For any c and $g<g_1$, both $(\varphi_-^0,0)$ and $(\varphi_+^0,0)$ are saddle points and $\varphi_-^0<q<\varphi_+^0$ (see (c) of Fig.3). If $\varphi_0\in(\varphi_-^0,q)$ or $\varphi_0\in(q,\varphi_+^0)$, then it follows that

$$d<\varphi_-^0<\varphi_0<q<\varphi_+^0<a \tag{3.8}$$

or

$$d<\varphi_-^0<q<\varphi_0<\varphi_+^0<a. \tag{3.9}$$

(7) When $c<c^*$ and $g_1<g<g_2$, $(\varphi_+^0,0)$ is a saddle point. $(\varphi_-^0,0)$ is a center point surrounded by a family of closed orbits. The boundary of the closed orbit passes the point $(\varphi_-^*,0)$. If φ_0 satisfies $\varphi_-^*<\varphi_0<\varphi_+^0$, then it follows that

$$d<q<\varphi_-^0<\varphi_-^*<\varphi_0<\varphi_+^0<a \text{ (see (d) of Fig.3)}. \tag{3.10}$$

(8) When $c<c^*$ and $g=g_1$, $(\varphi_+^0,0)$ is a saddle point, and $(\varphi_-^0,0)$ is a degenerate center point. If φ_0 satisfies $\varphi_-^*<\varphi_0<\varphi_+^0$, then it follows that

$$d<q=\varphi_-^0=\varphi_-^*<\varphi_0<\varphi_+^0<a \text{ (see (e) of Fig.3)}. \tag{3.11}$$

4 Proof of Theorem 2.1

Proof Let l_{φ_0} denote the orbit of (3.3) passing the point $(\varphi_0,0)$, where φ_0 belongs to one of the following cases: (i) $\varphi_0\in(\varphi_-^0,\varphi_+^*)$ when $c>c^*$ and $g_1\leqslant g<g_2$; (ii) $\varphi_0\in(\varphi_-^*,\varphi_+^0)$ when $c<c^*$ and $g_1\leqslant g<g_2$; (iii) $\varphi_0\in(\varphi_-^0,\varphi_+^0)$ and $\varphi_0\neq q$ when $g<g_1$. Assume that $\varphi=\varphi(\xi)$ and $y=y(\xi)$ are the parameter representations of l_{φ_0}. Then from Fig.3 we see that $\varphi(\xi)$ satisfies $\varphi_0\leqslant\varphi(\xi)<q$ or $q<\varphi(\xi)\leqslant\varphi_0$. Under the conditions of Case 1 of Theorem 2.1, on $\varphi-y$ plane the orbit l_{φ_0} has the representation

$$y=\pm\frac{1}{|\alpha|}\sqrt{\frac{(a-\varphi)(\varphi-\varphi_0)(\varphi-d)}{q-\varphi}},\ \text{where } \varphi_0\leqslant\varphi<q. \tag{4.1}$$

Under the conditions of Case 2 of Theorem 2.1, the representation of l_{φ_0} is

$$y=\pm\frac{1}{|\alpha|}\sqrt{\frac{(a-\varphi)(\varphi_0-\varphi)(\varphi-d)}{\varphi-q}},\ \text{where } q<\varphi\leqslant\varphi_0, \tag{4.2}$$

q is in (2.3), a and d are in (2.8) and (2.9).

Let $\varphi(0)=\varphi_0$ and suppose

$$\lim_{\xi\to\xi_1^0}\varphi(\xi)=q \text{ for Case 1}, \tag{4.3}$$

$$\lim_{\xi\to\xi_2^0}\varphi(\xi)=q \text{ for Case 2}, \tag{4.4}$$

where $\xi_1^0>0$ and $\xi_2^0>0$. Note that l_{φ_0} is symmetric on φ axis. Thus it follows that

$$\lim_{\xi\to-\xi_1^0}\varphi(\xi)=q \text{ and } \lim_{\xi\to-\xi_2^0}\varphi(\xi)=q. \tag{4.5}$$

Substituting (4.1) and (4.2) into the first equation of System (3.3) and integrating them along l_{φ_0}, we have

①For Case 1,

$$\int_\varphi^q\sqrt{\frac{q-s}{(a-s)(s-\varphi_0)(s-d)}}\mathrm{d}s=\int_\xi^{\xi_1^0}\frac{1}{|\alpha|}\mathrm{d}s,\ \text{where } \xi\geqslant 0 \tag{4.6}$$

and

$$-\int_q^\varphi\sqrt{\frac{q-s}{(a-s)(s-\varphi_0)(s-d)}}\mathrm{d}s=\int_{-\xi_1^0}^{\xi}\frac{1}{|\alpha|}\mathrm{d}s,\ \text{where } \xi<0. \tag{4.7}$$

②For Case 2,

$$\int_q^\varphi\sqrt{\frac{s-q}{(a-s)(\varphi_0-s)(s-d)}}\mathrm{d}s=\int_{-\xi_2^0}^{\xi}\frac{1}{|\alpha|}\mathrm{d}s,\ \text{where } \xi<0 \tag{4.8}$$

and

$$-\int_\varphi^q\sqrt{\frac{s-q}{(a-s)(\varphi_0-s)(s-d)}}\mathrm{d}s=\int_\xi^{\xi_2^0}\frac{1}{|\alpha|}\mathrm{d}s,\ \text{where } \xi\geqslant 0. \tag{4.9}$$

From the above four formulae we obtain

③For Case 1,

$$\int_\varphi^q\sqrt{\frac{q-s}{(a-s)(s-\varphi_0)(s-d)}}\mathrm{d}s=\frac{\xi_1^0-|\xi|}{|\alpha|},\ \text{where } a>q>\varphi\geqslant\varphi_0>d. \tag{4.10}$$

④For Case 2,

$$\int_q^\varphi\sqrt{\frac{s-q}{(a-s)(\varphi_0-s)(s-d)}}\mathrm{d}s=\frac{\xi_2^0-|\xi|}{|\alpha|},\ \text{where } a>\varphi_0\geqslant\varphi_0>q>d. \tag{4.11}$$

In (4.3) and (4.4) respectively, let

$$u_1 = \mathrm{sn}^{-1}\sqrt{\frac{(a-\varphi_0)(q-\varphi)}{(q-\varphi_0)(a-\varphi)}} \tag{4.12}$$

and

$$u_2 = \mathrm{sn}^{-1}\sqrt{\frac{(\varphi_0-d)(\varphi-q)}{(\varphi_0-q)(\varphi-d)}}. \tag{4.13}$$

Then (4.10) and (4.11) correspondingly become

$$\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{d}u}{1-\alpha_1^2\mathrm{sn}^2 u} = \frac{\sqrt{(a-\varphi_0)(q-d)}(\xi_1^0-|\xi|)}{2(a-q)|\alpha|\alpha_1^2} \tag{4.14}$$

and

$$\int_0^{u_2}\frac{\mathrm{sn}^2 u\,\mathrm{d}u}{1-\alpha_2^2\mathrm{sn}^2 u} = \frac{\sqrt{(a-q)(\varphi_0-d)}(\xi_2^0-|\xi|)}{2(q-d)|\alpha|\alpha_2^2}. \tag{4.15}$$

where α_1^2 and α_2^2 are in (2.11) and (2.16).

Note that

$$\int_0^{u_1}\frac{\mathrm{sn}^2 u\,\mathrm{d}u}{1-\alpha_1^2\mathrm{sn}^2 u} = \frac{\prod(u_1,\alpha_1^2)-u_1}{\alpha_1^2} \tag{4.16}$$

and

$$\int_0^{u_2}\frac{\mathrm{sn}^2 u\,\mathrm{d}u}{1-\alpha_2^2\mathrm{sn}^2 u} = \frac{\prod(u_2,\alpha_2^2)-u_2}{\alpha_2^2}. \tag{4.17}$$

From (4.14) ~ (4.17), we obtain the implicit expressions of $\varphi(\xi)$ as (2.10) and (2.15).

On the other hand, letting $\varphi(0)=\varphi_0$ in (4.10) and (4.11), we get

$$\xi_1^0 = |\alpha|\int_{\varphi_0}^{q}\sqrt{\frac{q-s}{(a-s)(s-\varphi_0)(s-d)}}\mathrm{d}s \tag{4.18}$$

and

$$\xi_2^0 = |\alpha|\int_{q}^{\varphi_0}\sqrt{\frac{s-q}{(a-s)(\varphi_0-s)(s-d)}}\mathrm{d}s. \tag{4.19}$$

From (4.18) and (4.19) we get the expressions of ξ_1^0 and ξ_2^0 as (2.14) and (2.19). This completes the proof.

5 Proof of Theorem 2.2

Proof From Fig. 3, it is seen that there are four orbits entering the singular point $(\varphi_-^0, 0)$ or $(\varphi_+^0, 0)$ when they are the saddle points. We are interested in $l^{\pm}_{\varphi_-^0}$ and $l^{\pm}_{\varphi_+^0}$ (see Fig.3) on which $\varphi(\xi)$ is bounded. Note that on $\varphi-y$ plane they have