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Vladimir I. Arnold Valery V. Kozlov Anatoly I. Neishtadt

Mathematical Aspects of Classical and Celestial Mechanics

Third Edition

经典力学与天体力学中的数学问题

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Contents

1

Basic Principles of Classical Mechanics

For describing the motion of a mechanical system various mathematical models are used based on different "principles" – laws of motion. In this chapter we list the basic objects and principles of classical dynamics. The simplest and most important model of the motion of real bodies is Newtonian mechanics, which describes the motion of a free system of interacting points in three-dimensional Euclidean space. In § 1.6 we discuss the suitability of applying Newtonian mechanics when dealing with complicated models of motion.

1.1 Newtonian Mechanics

1.1.1 Space, Time, Motion

The space where the motion takes place is three-dimensional and Euclidean with a fixed orientation. We shall denote it by E^3. We fix some point $o \in E^3$ called the "origin of reference". Then the position of every point s in E^3 is uniquely determined by its position vector $\vec{os} = \mathbf{r}$ (whose initial point is o and end point is s). The set of all position vectors forms the three-dimensional vector space $\mathbb{R}^3$, which is equipped with the scalar product $\langle \, , \rangle$.

Time is one-dimensional; it is denoted by t throughout. The set $\mathbb{R} = \{t\}$ is called the *time axis*.

A *motion* (or *path*) of the point s is a smooth map $\Delta \to E^3$, where Δ is an interval of the time axis. We say that the motion is defined on the interval Δ. If the origin (point o) is fixed, then every motion is uniquely determined by a smooth vector-function $\mathbf{r}\colon \Delta \to \mathbb{R}^3$.

The image of the interval Δ under the map $t \mapsto \mathbf{r}(t)$ is called the *trajectory* or *orbit of the point s*.

The *velocity* $\mathbf{v}$ of the point s at an instant $t \in \Delta$ is by definition the derivative $d\mathbf{r}/dt = \dot{\mathbf{r}}(t) \in \mathbb{R}^3$. Clearly the velocity is independent of the choice of the origin.

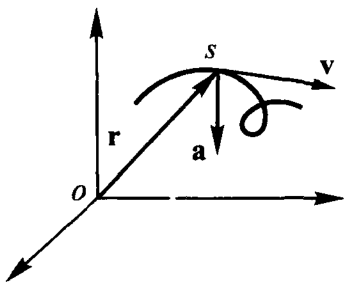

Fig. 1.1.

The *acceleration* of the point is by definition the vector $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \in \mathbb{R}^3$. The velocity and acceleration are usually depicted as vectors with initial point at the point s (see Fig. 1.1).

The set E^3 is also called the *configuration space* of the point s. The pair $(s, \mathbf{v})$ is called the state of the point, and the set $E^3 \times \mathbb{R}^3\{\mathbf{v}\}$, the *phase* (or *state*) *space*.

Now consider a more general case when there are n points $s_1, \dots, s_n$ moving in the space E^3. The set $E^{3n} = E^3\{s_1\} \times \cdots \times E^3\{s_n\}$ is called the configuration space of this "free" system. If it is necessary to exclude collisions of the points, then E^{3n} must be diminished by removing from it the union of diagonals

$$\Delta = \bigcup_{i<j} \{s_i = s_j\}.$$

Let $(\mathbf{r}_1, \dots, \mathbf{r}_n) = \mathbf{r} \in \mathbb{R}^{3n}$ be the position vectors of the points $s_1, \dots, s_n$. A motion of the free system is given by smooth vector-functions $\mathbf{r}(t) = (\mathbf{r}_1(t), \dots, \mathbf{r}_n(t))$. We define in similar fashion the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = (\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_n) = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathbb{R}^{3n}$$

and the acceleration

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{\mathbf{r}}_1, \dots, \ddot{\mathbf{r}}_n) = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{R}^{3n}.$$

The set $E^{3n} \times \mathbb{R}^{3n}\{\mathbf{v}\}$ is called the phase (or state) space, and the pair $(s, \mathbf{v})$, the state of the system.

1.1.2 Newton–Laplace Principle of Determinacy

This principle (which is an experimental fact) asserts that the state of the system at any fixed moment of time uniquely determines all of its motion (both in the future and in the past).

Suppose that we know the state of the system $(\mathbf{r}_0, \mathbf{v}_0)$ at an instant t_0. Then, according to the principle of determinacy, we know the motion $\mathbf{r}(t)$,

$t \in \Delta \subset \mathbb{R}$; $\mathbf{r}(t_0) = \mathbf{r}_0$, $\dot{\mathbf{r}}(t_0) = \dot{\mathbf{r}}_0 = \mathbf{v}_0$. In particular, we can calculate the acceleration $\ddot{\mathbf{r}}$ at the instant $t = t_0$.[1] Then $\ddot{\mathbf{r}}(t_0) = \mathbf{f}(t_0, \mathbf{r}_0, \dot{\mathbf{r}}_0)$, where $\mathbf{f}$ is some function whose existence follows from the Newton–Laplace principle. Since the time t_0 can be chosen arbitrarily, we have the equation

$$\ddot{\mathbf{r}} = \mathbf{f}(t, \mathbf{r}, \dot{\mathbf{r}})$$

for all t.

This differential equation is called the *equation of motion* or *Newton's equation*. The existence of Newton's equation (with a smooth vector-function $\mathbf{f}\colon \mathbb{R}\{t\} \times \mathbb{R}^{3n}\{\mathbf{r}\} \times \mathbb{R}^{3n}\{\dot{\mathbf{r}}\} \to \mathbb{R}^{3n}$) is equivalent to the principle of determinacy. This follows from the existence and uniqueness theorem in the theory of differential equations. The function $\mathbf{f}$ in Newton's equations is usually determined in experiments. The definition of a mechanical system includes specifying this function.

We now consider examples of Newton's equations.

a) The equation of a point in free fall in vacuum near the surface of the Earth (obtained experimentally by Galileo) has the form $\ddot{\mathbf{r}} = -g\mathbf{e}_z$, where $g \approx 9.8\ \mathrm{m/s^2}$ (the acceleration of gravity) and $\mathbf{e}_z$ is the vertical unit vector. The trajectory of a falling point is a parabola.

b) Hooke showed that the equation of small oscillations of a body attached to the end of an elastic spring has the form $\ddot{x} = -\alpha x$, $\alpha > 0$. The constant coefficient α depends on the choice of the body and spring. This mechanical system is called a *harmonic oscillator* (see Fig. 1.2).

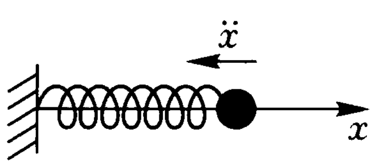

Fig. 1.2. Harmonic oscillator

It turned out that in experiments, rather than finding the acceleration $\mathbf{f}$ on the right-hand side of Newton's equations, it is more convenient to determine the product $m\mathbf{f} = \mathbf{F}$, where m is some positive number called the mass of the point (an instructive discussion of the physical meaning of the notion of mass can be found in [601, 401, 310]). For example, in Hooke's experiments the constant $m\alpha = c$ depends on the properties of the elastic spring, but not on the choice of the body. This constant is called the coefficient of elasticity.

The pair (s, m) (or $(\mathbf{r}, m)$, where $\mathbf{r}$ is the position vector of the point s) is called a *material point* of mass m. In what follows we shall often denote a point s and its mass m by one and the same symbol m. If a system of material

[1] We assume that all the functions occurring in dynamics are smooth.

points consists of n points with masses $m_1, \ldots, m_n$, then Newton's equations

$$\ddot{\mathbf{r}}_i = \mathbf{f}_i(t, \mathbf{r}_1, \ldots, \mathbf{r}_n, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_n), \qquad 1 \leqslant i \leqslant n,$$

can be rewritten as

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i(t, \mathbf{r}, \dot{\mathbf{r}}), \qquad 1 \leqslant i \leqslant n.$$

The vector $\mathbf{F}_i = m_i \mathbf{f}_i$ is called the *force* acting on the point m_i. "The word *force* does not occur in the principles of Dynamics, as we have just presented it. One can, in effect, bypass it."[2] The last equations are also called Newton's equations.

c) As established by Newton (in development of earlier ideas of Kepler), if there are n material points $(\mathbf{r}_1, m_1), \ldots, (\mathbf{r}_n, m_n)$ in space, then the ith point is acted upon by the force $\mathbf{F}_i = \sum_{i \neq j} \mathbf{F}_{ij}$, where

$$\mathbf{F}_{kl} = -\frac{\gamma m_k m_l}{|\mathbf{r}_{kl}|^3} \mathbf{r}_{kl}, \qquad \mathbf{r}_{kl} = \mathbf{r}_l - \mathbf{r}_k, \qquad \gamma = \text{const} > 0.$$

This is the *law of universal gravitation.*

d) When a body is moving fast through the air, the resistance force is proportional to the square of the velocity (Stokes' law). Hence the equation of a body falling in the air has the form $m\ddot{z} = mg - c\dot{z}^2$, $c > 0$. It turns out that there always exists the limit $\lim\limits_{t \to \infty} v(t)$ equal to $\sqrt{mg/c}$ and independent of the initial state.

When a body moves slowly in a resisting medium, the friction force is a linear function of the velocity. The idea of approximating the resistance force by the formula

$$F = -\alpha v - cv^2, \qquad \alpha, c = \text{const} > 0,$$

goes back to Huygens; this formula takes into account both limiting cases. The vertical fall of a heavy body is described by the equation

$$m\ddot{z} = mg - \alpha\dot{z} - c\dot{z}^2.$$

It is easy to show that

$$\lim_{t \to \infty} v(t) = \frac{\sqrt{\alpha^2 + 4mgc} - \alpha}{2c}.$$

For $\alpha > 0$ this quantity is clearly less than $\sqrt{mg/c}$.

---

[2] Appell ([5], p. 94). In Newton's time the word "force" (vis in Latin) was used for various objects, for example, the acceleration of a point. Leibnitz called the product of the mass of a point and the square of its velocity vis viva (live force). The modern term "force" corresponds to Newton's vis motrix (accelerating force).

Suppose that a material point $(\mathbf{r}, m)$ is moving under the action of a force $\mathbf{F}$. Let

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z, \qquad \mathbf{F} = X\mathbf{e}_x + Y\mathbf{e}_y + Z\mathbf{e}_z,$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ is a fixed orthonormal frame of reference. Then Newton's equation $m\ddot{\mathbf{r}} = \mathbf{F}$ is equivalent to the three scalar equations

$$m\ddot{x} = X, \qquad m\ddot{y} = Y, \qquad m\ddot{z} = Z.$$

This self-evident trick, which was suggested by Maclaurin for describing the motion of a point in three-dimensional space, was not evident to the classics. Before Maclaurin the so-called *natural equations of motion* were usually used.

Let s be the natural parameter along the trajectory of motion of the point. The trajectory is given by the correspondence $s \mapsto \mathbf{r}(s)$. The unit vector $\boldsymbol{\tau} = \mathbf{r}'$ (prime denotes differentiation with respect to the natural parameter) is tangent to the trajectory. The vector

$$\frac{\mathbf{r}''}{|\mathbf{r}''|} = \boldsymbol{\nu}$$

defines the normal, and the vector $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$, the binormal, to the trajectory. The vectors $\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}$ are functions of s. Their evolution is described by the *Frenet formulae*, which are well-known in geometry:

$$\begin{aligned} \boldsymbol{\tau}' &= \qquad k\boldsymbol{\nu} \\ \boldsymbol{\nu}' &= -k\boldsymbol{\tau} \qquad\qquad + \varkappa\boldsymbol{\beta} \\ \boldsymbol{\beta}' &= \qquad - \varkappa\boldsymbol{\nu}. \end{aligned}$$

The quantities k and $\varkappa$ depend on the point of the trajectory; they are called the curvature and the torsion of the trajectory at this point. The motion of the point $\mathbf{r}\colon \Delta \to E^3$ can be represented as the composition $t \mapsto \mathbf{r}(s(t))$. Then $\mathbf{v} = \mathbf{r}'\dot{s}$ and $\mathbf{a} = \mathbf{r}''\dot{s}^2 + \mathbf{r}'\ddot{s}$. Since $\mathbf{r}' = \boldsymbol{\tau}$ and $\mathbf{r}'' = \boldsymbol{\tau}' = k\boldsymbol{\nu}$ (Frenet formula), we have

$$\mathbf{a} = \ddot{s}\boldsymbol{\tau} + k\dot{s}^2\boldsymbol{\nu}.$$

This formula was essentially known already to Huygens. Multiplying it by m and setting $\mathbf{F} = F_\tau\boldsymbol{\tau} + F_\nu\boldsymbol{\nu} + F_\beta\boldsymbol{\beta}$ we arrive at the natural equations of motion

$$m\ddot{s} = F_\tau, \qquad mk\dot{s}^2 = F_\nu, \qquad F_\beta = 0. \tag{1.1}$$

Since s is the arc length, $\dot{s} = v$ is the speed of motion of the point. Then the first two equations (1.1) are usually written in the form

$$m\dot{v} = F_\tau, \qquad \frac{mv^2}{\rho} = F_\nu, \tag{1.2}$$

where $\rho = k^{-1}$ is the radius of curvature of the trajectory.

We now consider some more examples of application of Newton's equation.

e) It is known [4] that a charge e placed in an electro-magnetic field is acted upon by the force

$$\mathbf{F} = e\left(\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H})\right),$$

where $\mathbf{E}$, $\mathbf{H}$ are the strengths of the electric and magnetic fields (they satisfy the Maxwell system of equations) and c is the speed of light. This force is called the Lorentz force.

Consider a special case of motion where the electric field is absent. Then the Lorentz force is orthogonal to the velocity of the charge and therefore $F_\tau = 0$ in equations (1.2). Consequently, the charge is moving with constant speed.

Suppose in addition that the magnetic field is homogeneous ($\mathbf{H} = \text{const}$), and at the initial instant the velocity of the charge is orthogonal to the magnetic force lines. Then, as can be easily seen, the trajectory of the charge is a planar curve orthogonal to $\mathbf{H}$. Since

$$|F_\nu| = \frac{e|\mathbf{v}|H}{c}, \qquad \text{where } H = |\mathbf{H}|,$$

it follows from the second of equations (1.2) that the charge is moving along a circle of radius

$$\rho = \frac{mvc}{eH}.$$

This quantity is called the *Larmor radius.*

More interesting is the problem of motion of a charge in the field of a magnetic pole, which was considered by Poincaré. If $\mathbf{E} = 0$, then the magnetic field is stationary and satisfies the Maxwell equations

$$\operatorname{curl} \mathbf{H} = 0, \qquad \operatorname{div} \mathbf{H} = 0.$$

It follows from the first equation that $\mathbf{H}$ is locally conservative ($\mathbf{H} = \operatorname{grad} U$), and the second equation shows that the potential is a harmonic function ($\Delta U = 0$, where Δ is the Laplace operator). Poincaré considered the only potential depending only on the distance:

$$U = \frac{k}{|\mathbf{r}|}, \qquad k = \text{const}.$$

In this case,

$$\mathbf{H} = -\frac{k\mathbf{r}}{|r|^3}$$

and therefore the equation of motion of the charge has the form

$$\mu\ddot{\mathbf{r}} = \frac{\mathbf{r} \times \dot{\mathbf{r}}}{|\mathbf{r}^3|}, \qquad \mu = \frac{mc}{ek}.$$

It is equivalent to the following relation:

$$\mu(\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{\mathbf{r}}{|\mathbf{r}|} + \mathbf{a}, \qquad \mathbf{a} = \text{const}.$$

Consequently,

$$(\mathbf{a}, \mathbf{r}) = |\mathbf{r}|. \tag{1.3}$$

This is the equation of a cone of revolution whose symmetry axis is parallel to the vector $\mathbf{a}$. We demonstrate that the charged particle moves along the geodesics on this cone. Indeed, $\mathbf{r}$ and $\dot{\mathbf{r}}$ are tangent to the cone (1.3). Consequently, the acceleration vector is orthogonal to this cone. Since the speed of motion is constant, by Huygens' formula the normal to the trajectory coincides with the normal to the cone. Therefore the trajectories are geodesics.

This result of Poincaré explains the phenomenon of cathode rays being drawn in by a magnetic pole discovered in 1895 by Birkeland [501].

f) We consider in addition the problem of external ballistics: a material point $(\mathbf{r}, m)$ is moving along a curvilinear orbit near the surface of the Earth experiencing the air resistance. We assume that the resistance force $\mathbf{F}$ has opposite direction to the velocity and its magnitude can be represented in the form

$$|\mathbf{F}| = mg\varphi(v),$$

where φ is a monotonically increasing function such that $\varphi(0) = 0$ and $\varphi(v) \to +\infty$ as $v \to +\infty$.

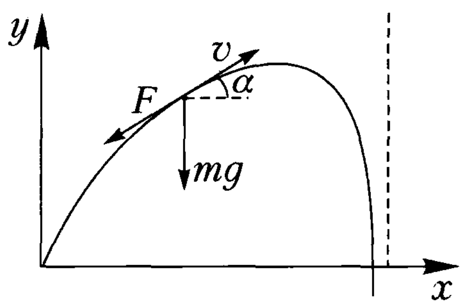

Fig. 1.3. Ballistic trajectory

Since at every moment of time the vectors of the velocity of the point, its weight, and the resistance force lie in the same vertical plane, the trajectory of the point is a planar curve. In the plane of the orbit we introduce Cartesian coordinates x, y such that the y-axis is directed vertically upwards. Let α be the angle between the velocity of the point $\mathbf{v}$ and the horizon (Fig. 1.3). The first of equations (1.2) gives the relation

$$\dot{v} = -g\big[\sin\alpha + \varphi(v)\big]. \tag{1.4}$$