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R. V. Gamkrelidze (Ed.)

Geometry I

Basic Ideas and Concepts of Differential Geometry

几何 I

微分几何基本思想与概念

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D.V. Alekseevskij, A.M. Vinogradov, V.V. Lychagin

Translated from the Russian
by E. Primrose

Contents

Preface	8
Chapter 1. Introduction: A Metamathematical View of Differential Geometry	9
§ 1. Algebra and Geometry – the Duality of the Intellect.	9
§ 2. Two Examples: Algebraic Geometry, Propositional Logic and Set Theory	11
§ 3. On the History of Geometry	14
§ 4. Differential Calculus and Commutative Algebra	18
§ 5. What is Differential Geometry?	22
Chapter 2. The Geometry of Surfaces	25
§ 1. Curves in Euclidean Space	25
1.1. Curves	25
1.2. The Natural Parametrization and the Intrinsic Geometry of Curves	25
1.3. Curvature. The Frenet Frame.	26
1.4. Affine and Unimodular Properties of Curves	27
§ 2. Surfaces in E^3	28
2.1. Surfaces. Charts	29
2.2. The First Quadratic Form. The Intrinsic Geometry of a Surface	29
2.3. The Second Quadratic Form. The Extrinsic Geometry of a Surface	30
2.4. Derivation Formulae. The First and Second Quadratic Forms	32
2.5. The Geodesic Curvature of Curves. Geodesics.	32

2.6.	Parallel Transport of Tangent Vectors on a Surface. Covariant Differentiation. Connection	33
2.7.	Deficiencies of Loops, the "Theorema Egregium" of Gauss and the Gauss-Bonnet Formula	35
2.8.	The Link Between the First and Second Quadratic Forms. The Gauss Equation and the Peterson-Mainardi-Codazzi Equations	37
2.9.	The Moving Frame Method in the Theory of Surfaces	38
2.10.	A Complete System of Invariants of a Surface	39
§3.	Multidimensional Surfaces	40
3.1.	n -Dimensional Surfaces in E^{n+p}	40
3.2.	Covariant Differentiation and the Second Quadratic Form	41
3.3.	Normal Connection on a Surface. The Derivation Formulae	42
3.4.	The Multidimensional Version of the Gauss-Peterson- Mainardi-Codazzi Equations. Ricci's Theorem	43
3.5.	The Geometrical Meaning and Algebraic Properties of the Curvature Tensor	45
3.6.	Hypersurfaces. Mean Curvatures. The Formulae of Steiner and Weyl	47
3.7.	Rigidity of Multidimensional Surfaces	48
Chapter 3.	The Field Approach of Riemann	50
§1.	From the Intrinsic Geometry of Gauss to Riemannian Geometry	50
1.1.	The Essence of Riemann's Approach	50
1.2.	Intrinsic Description of Surfaces	51
1.3.	The Field Point of View on Geometry	51
1.4.	Two Examples	52
§2.	Manifolds and Bundles (the Basic Concepts)	54
2.1.	Why Do We Need Manifolds?	54
2.2.	Definition of a Manifold	55
2.3.	The Category of Smooth Manifolds	57
2.4.	Smooth Bundles	58
§3.	Tensor Fields and Differential Forms	60
3.1.	Tangent Vectors	60
3.2.	The Tangent Bundle and Vector Fields	61
3.3.	Covectors, the Cotangent Bundle and Differential Forms of the First Degree	63
3.4.	Tensors and Tensor Fields	65
3.5.	The Behaviour of Tensor Fields Under Maps. The Lie Derivative	69
3.6.	The Exterior Differential. The de Rham Complex	70
§4.	Riemannian Manifolds and Manifolds with a Linear Connection	71
4.1.	Riemannian Metric	71
4.2.	Construction of Riemannian Metrics	71
4.3.	Linear Connections	72
4.4.	Normal Coordinates	75

4.5.	A Riemannian Manifold as a Metric Space. Completeness.	76
4.6.	Curvature	77
4.7.	The Algebraic Structure of the Curvature Tensor. The Ricci and Weyl Tensors and Scalar Curvature.	79
4.8.	Sectional Curvature. Spaces of Constant Curvature	81
4.9.	The Holonomy Group and the de Rham Decomposition.	82
4.10.	The Berger Classification of Holonomy Groups. Kähler and Quaternion Manifolds.	83
§5.	The Geometry of Symbols	85
5.1.	Differential Operators in Bundles	85
5.2.	Symbols of Differential Operators	86
5.3.	Connections and Quantization.	87
5.4.	Poisson Brackets and Hamiltonian Formalism	88
5.5.	Poissonian and Symplectic Structures.	89
5.6.	Left-Invariant Hamiltonian Formalism on Lie Groups	89
Chapter 4.	The Group Approach of Lie and Klein. The Geometry of Transformation Groups	92
§1.	Symmetries in Geometry.	92
1.1.	Symmetries and Groups	92
1.2.	Symmetry and Integrability	93
1.3.	Klein's Erlangen Programme	94
§2.	Homogeneous Spaces	95
2.1.	Lie Groups	96
2.2.	The Action of the Lie Group on a Manifold.	96
2.3.	Correspondence Between Lie Groups and Lie Algebras	97
2.4.	Infinitesimal Description of Homogeneous Spaces	98
2.5.	The Isotropy Representation. Order of a Homogeneous Space	99
2.6.	The Principle of Extension. Invariant Tensor Fields on Homogeneous Spaces	99
2.7.	Primitive and Imprimitive Actions.	100
§3.	Invariant Connections on a Homogeneous Space	101
3.1.	A General Description.	101
3.2.	Reductive Homogeneous Spaces	102
3.3.	Affine Symmetric Spaces	104
§4.	Homogeneous Riemannian Manifolds	106
4.1.	Infinitesimal Description.	106
4.2.	The Link Between Curvature and the Structure of the Group of Motions	107
4.3.	Naturally Reductive Spaces	107
4.4.	Symmetric Riemannian Spaces.	108
4.5.	Holonomy Groups of Homogeneous Riemannian Manifolds. Kählerian and Quaternion Homogeneous Spaces	110
§5.	Homogeneous Symplectic Manifolds	111
5.1.	Motivation and Definitions	111
5.2.	Examples.	111

5.3. Homogeneous Hamiltonian Manifolds	112
5.4. Homogeneous Symplectic Manifolds and Affine Actions	112
Chapter 5. The Geometry of Differential Equations	114
§1. Elementary Geometry of a First-Order Differential Equation	114
1.1. Ordinary Differential Equations	115
1.2. The General Case.	116
1.3. Geometrical Integration	117
§2. Contact Geometry and Lie's Theory of First-Order Equations	118
2.1. Contact Structure on J^1	118
2.2. Generalized Solutions and Integral Manifolds of the Contact Structure	119
2.3. Contact Transformations	121
2.4. Contact Vector Fields	122
2.5. The Cauchy Problem.	123
2.6. Symmetries. Local Equivalence	124
§3. The Geometry of Distributions	125
3.1. Distributions.	126
3.2. A Distribution of Codimension 1. The Theorem of Darboux.	128
3.3. Involutiv Systems of Equations	130
3.4. The Intrinsic and Extrinsic Geometry of First-Order Differential Equations	131
§4. Spaces of Jets and Differential Equations	132
4.1. Jets.	132
4.2. The Cartan Distribution	133
4.3. Lie Transformations	135
4.4. Intrinsic and Extrinsic Geometries.	136
§5. The Theory of Compatibility and Formal Integrability	137
5.1. Prolongations of Differential Equations	137
5.2. Formal Integrability	138
5.3. Symbols.	138
5.4. The Spencer δ -Cohomology	140
5.5. Involutivity.	141
§6. Cartan's Theory of Systems in Involution	142
6.1. Polar Systems, Characters and Genres	142
6.2. Involutivity and Cartan's Existence Theorems.	144
§7. The Geometry of Infinitely Prolonged Equations	145
7.1. What is a Differential Equation?	145
7.2. Infinitely Prolonged Equations	146
7.3. C -Maps and Higher Symmetries	147
Chapter 6. Geometric Structures	149
§1. Geometric Quantities and Geometric Structures.	149
1.1. What is a Geometric Quantity?	149
1.2. Bundles of Frames and Coframes	149
1.3. Geometric Quantities (Structures) as Equivariant Functions on the Manifold of Coframes	150

1.4. Examples. Infinitesimally Homogeneous Geometric Structures and G -Structures	151
1.5. Natural Geometric Structures and the Principle of Covariance . . .	153
§ 2. Principal Bundles.	154
2.1. Principal Bundles.	154
2.2. Examples of Principal Bundles.	155
2.3. Homomorphisms and Reductions	155
2.4. G -Structures as Principal Bundles	156
2.5. Generalized G -Structures	157
2.6. Associated Bundles	158
§ 3. Connections in Principal Bundles and Vector Bundles.	159
3.1. Connections in a Principal Bundle.	159
3.2. Infinitesimal Description of Connections	161
3.3. Curvature and the Holonomy Group	162
3.4. The Holonomy Group.	162
3.5. Covariant Differentiation and the Structure Equations.	163
3.6. Connections in Associated Bundles	164
3.7. The Yang-Mills Equations	166
§ 4. Bundles of Jets	167
4.1. Jets of Submanifolds	167
4.2. Jets of Sections	169
4.3. Jets of Maps	169
4.4. The Differential Group	170
4.5. Affine Structures.	171
4.6. Differential Equations and Differential Operators.	171
4.7. Spencer Complexes	172
Chapter 7. The Equivalence Problem, Differential Invariants and Pseudogroups.	174
§ 1. The Equivalence Problem. A General View	174
1.1. The Problem of Recognition (Equivalence).	174
1.2. The Problem of Triviality	175
1.3. The Equivalence Problem in Differential Geometry	176
1.4. Scalar and Non-Scalar Differential Invariants	177
1.5. Differential Invariants in Physics.	177
§ 2. The General Equivalence Problem in Riemannian Geometry	178
2.1. Preparatory Remarks	178
2.2. Two-Dimensional Riemannian Manifolds	178
2.3. Multidimensional Riemannian Manifolds.	179
§ 3. The General Equivalence Problem for Geometric Structures.	180
3.1. Statement of the Problem	180
3.2. Flat Geometry Structures and the Problem of Triviality	181
3.3. Homogeneous and Non-Homogeneous Equivalence Problems. . .	181
§ 4. Differential Invariants of Geometric Structures and the Equivalence Problem	182
4.1. Differential Invariants	182

4.2. Calculation of Differential Invariants	183
4.3. The Principle of n Invariants	184
4.4. Non-General Structures and Symmetries	184
§ 5. The Equivalence Problem for G -Structures	185
5.1. Three Examples	185
5.2. Structure Functions and Prolongations	186
5.3. Formal Integrability	188
5.4. G -Structures and Differential Invariants	189
§ 6. Pseudogroups, Lie Equations and Their Differential Invariants	189
6.1. Lie Pseudogroups	190
6.2. Lie Equations	190
6.3. Linear Lie Equations	191
6.4. Differential Invariants of Lie Pseudogroups	192
6.5. On the Structure of the Algebra of Differential Invariants	193
§ 7. On the Structure of Lie Pseudogroups	193
7.1. Representation of Isotropy	193
7.2. Examples of Transitive Pseudogroups	194
7.3. Cartan's Classification	194
7.4. The Jordan-Hölder-Guillemin Decomposition	195
7.5. Pseudogroups of Finite Type	195
Chapter 8. Global Aspects of Differential Geometry	197
§ 1. The Four Vertices Theorem	197
§ 2. Carathéodory's Problem About Umbilics	198
§ 3. Geodesics on Oval Surfaces	199
§ 4. Rigidity of Oval Surfaces	200
§ 5. Realization of 2-Dimensional Metrics of Positive Curvature (A Problem of H. Weyl)	201
§ 6. Non-Realizability of the Lobachevskij Plane in \mathbb{R}^3 and a Theorem of N.V. Efimov	202
§ 7. Isometric Embeddings in Euclidean Spaces	203
§ 8. Minimal Surfaces. Plateau's Problem	206
§ 9. Minimal Surfaces. Bernstein's Problem	208
§ 10. de Rham Cohomology	209
§ 11. Harmonic Forms. Hodge Theory	211
§ 12. Application of the Maximum Principle	214
§ 13. Curvature and Topology	216
§ 14. Morse Theory	219
§ 15. Curvature and Characteristic Classes	223
15.1. Bordisms and Stokes's Formula	223
15.2. The Generalized Gauss-Bonnet Formula	226
15.3. Weil's Homomorphism	227
15.4. Characteristic Classes	228
15.5. Characteristic Classes and the Gaussian Map	228
§ 16. The Global Geometry of Elliptic Operators	229
16.1. The Euler Characteristic as an Index	229

16.2. The Chern Character and the Todd Class	230
16.3. The Atiyah-Singer Index Theorem	230
16.4. The Index Theorem and the Riemann-Roch-Hirzebruch Theorem.....	231
16.5. The Dolbeault Cohomology of Complex Manifolds.....	231
16.6. The Riemann-Roch-Hirzebruch Theorem	233
§ 17. The Space of Geometric Structures and Deformations	234
17.1. The Moduli Space of Geometric Structures.....	234
17.2. Examples	235
17.3. Deformation and Supersymmetries.....	237
17.4. Lie Superalgebras	237
17.5. The Space of Infinitesimal Deformations of a Lie Algebra. Rigidity Conditions.....	239
17.6. Deformations and Rigidity of Complex Structures.....	240
§ 18. Minkowski's Problem, Calabi's Conjecture and the Monge-Ampère Equations	241
§ 19. Spectral Geometry.....	244
Commentary on the References.....	248
References	249
Author Index.....	257
Subject Index.....	259

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Preface	8
Chapter 1. Introduction: A Metamathematical View of Differential Geometry	9
§ 1. Algebra and Geometry – the Duality of the Intellect.	9
§ 2. Two Examples: Algebraic Geometry, Propositional Logic and Set Theory	11
§ 3. On the History of Geometry	14
§ 4. Differential Calculus and Commutative Algebra	18
§ 5. What is Differential Geometry?	22
Chapter 2. The Geometry of Surfaces	25
§ 1. Curves in Euclidean Space	25
1.1. Curves	25
1.2. The Natural Parametrization and the Intrinsic Geometry of Curves	25
1.3. Curvature. The Frenet Frame.	26
1.4. Affine and Unimodular Properties of Curves	27
§ 2. Surfaces in E^3	28
2.1. Surfaces. Charts	29
2.2. The First Quadratic Form. The Intrinsic Geometry of a Surface	29
2.3. The Second Quadratic Form. The Extrinsic Geometry of a Surface	30
2.4. Derivation Formulae. The First and Second Quadratic Forms	32
2.5. The Geodesic Curvature of Curves. Geodesics	32

2.6.	Parallel Transport of Tangent Vectors on a Surface. Covariant Differentiation. Connection	33
2.7.	Deficiencies of Loops, the "Theorema Egregium" of Gauss and the Gauss-Bonnet Formula	35
2.8.	The Link Between the First and Second Quadratic Forms. The Gauss Equation and the Peterson-Mainardi-Codazzi Equations	37
2.9.	The Moving Frame Method in the Theory of Surfaces.....	38
2.10.	A Complete System of Invariants of a Surface	39
§ 3.	Multidimensional Surfaces	40
3.1.	n -Dimensional Surfaces in E^{n+p}	40
3.2.	Covariant Differentiation and the Second Quadratic Form	41
3.3.	Normal Connection on a Surface. The Derivation Formulae.....	42
3.4.	The Multidimensional Version of the Gauss-Peterson- Mainardi-Codazzi Equations. Ricci's Theorem	43
3.5.	The Geometrical Meaning and Algebraic Properties of the Curvature Tensor.....	45
3.6.	Hypersurfaces. Mean Curvatures. The Formulae of Steiner and Weyl.....	47
3.7.	Rigidity of Multidimensional Surfaces.....	48
Chapter 3.	The Field Approach of Riemann.....	50
§ 1.	From the Intrinsic Geometry of Gauss to Riemannian Geometry.....	50
1.1.	The Essence of Riemann's Approach.....	50
1.2.	Intrinsic Description of Surfaces	51
1.3.	The Field Point of View on Geometry	51
1.4.	Two Examples	52
§ 2.	Manifolds and Bundles (the Basic Concepts).....	54
2.1.	Why Do We Need Manifolds?.....	54
2.2.	Definition of a Manifold	55
2.3.	The Category of Smooth Manifolds	57
2.4.	Smooth Bundles.....	58
§ 3.	Tensor Fields and Differential Forms	60
3.1.	Tangent Vectors.....	60
3.2.	The Tangent Bundle and Vector Fields.....	61
3.3.	Covectors, the Cotangent Bundle and Differential Forms of the First Degree	63
3.4.	Tensors and Tensor Fields	65
3.5.	The Behaviour of Tensor Fields Under Maps. The Lie Derivative.....	69
3.6.	The Exterior Differential. The de Rham Complex	70
§ 4.	Riemannian Manifolds and Manifolds with a Linear Connection.....	71
4.1.	Riemannian Metric	71
4.2.	Construction of Riemannian Metrics	71
4.3.	Linear Connections	72
4.4.	Normal Coordinates	75

4.5.	A Riemannian Manifold as a Metric Space. Completeness.	76
4.6.	Curvature	77
4.7.	The Algebraic Structure of the Curvature Tensor. The Ricci and Weyl Tensors and Scalar Curvature.	79
4.8.	Sectional Curvature. Spaces of Constant Curvature	81
4.9.	The Holonomy Group and the de Rham Decomposition.	82
4.10.	The Berger Classification of Holonomy Groups. Kähler and Quaternion Manifolds.	83
§ 5.	The Geometry of Symbols	85
5.1.	Differential Operators in Bundles	85
5.2.	Symbols of Differential Operators	86
5.3.	Connections and Quantization.	87
5.4.	Poisson Brackets and Hamiltonian Formalism	88
5.5.	Poissonian and Symplectic Structures.	89
5.6.	Left-Invariant Hamiltonian Formalism on Lie Groups	89
Chapter 4.	The Group Approach of Lie and Klein. The Geometry of Transformation Groups	92
§ 1.	Symmetries in Geometry.	92
1.1.	Symmetries and Groups	92
1.2.	Symmetry and Integrability	93
1.3.	Klein's Erlangen Programme	94
§ 2.	Homogeneous Spaces	95
2.1.	Lie Groups	96
2.2.	The Action of the Lie Group on a Manifold.	96
2.3.	Correspondence Between Lie Groups and Lie Algebras.	97
2.4.	Infinitesimal Description of Homogeneous Spaces	98
2.5.	The Isotropy Representation. Order of a Homogeneous Space	99
2.6.	The Principle of Extension. Invariant Tensor Fields on Homogeneous Spaces	99
2.7.	Primitive and Imprimitive Actions.	100
§ 3.	Invariant Connections on a Homogeneous Space	101
3.1.	A General Description.	101
3.2.	Reductive Homogeneous Spaces	102
3.3.	Affine Symmetric Spaces	104
§ 4.	Homogeneous Riemannian Manifolds	106
4.1.	Infinitesimal Description.	106
4.2.	The Link Between Curvature and the Structure of the Group of Motions	107
4.3.	Naturally Reductive Spaces	107
4.4.	Symmetric Riemannian Spaces.	108
4.5.	Holonomy Groups of Homogeneous Riemannian Manifolds. Kählerian and Quaternion Homogeneous Spaces	110
§ 5.	Homogeneous Symplectic Manifolds	111
5.1.	Motivation and Definitions	111
5.2.	Examples.	111

5.3. Homogeneous Hamiltonian Manifolds	112
5.4. Homogeneous Symplectic Manifolds and Affine Actions	112
Chapter 5. The Geometry of Differential Equations	114
§ 1. Elementary Geometry of a First-Order Differential Equation	114
1.1. Ordinary Differential Equations	115
1.2. The General Case.	116
1.3. Geometrical Integration	117
§ 2. Contact Geometry and Lie's Theory of First-Order Equations	118
2.1. Contact Structure on J^1	118
2.2. Generalized Solutions and Integral Manifolds of the Contact Structure	119
2.3. Contact Transformations	121
2.4. Contact Vector Fields	122
2.5. The Cauchy Problem.	123
2.6. Symmetries. Local Equivalence	124
§ 3. The Geometry of Distributions	125
3.1. Distributions.	126
3.2. A Distribution of Codimension 1. The Theorem of Darboux.	128
3.3. Involutive Systems of Equations	130
3.4. The Intrinsic and Extrinsic Geometry of First-Order Differential Equations	131
§ 4. Spaces of Jets and Differential Equations	132
4.1. Jets.	132
4.2. The Cartan Distribution	133
4.3. Lie Transformations	135
4.4. Intrinsic and Extrinsic Geometries.	136
§ 5. The Theory of Compatibility and Formal Integrability	137
5.1. Prolongations of Differential Equations	137
5.2. Formal Integrability	138
5.3. Symbols.	138
5.4. The Spencer δ -Cohomology	140
5.5. Involutivity	141
§ 6. Cartan's Theory of Systems in Involution	142
6.1. Polar Systems, Characters and Genres	142
6.2. Involutivity and Cartan's Existence Theorems.	144
§ 7. The Geometry of Infinitely Prolonged Equations	145
7.1. What is a Differential Equation?	145
7.2. Infinitely Prolonged Equations	146
7.3. C -Maps and Higher Symmetries	147
Chapter 6. Geometric Structures	149
§ 1. Geometric Quantities and Geometric Structures	149
1.1. What is a Geometric Quantity?	149
1.2. Bundles of Frames and Coframes	149
1.3. Geometric Quantities (Structures) as Equivariant Functions on the Manifold of Coframes	150

1.4. Examples. Infinitesimally Homogeneous Geometric Structures and G -Structures	151
1.5. Natural Geometric Structures and the Principle of Covariance.	153
§2. Principal Bundles.	154
2.1. Principal Bundles.	154
2.2. Examples of Principal Bundles.	155
2.3. Homomorphisms and Reductions	155
2.4. G -Structures as Principal Bundles	156
2.5. Generalized G -Structures	157
2.6. Associated Bundles	158
§3. Connections in Principal Bundles and Vector Bundles.	159
3.1. Connections in a Principal Bundle.	159
3.2. Infinitesimal Description of Connections	161
3.3. Curvature and the Holonomy Group	162
3.4. The Holonomy Group.	162
3.5. Covariant Differentiation and the Structure Equations.	163
3.6. Connections in Associated Bundles	164
3.7. The Yang-Mills Equations	166
§4. Bundles of Jets	167
4.1. Jets of Submanifolds	167
4.2. Jets of Sections	169
4.3. Jets of Maps	169
4.4. The Differential Group	170
4.5. Affine Structures.	171
4.6. Differential Equations and Differential Operators.	171
4.7. Spencer Complexes	172
Chapter 7. The Equivalence Problem, Differential Invariants and Pseudogroups.	174
§1. The Equivalence Problem. A General View	174
1.1. The Problem of Recognition (Equivalence).	174
1.2. The Problem of Triviality	175
1.3. The Equivalence Problem in Differential Geometry	176
1.4. Scalar and Non-Scalar Differential Invariants	177
1.5. Differential Invariants in Physics	177
§2. The General Equivalence Problem in Riemannian Geometry	178
2.1. Preparatory Remarks	178
2.2. Two-Dimensional Riemannian Manifolds	178
2.3. Multidimensional Riemannian Manifolds.	179
§3. The General Equivalence Problem for Geometric Structures.	180
3.1. Statement of the Problem	180
3.2. Flat Geometry Structures and the Problem of Triviality	181
3.3. Homogeneous and Non-Homogeneous Equivalence Problems.	181
§4. Differential Invariants of Geometric Structures and the Equivalence Problem	182
4.1. Differential Invariants	182

4.2. Calculation of Differential Invariants	183
4.3. The Principle of n Invariants	184
4.4. Non-General Structures and Symmetries	184
§ 5. The Equivalence Problem for G -Structures	185
5.1. Three Examples	185
5.2. Structure Functions and Prolongations	186
5.3. Formal Integrability	188
5.4. G -Structures and Differential Invariants	189
§ 6. Pseudogroups, Lie Equations and Their Differential Invariants	189
6.1. Lie Pseudogroups	190
6.2. Lie Equations	190
6.3. Linear Lie Equations	191
6.4. Differential Invariants of Lie Pseudogroups	192
6.5. On the Structure of the Algebra of Differential Invariants	193
§ 7. On the Structure of Lie Pseudogroups	193
7.1. Representation of Isotropy	193
7.2. Examples of Transitive Pseudogroups	194
7.3. Cartan's Classification	194
7.4. The Jordan-Hölder-Guillemín Decomposition	195
7.5. Pseudogroups of Finite Type	195
Chapter 8. Global Aspects of Differential Geometry	197
§ 1. The Four Vertices Theorem	197
§ 2. Carathéodory's Problem About Umbilics	198
§ 3. Geodesics on Oval Surfaces	199
§ 4. Rigidity of Oval Surfaces	200
§ 5. Realization of 2-Dimensional Metrics of Positive Curvature (A Problem of H. Weyl)	201
§ 6. Non-Realizability of the Lobachevskij Plane in \mathbb{R}^3 and a Theorem of N.V. Efimov	202
§ 7. Isometric Embeddings in Euclidean Spaces	203
§ 8. Minimal Surfaces. Plateau's Problem	206
§ 9. Minimal Surfaces. Bernstein's Problem	208
§ 10. de Rham Cohomology	209
§ 11. Harmonic Forms. Hodge Theory	211
§ 12. Application of the Maximum Principle	214
§ 13. Curvature and Topology	216
§ 14. Morse Theory	219
§ 15. Curvature and Characteristic Classes	223
15.1. Bordisms and Stokes's Formula	223
15.2. The Generalized Gauss-Bonnet Formula	226
15.3. Weil's Homomorphism	227
15.4. Characteristic Classes	228
15.5. Characteristic Classes and the Gaussian Map	228
§ 16. The Global Geometry of Elliptic Operators	229
16.1. The Euler Characteristic as an Index	229

16.2. The Chern Character and the Todd Class	230
16.3. The Atiyah-Singer Index Theorem	230
16.4. The Index Theorem and the Riemann-Roch-Hirzebruch Theorem.	231
16.5. The Dolbeault Cohomology of Complex Manifolds.	231
16.6. The Riemann-Roch-Hirzebruch Theorem	233
§ 17. The Space of Geometric Structures and Deformations.	234
17.1. The Moduli Space of Geometric Structures.	234
17.2. Examples	235
17.3. Deformation and Supersymmetries.	237
17.4. Lie Superalgebras	237
17.5. The Space of Infinitesimal Deformations of a Lie Algebra. Rigidity Conditions.	239
17.6. Deformations and Rigidity of Complex Structures	240
§ 18. Minkowski's Problem, Calabi's Conjecture and the Monge-Ampère Equations.	241
§ 19. Spectral Geometry.	244
Commentary on the References.	248
References	249
Author Index.	257
Subject Index.	259