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A. L. Onishchik (Ed.)

Lie Groups and Lie Algebras I

Foundations of Lie Theory, Lie Transformation Groups

李群与李代数 I

李理论基础,李交换群

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I. Foundations of Lie Theory

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Translated from the Russian
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Introduction

The theory of Lie groups, to which this volume is devoted, is one of the classical well established chapters of mathematics. There is a whole series of monographs devoted to it (see, for example, Pontryagin 1984, Postnikov 1982, Bourbaki 1947, Chevalley 1946, Helgason 1962, Sagle and Walde 1973, Serre 1965, Warner 1983). This theory made its first appearance at the end of the last century in the works of S. Lie, whose aim was to apply algebraic methods to the theory of differential equations and to geometry. During the past one hundred years the concepts and methods of the theory of Lie groups entered into many areas of mathematics and theoretical physics and became inseparable from them.

The first three chapters of the present work contain a systematic exposition of the foundations of the theory of Lie groups. We have tried to give here brief proofs of most of the more important theorems. Certain more complex theorems, not used in the text, are stated without proof. Chapter 4 is of a special character: it contains a survey of certain contemporary generalizations of Lie groups.

The authors deliberately have not touched upon structural questions of the theory of Lie groups and algebras, in particular, the theory of semi-simple Lie groups. To these questions will be devoted a separate study in one of the future volumes of this series.

In this entire work Lie groups, as a rule, will be denoted with capital Latin letters, and their tangent algebras with the corresponding small Gothic letters, In addition the following notation will be used:

G^0 – connected component of the identity of a Lie group (or a topological group) G;
$G' = (G, G)$ – the commutator subgroup of a group G; $G^{(p)} = (G^{(p-1)}, G^{p-1)})$;
$\mathrm{Rad}\, G$ – the radical of a Lie group G;
$\mathrm{rad}\, \mathfrak{g}$ – the radical of a Lie algebra $\mathfrak{g}$;
$\rtimes$ – the semidirect product of groups (normal subgroup on the left);
$\oplus$ – the semidirect sum of Lie algebras (ideal on the left);
$\mathbb{T}$ – the group of complex numbers of modulus 1;
exp – the exponential mapping;
Ad – the adjoint representation of a Lie group;
ad – the adjoint representation of a Lie algebra;
$\mathrm{Aut}\, A$ – the group of automorphisms of a group or algebra A;
$\mathrm{Int}\, G$ – the group of inner automorphisms of a group G;
$\mathrm{Der}\, A$ – the Lie algebra of derivations of an algebra A;
$\mathrm{Int}\, \mathfrak{g}$ – the group of inner automorphisms of a Lie algebra $\mathfrak{g}$;
$\mathrm{GL}(V)$ – the group of all automorphisms (invertible linear transformations) of a vector space V;

$L_n(K)$ – the associative algebra of all square matrices of order n over a field K;
$GL_n(K)$ – the group of all non singular matrices of order n over K;
$SL_n(K)$ – the group of all matrices of order n with determinant 1;
$PGL_n(K) = GL(K)/\{\lambda E\}$ – the projective linear group;
$GL_n^+(\mathbb{R})$ – the group of all real matrices of order n with positive determinant;
$O_n(K)$ – the group of all orthogonal matrices of order n over K;
$SO_n(K) = O_n(K) \cap SL_n(K)$;
$Sp_n(K)$ – the group of all symplectic matrices of order n over K (n even):
$O_{k,l}$ – the group of all pseudo-orthogonal real matrices of signature (k, l);
$SO_{k,l} = O_{k,l} \cap SL_n(\mathbb{R})$;
$O'_{k,l}$ – the group of pseudo-orthogonal matrices of signature (k, l) whose minor of order k at the top left corner is positive;
U_n – the group of unitary complex matrices of order n;
$U_{k,l}$ – the group of pseudo-unitary complex matrices of signature (k, l);
$SU_n = U_n \cap SL_n(\mathbb{C})$; $SU_{k,l} = U_{k,l} \cap SL_{k+l}(\mathbb{C})$.

Finally we would like to mention a piece of non-standard terminology: we use the term "the tangent algebra of a Lie group" instead of the usual "the Lie algebra of a Lie group". We do so with a view to emphasise the construction of this Lie algebra as the tangent space to the Lie group. This seems to be appropriate here since, in particular, the tangent algebra of an analytic loop is not, in general, a Lie algebra. We reserve the term "Lie algebra" for its algebraic context.

Chapter 1
Basic Notions

We will assume familiarity with the basic concepts of manifold theory. However in order to avoid misunderstandings some of them will be defined in the text. The basic field, by which we mean either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, will be denoted by K. Unless stated otherwise, differentiability of functions will be understood in the following sense: in every case there exist as many derivatives as are needed. Differentiability of manifolds and maps is understood in the same sense. The Jacobian matrix of a system of differentiable functions $f_1, \dots, f_m$ of variables $x_1, \dots, x_n$ will be denoted by $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_n)}$. For $m = n$ its determinant will be denoted by $\frac{D(f_1,\dots,f_n)}{D(x_1,\dots,x_n)}$.

The tangent space of a manifold X at a point x will be denoted by $T_x(X)$ and the differential of a map $f: X \to Y$ at a point x by $d_x f$. In many cases, when it is clear which point is being considered, the subscript will be omitted in denoting a tangent space or a differential.

All differentiable manifolds will be assumed to possess a countable base of open sets.

§1. Lie Groups, Subgroups and Homomorphisms

1.1. Definition of a Lie Group. A Lie group over the field K is a group G equipped with the structure of a differentiable manifold over K in such a way that the map

$$\mu : G \times G \to G, \ (x, y) \mapsto xy$$

is differentiable. In other words, the coordinates of the product of two elements have to be differentiable functions of the coordinates of the factors.

With the aid of the implicit function theorem it is easy to show that in any Lie group the inverse

$$\iota : G \to G, \ x \mapsto x^{-1}$$

is also a differentiable map. Lie groups over $\mathbb{C}$ are called *complex Lie groups* and Lie groups over $\mathbb{R}$ – *real Lie groups.* Any complex Lie group can be viewed as a real Lie group of twice the dimension.

One can also consider analytic groups by requiring that the manifold G and the map μ be analytic over the field K. Clearly, every complex Lie group is analytic, but even in the real case it turns out that in any Lie group there exists an atlas with analytic transition functions, in which the map μ is expressed in terms of analytic functions (see 3.3 of Chap. 3).

Examples. 1. The additive group of the field K (we will denote it also by K).

2. The multiplicative group $K^\times$ of the field K.

3. 'The circle' $\mathbb{T} = \{z \in \mathbb{C}^\times : |z| = 1\}$ is a real Lie group.

4. The group $\mathrm{GL}_n(K)$ of invertible matrices of order n over the field K, with the differentiable structure of an open subset of the vector space $\mathrm{L}_n(K)$ of all matrices, i.e. (global) coordinates are given by the matrix entries.

5. The group $\mathrm{GL}(V)$ of invertible linear transformations of an n-dimensional vector space over the field K can be regarded as a Lie group in view of the isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}_n(K)$, which assigns to each linear transformation its matrix with respect to some fixed basis.

6. The group $\mathrm{GA}(S)$ of (invertible) affine transformations of an n-dimensional affine space S over the field K possesses also a canonical differentiable structure, which turns it into a Lie group. Namely, with respect to the affine coordinate system of the space S affine transformations can be written in the form $X \mapsto AX + B$, where X is a column vector of coordinates of a point, A an invertible square matrix and B a column vector. The entries of the matrix A and the column vector B can be taken as (global) coordinates in the group $\mathrm{GA}(S)$.

7. Any finite or countable group equipped with the discrete topology and the structure of a 0-dimensional differentiable manifold.

The *direct product of Lie groups* is the direct product of the corresponding abstract groups endowed with the differentiable structure of the direct product of differentiable manifolds.

The Lie group K^n (the direct product of n copies of the additive group of the field K) is called the n-dimensional *vector Lie group*. The Lie group $\mathbb{T}^n$ (the direct product of n copies of the group $\mathbb{T}$) is called the *n-dimensional torus*.

1.2. Lie Subgroups. A subgroup H of a Lie group G is said to be a *Lie subgroup* if it is a submanifold of the underlying manifold of G.

Let us recall that by a m-dimensional submanifold of an n-dimensional manifold X we mean a subset $Y \subset X$ such that for each of its points y one of the following equivalent conditions is satisfied:

(1) in a local coordinate system in some neighbourhood U of the point y the subset $Y \cap U$ can be described parametrically in the form

$$x_i = \phi_i(t_1, \dots, t_m) \qquad (i = 1, \dots, n)$$

where $\phi_1, \dots, \phi_n$ are differentiable functions defined in some domain of the space K^m and the rank of the matrix $\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(t_1, \dots, t_m)}$ at all points of this domain is equal to m.

(2) in a local coordinate system in some neighbourhood U of the point y the set $Y \cap U$ can be given by equations of the form

$$f_i(x_1, \dots, x_n) = 0 \qquad (i = 1, \dots, n - m),$$

where $f_1, \dots, f_{n-m}$ are differentiable functions and the rank of the matrix $\frac{\partial(f_1,\dots,f_{n-m})}{\partial(x_1,\dots,x_n)}$ at all points of the neighbourhood U is $n-m$.

(3) in a suitable local coordinate system in some neighbourhood U of the point y the subset $Y \cap U$ is given by equations

$$x_{m+1} = \dots = x_n = 0.$$

(Sometimes the terms 'submanifold' and correspondingly 'Lie subgroup' are understood in a wider sense. In this book this wider meaning is referred to by the term 'virtual Lie subgroup'; see 2.3 of Chap. 2. Lie subgroups in our sense are also known as 'closed Lie subgroups'.)

Every m-dimensional submanifold of a differentiable manifold carries the structure of a m-dimensional differentiable manifold, as local coordinates on which we can take, for example, the parameters $t_1, \dots, t_m$ from condition (1). Every Lie subgroup, endowed with this differentiable structure is itself a Lie group.

From the topological and the differential geometric viewpoints every subgroup H of a Lie group G looks at any point $h \in H$ the same as at the identity, since it is transformed into itself by a translation (left or right) by h, which is a diffeomorphism of the manifold G. Therefore in order to verify that a subgroup H is a Lie subgroup it suffices to establish that it is a submanifold in some neighbourhood of the identity.

Examples. 1. Any subspace of a vector space is a Lie subgroup of the corresponding Lie group.

2. The group $\mathbb{T}$ (see Example 3 of 1.1) is a Lie subgroup of the group $\mathbb{C}^\times$, viewed as a real Lie group.

3. Any discrete subgroup is a Lie subgroup.

4. The group of non-singular diagonal matrices is a Lie subgroup of the Lie group $\mathrm{GL}_n(K)$.

5. The group of non-singular triangular matrices is a Lie subgroup of the Lie group $\mathrm{GL}_n(K)$.

6. The group $\mathrm{SL}_n(K)$ of unimodular matrices is a codimension 1 Lie subgroup of the Lie group $\mathrm{GL}_n(K)$.

7. The group $\mathrm{O}_n(K)$ of orthogonal matrices is a Lie subgroup of dimension $\frac{n(n-1)}{2}$ of the Lie group $\mathrm{GL}_n(K)$.

8. The group $\mathrm{Sp}_n(K)$ (n even) of symplectic matrices is a Lie subgroup of dimension $\frac{n(n+1)}{2}$ of the Lie group $\mathrm{GL}_n(K)$.

9. The group U_n of unitary matrices is a real Lie subgroup of dimension n^2 of the Lie group $\mathrm{GL}_n(\mathbb{C})$.

A Lie subgroup of the Lie group $\mathrm{GL}_n(V)$ (and in particular of $\mathrm{GL}_n(K) = \mathrm{GL}(K^n)$) is called a *linear Lie group*.

As any submanifold, a Lie subgroup is an open subset of its closure. However, any open subgroup of a topological group is at the same time closed, since it is the complement of the union of its own cosets, which, like the subgroup itself, are open subsets. Hence any Lie subgroup is closed. For real Lie groups the converse is also valid, see Theorem 3.6 of Chap. 2.

1.3. Homomorphisms of Lie Groups. Let G and H be Lie groups. A map $f : G \to H$ is a *homomorphism* if it is simultaneously a homomorphism of abstract groups and a differentiable map. A homomorphism $f : G \to H$ is called an *isomorphism* if there exists an inverse $f^{-1} : H \to G$, i.e. if f is simultaneously an isomorphism of abstract groups and a diffeomorphism of manifolds (however, in connection with this, see the corollary to Theorem 3.4).

Examples. 1. The exponential map $x \mapsto e^x$ is a homomorphism from the additive Lie group K to the Lie group $K^\times$

2. The map $A \mapsto \det A$ is a homomorphism from the Lie group $\mathrm{GL}_n(K)$ to the Lie group $K^\times$

3. For any element g of a Lie group G the inner automorphism $a(g) : x \mapsto gxg^{-1}$ is a Lie group automorphism.

4. The map $x \mapsto e^{ix}$ is a homomorphism from the Lie group $\mathbb{R}$ to the Lie group $\mathbb{T}$.

5. The map assigning to each affine transformation of an affine space its differential (linear part) is a homomorphism from the Lie group $\mathrm{GA}(S)$ (see Example 6 of 1.1) to the Lie group $\mathrm{GL}(V)$, where V is the vector space associated with S.

6. Any homomorphism from a finite or a countable group to a Lie group is a homomorphism in the sense of the theory of Lie groups.

Obviously the composition of homomorphisms of Lie groups is also a homomorphism of Lie groups.

1.4. Linear Representations of Lie Groups. A homomorphism from a Lie group G to the Lie group $\mathrm{GL}(V)$ is called its *linear representation* in the space V.

For example, if to each matrix $A \in \mathrm{GL}_n(K)$ we assign the transformations $\mathrm{Ad}(A)$ and $\mathrm{Sq}(A)$ of the space $\mathrm{L}_n(K)$, defined by the formulas

$$\mathrm{Ad}(A)X = AXA^{-1}, \ \mathrm{Sq}(A)X = AXA^T, \tag{1}$$

then we obtain linear representations Ad and Sq of the Lie group $\mathrm{GL}_n(K)$ in the space $\mathrm{L}_n(K)$.

Sometimes one considers complex linear representations of real Lie groups or real linear representations of complex Lie groups. In the former case, it is understood that the group of linear transformations of a complex vector space is being considered as a real Lie group, in the latter – that the given complex Lie group is being considered as a real one.

Let R and S be linear representations of some group G in spaces V and U respectively. Recall that, by the sum of representations R and S, is meant the linear representation $R + S$ of the group G in the space $V \oplus U$, defined by the formula

$$(R+S)(g)(v+u) = R(g)v + S(g)u \tag{2}$$

by the product of the representations R and S the linear representation RS of the group G in the space $V \otimes U$, defined on decomposable elements by the formula

$$(RS)(g)(u \otimes v) = R(g)v \otimes S(g)u \tag{3}$$

The sum and product of an arbitrary number of representations are defined analogously.

By the dual representation of a representation R we mean the representation R^* of the group G in the space V^* – the dual of V, given by the formula

$$(R^*(g)f)(v) = f(R(g)^{-1}v) \tag{4}$$

It is easy to see that, if R and S are linear representations of a Lie group G, then the representations $R+S$, RS and R^* are also linear representations of it as a Lie group (i.e. they are differentiable).

For any integers $k, l \geq 0$ the identity linear representation Id of the group $\mathrm{GL}(V)$ in the space V generates its linear representation $T_{k,l} = Id^k(Id^*)^l$ in the space $\underbrace{V \otimes \ldots \otimes V}_{k} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{l}$ of tensors of type (k,l) on V. We will give convenient interpretations of representations $T_{k,l}$ in the two most commonly met cases: $k=0$ and $k=1$. Tensors of type $(0,l)$ can be viewed as l-linear forms on V. For any such form f we have

$$(T_{0,l}(A)f)(v_1, \ldots, v_l) = f(A^{-1}v_1 \ldots, A^{-1}v_l) \tag{5}$$

Tensors of type $(1,l)$ can be viewed as l-linear maps $V \times \ldots \times V \to V$. For any such map F we have

$$(T_{1,l}(A)F)(v_1, \ldots, v_l) = AF(A^{-1}v_1 \ldots, A^{-1}v_l) \tag{6}$$

The representations Ad and Sq of the group $\mathrm{GL}_n(K)$ considered above, are just its representations in the spaces of tensors (on K^n) of type $(1,1)$ and $(2,0)$ respectively, expressed in the matrix form.

If R is a linear representation of some group G in a space V and $U \subset V$ is an invariant subspace, there is a natural way to define the subrepresentation $R_U : G \to \mathrm{GL}(U)$ and the quotient representation $R_{V/U} : G \to \mathrm{GL}(V/U)$. Clearly, every subrepresentation and every quotient representation of a linear representation of a Lie group G are linear representations of it as a Lie group.

A special role in group theory is played by one-dimensional representations, which are precisely the homomorphisms from the given group to the multiplicative group of the base field. They are referred to as characters [1] of the group G. Characters form a group with respect to the operation of multiplication of representations; the inverse of an element in this group is its dual representation. We will denote the group of characters of a group G by

[1] Here the word character is being used in its narrower sense. In its wider sense character refers to the trace of any (not necessarily one-dimensional) linear representation.

$\mathcal{X}(G)$. Traditionally additive notation is used to denote its group operation, thus by definition

$$(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g) \quad (\chi_1, \chi_2 \in \mathcal{X}(G)).$$

In the context of the theory of Lie groups characters are assumed to be differentiable.

1.5. Local Lie Groups. In certain situations it turns out to be useful to have a local version of the concept of a Lie group. By a *local Lie group* we mean a differentiable manifold U together with a base point e, its neighbourhood V and a differentiable map (multiplication)

$$\mu : V \times V \to U, \quad (x, y) \mapsto xy$$

satisfying the conditions $ex = xe = x$ and $(xy)z = x(yz)$ for $x, y, z, xy, yz \in V$. These conditions imply the existence of a neighbourhood of the identity $W \subset V$ and a differentiable map (inversion)

$$\iota : W \to W, \quad x \mapsto x^{-1}$$

such that $xx^{-1} = x^{-1}x = e$ for $w \in W$. Every Lie group G can be viewed as a local Lie group by taking $V = U = G$.

Replacing U and V by neighbourhoods of the identity U_1 and $V_1 \subset V \cap U_1$, satisfying the condition $V_1 V_1 \subset U_1$, one obtains also a local Lie group, called a *restriction* of the original one. By transitivity restriction generates a certain equivalence relation of local Lie groups. Strictly speaking, by a local Lie group one understands an equivalence class defined in this way. Two local Lie groups are said to be *isomorphic*, if for some of their restrictions (U_1, e_1, V_1, μ_1) and (U_2, e_2, V_2, μ_2) there is a diffeomorphism $f : U_1 \to U_2$ satisfying the conditions $f(e_1) = e_2, f(V_1) = V_2$ and $f(xy) = f(x)f(y)$ for $x, y \in V_1$. One can easily see that isomorphism of local Lie groups is an equivalence relation.

The concepts of Lie subgroup, homomorphism of Lie groups, etc. have natural local analogues and many theorems from the theory of Lie groups can be formulated for local Lie groups (some of them even turn out to be simpler). However the theory of local Lie groups does not have an independent status for the reason that every local Lie group a posteriori turns out to be a restriction of some Lie group. (This is a corollary of the theorem on the existence of a Lie group with a given tangent algebra: see Theorem 2.11 of Chap. 2).

Within the theory of Lie groups the significance of the concept of a local Lie group lies basically in that it enables us to use local terminology. For example, two Lie groups are said to be *locally isomorphic* if they are isomorphic as local Lie groups. This definition is a precise interpretation of the intuitive notion that two given Lie groups "look the same in a neighbourhood of the identity".

§2. Actions of Lie Groups

2.1. Definition of an Action. A homomorphism α from a Lie group G to the group $\operatorname{Diff} X$ of diffeomorphisms of a differentiable manifold X is called its *action* on X if the map $G \times X \to X$, $(g, x) \mapsto \alpha(g)x$ is differentiable.

Examples. 1. For any Lie group G one can define the following three actions l, r, a on itself:

$$l(g)x = gx, \ r(g)x = xg^{-1}, \ a(g)x = gxg^{-1}$$

2. The natural action of the group $\mathrm{GL}_n(K)$ on the projective space $P(K^n)$ is a Lie group action.

3. Every linear representation $R : G \to \mathrm{GL}(V)$ of a Lie group G can be viewed as its action on the space V. This kind of action is called *linear*.

4. Analogously, every homomorphism $f : G \to GA(S)$ can be viewed as an action of the Lie group G on the affine space S. Such an action is called *affine*.

Clearly, the composition of an action $f : G \to \operatorname{Diff} X$ and a homomorphism $f : H \to G$ is an action of the Lie group H on the manifold X.

In cases where there is no danger of confusion we will write simply gx in place of $\alpha(g)x$.

Actions of Lie groups will be considered in detail in the second part of this volume. We will use without any additional explanations certain common terms which are defined there.

2.2. Orbits and Stabilizers. Suppose we are given an action α of a Lie group G on a manifold X and let x be a point of this manifold. Consider the map $\alpha_x : G \to X$, $g \mapsto \alpha(g)x$. Its image is precisely the orbit $\alpha(G)x$ of the point x, and the pre-image of the point x is its stabilizer

$$G_x = \{g \in G : \alpha(g)x = x\}$$

The pre-images of the other points of the orbit are the cosets of G_x.

From the definition of a Lie group action it follows that the map α_x is differentiable, and from the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha_x} & X \\ {\scriptstyle l(g)}\downarrow & & \downarrow{\scriptstyle \alpha(g)} \\ G & \xrightarrow{\alpha_x} & X \end{array}$$

for any $g \in G$, that it has constant rank.

It is known (see, for example, Dieudonné 1960), that a differentiable map $f : X \to Y$ of constant rank k is linearizable in a neighbourhood of any point of the manifold X. From this it follows that:

1) the pre-image of any point $y = f(x)$ is a submanifold of codimension k in X, with $T_x(f^{-1}(y)) = \operatorname{Ker} d_x f$;

2) for each point $x \in X$ there is some neighbourhood U such that its image is a submanifold of dimension k in Y, with $T_{f(x)}(f(U)) = d_x f(T_x(X))$;

3) if $f(X)$ is a submanifold in Y, then $\dim f(X) = k$

Proof. The last part is proved in the following way: if we had $\dim f(X) > k$, then in view of (2) the manifold $f(X)$ would be covered by a countable number of submanifolds of lower dimension, which is impossible. □

Applying this to the map α_x constructed above we obtain the following theorem:

Theorem 2.1. *Let α be an action of a Lie group G on a differentiable manifold X. For any point $x \in X$ the map α_x has a constant rank and if this constant rank is k, then:*

1) the stabilizer G_x is a Lie subgroup of codimension k in G and $T_e(G_x) =$ $\operatorname{Ker} d_e\alpha_x$;

2) for some neighbourhood U of the identity in the group G the set $\alpha(U)x$ is a submanifold of dimension k in X, and $T_x(\alpha(U)x) = d_e\alpha_x(T_e(G))$;

3) if the orbit $\alpha(G)x$ is a submanifold in X, then $\dim \alpha(G)x = k$.

We remark that the orbit is not always a submanifold. (A counter-example will be given below).

Assertion 1) of the theorem can be used to prove that a given subgroup H of a Lie group G is a Lie subgroup. For this purpose it suffices to realize H as the stabilizer of some point for a certain action of the Lie group G. Moreover, if the orbit of the point turns out to be a manifold of known dimension, then assertion 3) makes it possible to compute the dimension of the subgroup H.

Applying these considerations to the representations $T_{k,l}$ of the group $\mathrm{GL}(V)$ in tensor spaces (see 1.4) we find, in particular, that the group of non-singular linear transformations, preserving some given tensor, is a linear Lie group.

Examples. 1. By considering the representation of the group $\mathrm{GL}(V)$ in the space $B_+(V)$ of symmetric bilinear forms (symmetric tensors of type $(0,2)$) we see that the group $O(V,f)$ of non-singular linear transformations preserving a given symmetric bilinear form f is a linear Lie group. If the form f is non-degenerate, then its orbit is open in $B_+(V)$ and, therefore,

$$\dim \mathrm{O}(V,f) = \dim \mathrm{GL}(V) - \dim B_+(V) = \frac{n(n-1)}{2}$$

where $n = \dim V$.

2. Analogously, by considering the representation of the group $\mathrm{GL}(V)$ in the space $B_-(V)$ of alternating bilinear forms, we see that the group $\mathrm{Sp}(V,f)$ of non-singular linear transformations preserving a given alternating bilinear form f is a linear Lie group. If the form f is non-degenerate, then

$$\dim \mathrm{Sp}(V,f) = \dim \mathrm{GL}(V) - \dim B_-(V) = \frac{n(n+1)}{2}$$

3. By considering the representation of the group $\mathrm{GL}(V)$ in the space of bilinear operations on V (tensors of type $(1,2)$) we see that the group of automorphisms of any algebra is a linear Lie group.

2.3. Images and Kernels of Homomorphisms. Let $f : G \to H$ be a homomorphism of Lie groups. Define an action α of G on the manifold H by the formula

$$\alpha(g)h = f(g)h,$$

where the right hand side is the product of elements in H. In other words, α is the composite of the homomorphism f and the action l of H on itself by left translations.

Let e be the identity of the group H. Then $\alpha_e = f$, $\alpha(G)e = f(G)$ and the stabilizer of the point e under the action α is just the kernel $\mathrm{Ker}\, f$ of the homomorphism f. Applying Theorem 2.1 to the action α and the point $e \in H$, we obtain the following theorem

Theorem 2.2. *Let $f : G \to H$ be a homomorphism of Lie groups. Then f is a map of constant rank and if this rank is equal to k, then*

1) $\mathrm{Ker}\, f$ *is a Lie subgroup of codimension k in G, and* $T_e(\mathrm{Ker}\, f) = \mathrm{Ker}\, d_e f$.

2) For some neighbourhood U of the identity in the group G the set $f(U)$ is a submanifold of dimension k in H and $T_e(f(U)) = d_e f(T_e(G))$.

3) if $f(G)$ is a Lie subgroup of H, then $\dim f(G) = k$.

Example. Consider the homomorphism $det : \mathrm{GL}_n(K) \to K^{\times}$. Its kernel is the group $\mathrm{SL}_n(K)$ of unimodular matrices. Since $det\, \mathrm{GL}_n(K) = K^{\times}$ we have $\mathrm{rk}\, det = 1$ and hence $\mathrm{SL}_n(K)$ is a Lie subgroup of codimension 1 in $\mathrm{GL}_n(K)$.

Clearly, if $f(G)$ is a submanifold, then $f(G)$ is a Lie subgroup in H. The following example shows that $f(G)$ is not always a submanifold. Let $f : \mathbb{R} \to \mathbb{T}^n$ be a homomorphism given by the formula

$$f(x) = (e^{ia_1 x}, \dots, e^{ia_n x}) \quad (a_1, \dots, a_n \in \mathbb{R})$$

It is known (see, for example, Bourbaki 1947), that if the numbers $a_1, \dots, a_n$ are linearly independent over $\mathbb{Q}$, then the set $f(\mathbb{R})$ is dense in $\mathbb{T}^n$ (this is the so called *dense winding of the torus*), and therefore, for $n > 1$ is not a submanifold. In order that the set $f(\mathbb{R})$ be a submanifold it is necessary and sufficient for the numbers $a_1, \dots, a_n$ to be commensurable.

2.4. Orbits of Compact Lie Groups. The preceding example makes the following assertion particularly interesting.

Theorem 2.3. *Every orbit of an action of a compact Lie group is a submanifold.*

Proof. Let α be an action of a compact Lie group G on a manifold X and let $x \in X$. We will prove that the orbit $\alpha(G)x$ is a submanifold in X. For this purpose it is enough to verify that it is a submanifold in a neighbourhood of