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（影印版）54

V. V. Kozlov

# Dynamical Systems X
## General Theory of Vortices

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## 旋涡的一般理论

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# Contents

# Introduction

> The English teach mechanics as an experimental science, while on the Continent, it has always been considered a more deductive and a priori science. Unquestionably, the English are right.*
>
> H. Poincaré, *Science and Hypothesis*

## Descartes, Leibnitz, and Newton

As is well known, the basic principles of dynamics were stated by Newton in his famous work *Philosophiae Naturalis Principia Mathematica*, whose publication in 1687 was paid for by his friend, the astronomer Halley. In essence, this book was written with a single purpose: to prove the equivalence of Kepler's laws and the assumption, suggested to Newton by Hooke, that the acceleration of a planet is directed toward the center of the Sun and decreases in inverse proportion to the square of the distance between the planet and the Sun. For this, Newton needed to systematize the principles of dynamics (which is how Newton's famous laws appeared) and to state the "theory of fluxes" (analysis of functions of one variable). The principle of the equality of an action and a counteraction and the inverse square law led Newton to the theory of gravitation, the interaction at a distance. In addition, Newton discussed a large number of problems in mechanics and mathematics in his book, such as the laws of similarity, the theory of impact, special variational problems, and algebraicity conditions for Abelian integrals. Almost everything in the *Principia* subsequently became classic. In this connection, A. N. Krylov, who translated the *Principia* into Russian, said that each sentence from Newton's book "was not forgotten but grew into large libraries of manuals, treatises, dissertations, and thousands of journals."

After these words, the modern reader will probably find it strange that Newton's *Principia* at first provoked a rather poor reaction in Continental centers of science. For example, in 1689 (i.e., two years after the *Principia* appeared), Leibnitz published an article explaining the planetary motions in the spirit of Descartes's vortex theory: not only a force directed toward the Sun but also a circular motion of a tenuous matter (ether) influences the planetary motion. He later returned to this question several times to describe the model in more detail. We should not think that his ideas resulted from a lack of knowledge: Newton and Leibnitz had a long correspondence with each other. We must also note that Leibnitz was not an orthodox Cartesian. In addition to a dispute with Newton over the priority in discovering the method of infinitesimals, Leibnitz had a long polemic with people sharing Descartes's ideas. He insisted on the opinion that the true measure of motion was the

*[Translated from the Russian translation.]

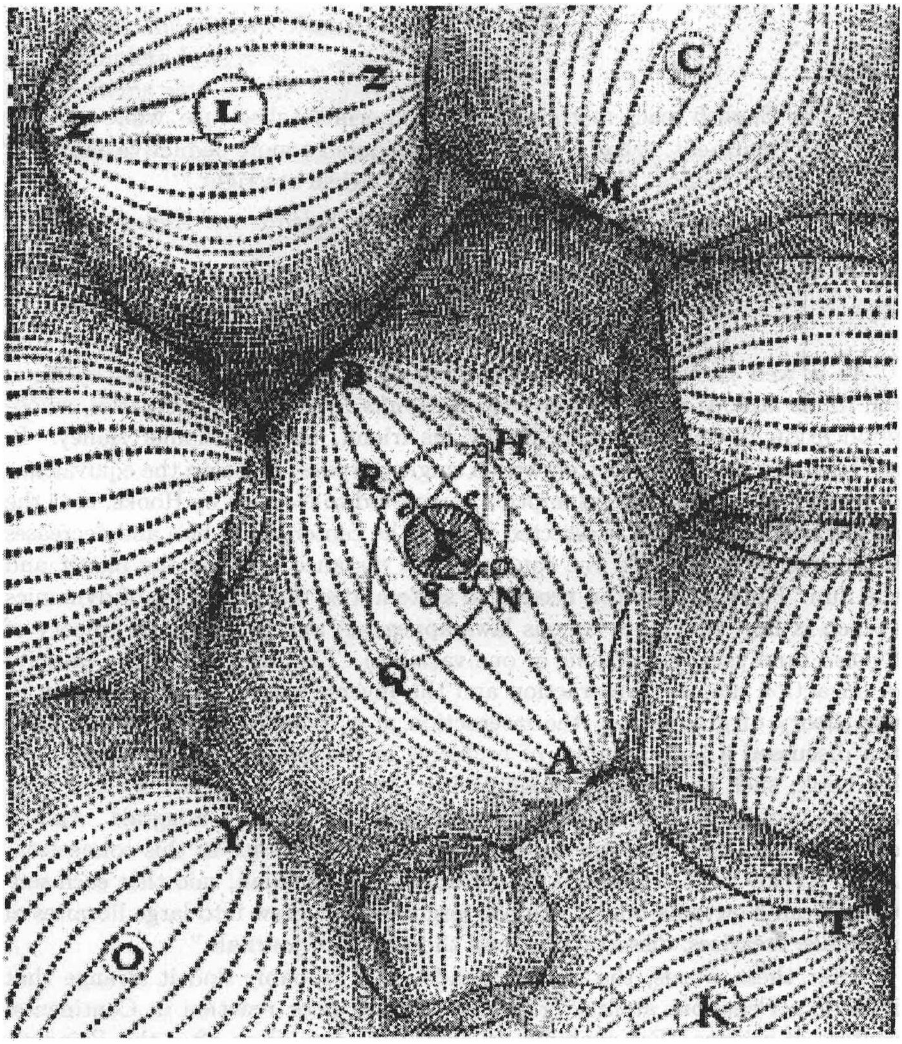


Fig. 1. Detail of a diagram from the 1644 *Principia philosophiae* of Rene Descartes, depicting his conception of the cosmos as an aggregate of contiguous vortices, most with a star at their center. The Sun is denoted by $S$.

weight of a body multiplied by the square of its velocity (*vis viva* according to Leibnitz) and not the product of weight and velocity, as stated by Descartes.

In 1713, after the second edition of the *Principia*, the situation changed abruptly. The new edition of Newton's book was supplemented with some added parts such as a well-developed foreword written by Cotes (a publisher and a member of the Cambridge Trinity board) and also a general *Precept* added by Newton at the end of the *Principia*. In these materials, the vortex theory was criticized rather sharply. In defining this Cartesian theory, Cotes's derisive style used expressions like "ridiculous invention," "nonsense," etc. Newton's style was more modest, although it is now known that Newton himself carefully edited the foreword written by Cotes. This foreword offended Leibnitz, who shared Descartes's opinion on cosmogony and made a number of essential additions, and it added fuel to the dispute over the priority in discovering differential and integral calculus. The priority dispute is widely known (see, e.g., the fascinating book by Arnold [8]). In contrast, the vortex theory dispute has been almost forgotten today and is only briefly mentioned in books on the history of mechanics. On the Continent, however, Newton's works met strong opposition, which lasted for decades. Leibnitz was not alone in opposing it; there were also such outstanding scientists as Huygens, Varignon, J. and D. Bernoulli, etc. "The German and French scientists are furiously attacking Newton's philosophy and agree with Descartes's," Jones (a supporter of Newton's philosophy) wrotes to Cotes in 1711 (see [34]). Among all the arguments for Newton's theory, there was also a thesis proposing a freedom of opinion: "They [the Cartesians – V. K.] do have the right to differ, but they should be fair enough to others and let them wish the same freedom that they desire to be given. So, let us hold to the philosophy of *Newton*, which we consider to be more correct"* (a passage from the foreword written by Cotes for the second edition of the *Principia*).

For a better understanding of the dispute, we recall some general ideas in Descartes's theory. These ideas were stated in *Discours de la méthode* (1637) and in *Principia philosophiae* (1644). According to Descartes, the understanding of cosmology starts from acceptance of the initial *chaos*, whose moving elements are ordered according to certain fixed laws and form the Cosmos. (As we see, these ideas have very much in common with those in contemporary synergetics!) According to Descartes, the Universe is filled with a tenuous fluid matter (prototype of the *ether*), which is constantly in a vortex motion. This motion moves the largest particles of matter off the vortex axis, and they subsequently form planets. Then, according to what Descartes wrote in his *Treatise on Light*, "the material of the Heaven must rotate the planets not only about the Sun but also about their own centers... and this will hence form several small Heavens rotating in the same direction as the great Heaven."* The term *vortex* (tourbillon) originated from a comparison with a river current swirling around objects carried by the water.

Huygens used a simple example to explain the main idea of the vortex theory: vortex motions of water in a bucket. Two identical bodies placed at different distances from the vortex axis rotate with different velocities. The closer a body is to the axis, the greater its velocity. This observation qualitatively corresponds to the law of the diminishing velocity of a planet in

*[Translated from the Russian translation.]

accordance with its increasing distance from the Sun. But does it correspond to Kepler's law?

## Newton and Bernoulli

Newton himself considered this question in the *Principia* (Chap. 9). According to Newton, if a homogeneous viscous liquid is moved by a cylinder or sphere uniformly rotating about its axis, then in the stationary case, the rotation period of liquid particles is respectively proportional to their distance or squared distance from the axis of rotation. However, according to Kepler's third law, this should result in a semicubical function of distance. Newton concludes the *Precept* for his theorems with the following words: "I would like the philosophers to think of a condition by which it would be possible to explain a phenomenon based on the sesquitripilicate proportion by the vortex theory."*

In his work (1730) awarded a prize by the Paris Academy of Sciences, J. Bernoulli judged Newton's ideas to be flawed. The subject for the competition announced by the Academy was to explain the elliptical shape of planetary orbits. (This occurred 40 years after Newton published his book!) Incidentally, in this work, Bernoulli proposed the analytic method for obtaining the elliptical shape of planetary orbits by applying the gravitation law (which we can now find in mechanics textbooks). Newton used labored geometric expressions in imitation of ancient authors. According to Bernoulli, the dependence of the rotation period of a particle on distance also never corresponded to Kepler's law. Moreover, his conclusions turned out to be not quite correct. The first correct solution of this hydrodynamic problem was obtained by Stokes in 1845: when a cylinder or sphere rotates, the power of the distance is respectively equal to two or three.

In the competition papers of 1732 and 1734 on the reason for the inclination of planetary orbits to the Sun's equator, J. Bernoulli gradually moved away from the vortex theory. The ideas of D. Bernoulli (who shared the Paris Academy prize with his father) evolved similarly.

Newton and Bernoulli both studied the viscous liquid. Incidentally, if we consider Descartes's ether an ideal fluid, we can simply obtain the semicubical relation. For simplicity, we consider a plain-parallel flow and obtain conditions for uniform rotatory motion of flow particles. Let $r$ be the distance from the axis of rotation, $v$ be the particle velocity, $\rho$ be the liquid density, and $p$ be the pressure ($v$, $\rho$, and $p$ are functions of $r$). It is easy to show that the continuity equation

$$\operatorname{div}(\rho v) = 0$$

*[Translated from the Russian translation.]

automatically holds, and two other dynamic equations reduce to the single relation

$$\rho \frac{v^2}{r} = p', \tag{1}$$

where the prime denotes the derivative with respect to $r$. This equation was actually presented by Huygens in his theory of centrifugal forces. According to Kepler's law, $v = cr^{1/2}$, $c =$ const. Taking, for example, $\rho = \rho_0 =$ const (a homogeneous liquid), we obtain

$$p = p_0 - \frac{\rho_0 c^2}{r}, \qquad p_0 = \text{const},$$

from (1). However, at small values of $r$, the pressure is always negative; this does not correspond to the properties of real liquids in normal conditions. We must also note that the curl of the velocity field in rotatory motion is orthogonal to the plane of the flow and equals $[rv']/r$. According to Newton (with Stokes's clarifications), $v = C_1/r$, $C_1 =$ const. Hence, the liquid in this case undergoes a vortex-free motion (with a multivalued potential). If we accept Kepler's law, we obtain the vortex flow (as it should be according to Descartes).

Newton gave a simpler, but stronger, argument against Descartes's theory. According to Newton, the motion of celestial bodies is described by second-order differential equations: to define a trajectory of a body, it is necessary to specify not only its position but also its velocity at a certain instant. If Descartes's theory is in fact correct, bodies are carried by the ether, and the equations of motion are consequently of first order: the velocity of a particle depends only on its position. However, Newton noted that some of the observed comets move in a direction opposite to that of all the planets.

## Voltaire, Maupertuis, and Clairaut

It is necessary to say that far from all Continental scientists shared Descartes's ideas about the vortex theory. A number of famous French scientists (Pascal, Fermat, Roberval, etc.) accepted his ideas rather guardedly. The main role in promoting Newton's theory belonged not to scientists but to the writer and philosopher Voltaire. As we would say now, Voltaire was a dissident. His *Lettres philosophiques* were based on comparing and contrasting the situations in England and France. According to Voltaire, England is the homeland of human reason: everything is fine on the blessed island with the citizens enjoying their political freedom and freedom of thought. And all is bad back home in France. The comparison of the London Royal Society and the Paris Academy of Sciences, certainly, does not speak well for the latter. According to Voltaire, the Society is independent, free of charge, and involved in work, while the Academy is isolated from practical affairs and only publishes volumes of compliments. Voltaire compared Newton and Descartes in the same spirit. Newton is a wise man and modest besides, venturing to explain Nature, while Descartes is a dreamer and all his philosophy is a novel. "Arriving in London, a Frenchman finds everything different including philosophy. He left a full Universe and finds emptiness. In Paris, the Universe is viewed as consisting of ether vortices; here, in the same world space, mysterious forces direct the play. We think that pressure from the moon causes the tides, but the English think that the sea is attracted to the moon. In Paris, the Earth presents itself in the form of a melon; in London, it is flattened from two sides"* (Letter 14). Voltaire's ridicule in comparing the two systems was ostensibly directed equally to both sides. However, the reactions in Paris and in London were different: his book was banned in France but met with approval in England.

The Paris Academy of Sciences organized several expeditions to determine the length of arcs of meridian with the objective of clarifying the shape of the Earth. The expedition to Lapland (1735–1742) led by Maupertuis was successful: the Earth appeared flattened at the poles, as predicted by Newton's theory. Maupertuis was a Newtonian and was the one who explained the meaning of his theory to Voltaire. But this did not save Maupertuis from Voltaire's malicious gibes concerning the theological aspects of the variational principle of dynamics, which is named for Maupertuis today.

Gradually, primarily because of the works of Clairaut (incidentally, a participant in the Maupertuis expedition), Newton's theory of gravitation gained a wide acceptance. First was his *Theory of the Shape of the Earth* based on the law of gravitation. Second was Clairaut's prediction of the appearance of Halley's comet in 1759 based on the perturbation theory. It is worth recalling that the French translation of the *Principia* by the Marquise Emily du Chatlé published in 1759 in Paris was edited by Clairaut. The initiator of this edition was the same Voltaire.

According to Poincaré, each Truth has its instant of celebration between the eternity when it is considered untrue and the eternity when it is considered trivial. True, for Newton's theory of gravitation, this instant lasted the life of an entire generation. Moreover, our story does not end here.

## Helmholtz and Thomson

The interest in the vortex theory revived in the middle of the 19th century because of the works of Helmholtz and Thomson (Lord Kelvin) on the vortex motion of an ideal fluid. It was proved that the circulation of velocity along a closed contour moving together with the fluid particles is constant, and, as a consequence, the law of the freezing-in of vortex lines was established. (We recall Descartes's ideas about the ether vortex transferring material bodies!)

The theory of vortex motion attracted even greater interest when Kelvin proposed his vortex theory of atoms ("On vortex atoms," *Phil. Mag.*, 1867).

*[Translated from the Russian translation.]

According to Kelvin, the Universe should be considered as a pure fluid (ether) containing separate, indissolubly linked Helmholtz vortices (atoms grouped into molecules). From this standpoint, gravitation should be explained statistically (in the spirit of Lesage's theory of 1764) as the impacts from a large number of small, rapidly moving vortices. Thomson proposed the beautiful name *ichthyodes* for them. As Klein wrote in his book *Development of Mathematics in the 19th Century*, "the theory did not go beyond the bounds of a remark, leading to nothing of substance, but it retains a known charm for the susceptible imagination."* In spite of this, Kelvin's theory stimulated a range of important research in studying the stability and fluctuations of various vortex structures.

The leading idea of Thomson's research program was the desire to find a mechanical model of complex physical phenomena in which action at a distance would be replaced with a direct contact (as in Descartes's theory). At that time, it was very popular to think that mechanics was a basis for physics. For example, Maxwell in his earlier works considered an electromagnetic model in which the induction currents of a magnet are caused by the medium rotating about magnetic force lines and there are small frictional balls between the rotating parts of the medium to avoid friction. Maxwell considered these balls the true state of electricity. Despite significant efforts, Maxwell could not move far in constructing adequate mechanical models of electromagnetism. Consequently, he accepted the now-usual idea of fields.

We also recall one more attempt to solve the problem of "action at a distance" using the theory of "latent motions." The main idea of this theory can be explained with the example of a rotating symmetric top. Because the rotation of the top about its axis of symmetry cannot be observed, we can suppose that the top does not rotate at all and explain its behavior by the influence of additional gyroscopic and potential forces. In the general case, this idea can be realized only within the framework of Routh's theory of the decrease of the order of systems with symmetries. We assume that a mechanical system with $n+1$ degrees of freedom moves by inertia and its Lagrangian, which equals the kinetic energy, has a one-parameter group of symmetries. If we decrease the order of the system by factoring with respect to orbits of this group, we see that the Routh function, which equals the Lagrangian of the reduced system with $n$ degrees of freedom, contains a term independent of velocity. This term can be interpreted as a potential force affecting the reduced system. Helmholtz, W. Thomson (Lord Kelvin), J. J. Thomson, and Hertz insisted on the idea that all the mechanical quantities appearing as "potential energies" were in fact caused by latent "cyclic" motions. This concept of the kinetic theory was most fully detailed by Hertz in his book *The Principles of Mechanics, Presented in a New Form.* It turns out that a system with a compact configuration space can really be obtained from geodesic flows by using Routh's method. However, in the case of a noncompact space (the most

*[Translated from the Russian translation.]

interesting from the standpoint of the theory of gravitation), it is no longer so.

## About the Book

In the present book, one more attempt is made to "rehabilitate" Descartes's vortex theory. Certainly, this book does not develop the theory of action at a distance in the spirit of Helmholtz and Thomson. Its main object is to systematize analogies between the usual mechanics of conservative systems and ideal fluid dynamics. It turns out that the family of phase trajectories composing an invariant manifold uniquely projected on the configuration space of a mechanical system admits a natural and convenient description in the terms of multidimensional hydrodynamics. On the other hand, in a number of problems, it is necessary to study not separate trajectories but families of them. For example, in geometric optics, the main object for constructing images is the *ray system*, not the separate light rays. If we also take the deep analogy between optics and mechanics opened by J. Bernoulli and developed by Hamilton into account, then the general theory of vortices stated in this book allows comprehending the basic results in mechanics, geometric optics, and hydrodynamics from a single standpoint. This theory reveals some general mathematical ideas that appeared in mechanics, optics, and hydrodynamics at different times under different names, and this gives a certain aesthetic pleasure. In addition, the general theory of vortices has interesting applications in numerical calculations, stability theory, and the theory of exact integration of dynamic equations.

Paraphrasing Newton, this book could be called

PHILOSOPHIAE CARTESIAN PRINCIPIA MATHEMATICA.

The author dedicates this book to the 400th birthday of Rene Descartes, a great scientist and human being, who, as Voltaire said, taught his contemporaries to reason.

# Chapter 1
# Hydrodynamics, Geometric Optics, and Classical Mechanics

## §1. Vortex Motions of a Continuous Medium

**1.1.** In the investigation of the general properties of vortex lines, a significant role is played by the equation

$$\frac{\partial u}{\partial t} = \operatorname{rot}(u \times v), \tag{1.1}$$

where $v(x,t)$ is the velocity of the particles of a medium in the three-dimensional Euclidean space $E^3 = \{x\}$ and $u(x,t)$ is a solenoidal vector field, $\operatorname{div} u = 0$. The physical meaning of the field $u$ is determined by the specific problem under investigation. Integral curves of the vector field $u$ (at a fixed instant $t$) are called *vortex lines*.

For example, in magnetohydrodynamics, which deals with media of infinite conductivity, Eq. (1.1) describes the magnetic field strength; in this case, the vortex lines coincide with the magnetic field lines.

The barotropic flow of an ideal fluid in a potential force field is a more fundamental example. We recall that fluid motion is described by the *Euler equation*

$$\rho\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}v\right) = -\frac{\partial p}{\partial x} + \rho F, \qquad F = -\frac{\partial V}{\partial x}, \tag{1.2}$$

where $\rho$ is the fluid density, $p$ is the pressure, $F$ is the external mass force density, and $V$ is the potential energy density. For a barotropic fluid, there exists a pressure function $P(x,t)$ such that

$$dP = \frac{dp}{\rho}.$$

In particular, a homogeneous fluid ($\rho = \text{const}$) is barotropic. To obtain a closed system of equations, we must add the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \tag{1.3}$$

to Euler equation (1.2); this equation expresses the mass conservation of the moving volume.

Under these assumptions, Eq. (1.2) can be transformed into the form of the Lamb equation

$$\frac{\partial v}{\partial t} + \left[\frac{\partial v}{\partial x} - \left(\frac{\partial v}{\partial x}\right)^{\mathrm{T}}\right] v = -\frac{\partial f}{\partial x}, \qquad f = \frac{v^2}{2} + P + V, \tag{1.4}$$

where the superscript T denotes transposition of the Jacobi matrix

$$\frac{\partial v}{\partial x} = \left\| \frac{\partial v^i}{\partial x^j} \right\| .$$

In hydrodynamics, the function $f$ is usually called the *Bernoulli function.*

As is known, multiplying a skew-symmetric matrix $\partial v/\partial x - (\partial v/\partial x)^{\mathrm{T}}$ by a vector $v$ in the three-dimensional Euclidean space, we obtain the vector cross product $(\operatorname{rot} v) \times v$; therefore, Eq. (1.4) can be rewritten in the equivalent form

$$\frac{\partial v}{\partial t} = v \times \operatorname{rot} v - \frac{\partial f}{\partial x}. \tag{1.5}$$

Taking the curl of both sides and using the relation $\operatorname{rot} \operatorname{grad} f = 0$, we obtain Eq. (1.1) with $u = \operatorname{rot} v$; the field of a curl is always solenoidal because $\operatorname{div} \operatorname{rot} = 0$. Vortex lines are integral curves of the field of the velocity curl (vortex); this explains the choice of the term in the general case.

The motion of fluid particles in $E^3$ is described by the differential equation

$$\dot{x} = v(x, t), \tag{1.6}$$

where the dot denotes differentiation with respect to $t$. Let $x(t, x_0)$ be its solution satisfying the initial condition $x(0, x_0) = x_0$. The family of mappings $E^3 \to E^3$ defined by the formula

$$x_0 \to x(t, x_0) \tag{1.7}$$

is called the *flow* of system (1.6). In the stationary case, where $v$ is independent of $t$, the family of transformation (1.7) is a group. Transformations (1.7) are usually denoted by $g_v^t$ (or simply $g^t$ if this does not lead to confusion).

**1.2.** Let $D$ be a measurable domain in $E^3$ and $g^t(D)$ be its image under transformation (1.7). By (1.3), the mass of the moving domain $g^t(D)$ is constant,

$$\int_{g^t(D)} \rho \, d^3x = \text{const} .$$

Now let $\Sigma$ be a two-dimensional bounded surface and $\Gamma = \partial\Sigma$ be its boundary. In fluid mechanics, the following formula for the flow of a solenoidal field is well known:

$$\frac{d}{dt} \int_{g^t(\Sigma)} (u, n) \, d\sigma = \int_{g^t(\Sigma)} \left( \frac{\partial u}{\partial t} + \operatorname{rot}(v \times u), n \right) d\sigma, \tag{1.8}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in the Euclidean space $E^3$, $n$ is a unit normal vector, and $d\sigma$ is the surface element of the surface $\Sigma$. Using (1.1), we obtain the conservation law for the flow of the field $u$ through a moving surface from (1.8):

$$\int_{g^t(\Sigma)} (u, n)\, d\sigma = \text{const}. \tag{1.9}$$

This implies the *Helmholtz–Thomson theorem* on the freezing-in of vortex lines: the flow of system (1.5) transforms vortex lines into vortex lines. This result explains the appearance of magnetic storms on the Earth. The Sun is a sphere of turbulent plasma with almost infinite conductivity. From time to time, prominences appear on the Sun: matter is thrown to the surface of the Sun with enormous speed and then dissipates and moves away. By the Helmholtz–Thomson theorem, this matter transfers the magnetic field and creates magnetic storms upon reaching the Earth.

Now let $\Gamma$ be a closed contour, the boundary of a bounded surface $\Sigma$. We consider the 1-form

$$(v, dx) = \sum v_i\, dx^i.$$

The integral of this form over $\Gamma$ is called the *circulation of the velocity* along the contour $\Gamma$ (Thomson). Applying the Stokes formula

$$\oint_\Gamma (v, dx) = \int_\Sigma (u, n) d\sigma, \qquad u = \operatorname{rot} v,$$

and taking (1.9) into account, we obtain the theorem on the constancy of the circulation of the velocity along the "fluid" contour:

$$\oint_{g^t(\Sigma)} (v, dx) = \text{const}. \tag{1.10}$$

The *Lagrange theorem* on potential flows is an important consequence of this result. We recall that the velocity field $v(x, t)$ is said to be a *potential* field if

$$v = \frac{\partial \varphi}{\partial x}. \tag{1.11}$$

The function $\varphi(x, t)$ is called the *potential.* The Lagrange theorem states that if the velocity field of a barotropic ideal fluid in a potential force field is a potential field at the initial instant (e.g., $t = 0$), then it is a potential field for all $t$.

Substituting (1.11) in Lamb equation (1.5) and using the obvious identity

$$\operatorname{rot} \frac{\partial \varphi}{\partial x} = 0,$$

we obtain the relation

$$\frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial t} + f \right) = 0.$$

Therefore, the expression in parentheses is a function depending only on time:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + P + V = g(t). \tag{1.12}$$

This relation is called the *Lagrange–Cauchy integral.* Performing the gauge transformation

$$\varphi \to \varphi - \int g(t)\, dt,$$

which preserves the velocity field, we obtain Eq. (1.12) with $g = 0$.

**1.3.** We return to the investigation of Eqs. (1.1) and (1.3). We set $w = u/\rho$. It is clear that integral curves of the field $w$ are exactly the vortex lines introduced above.

**Theorem 1.** *The field $w(x,t)$ satisfies the equation*

$$\frac{\partial w}{\partial t} = [v, w]. \tag{1.13}$$

The bracket $[\cdot, \cdot]$ is the commutator of vector fields. We recall its definition. Let $a = \{a_i\}$ and $b = \{b_i\}$ be two vector fields. The field $c = \{c_i\}$ defined as

$$c = \frac{\partial a}{\partial x} b - \frac{\partial b}{\partial x} a$$

with the components

$$c_j = \sum_i \left( b_i \frac{\partial a_j}{\partial x^i} - a_i \frac{\partial b_j}{\partial x_i} \right)$$

is called the *commutator* of these fields. If $L_a$, $L_b$, and $L_c$ are the differentiation operators along the respective fields $a$, $b$, and $c$, then

$$L_c = L_b L_a - L_a L_b.$$

The expression in the right-hand side of this relation is the commutator of the operators $L_a$ and $L_b$.

The fields $a$ and $b$ *commute* if $[a, b] = 0$. This property occurs if and only if the phase flows of the fields $a$ and $b$ commute, i.e.,

$$g_a^p g_b^q = g_b^q g_a^p$$

for all $p, q \in \mathbb{R}$.

Theorem 1 was initially obtained by Arnol'd [3] for the case of a homogeneous ideal fluid, where $\rho = \text{const}$; in this case, we can take $w = \operatorname{rot} v$. The general case is considered in [40]. If the medium motion is stationary, then the fields $v$ and $u/\rho$ commute (at first glance, the fields $u$ and $\rho v$, the momentum density, seem to commute). Equation (1.13) is usually called the *Euler*