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A. L. Onishchik E. B. Vinberg (Eds.)

Lie Groups and Lie Algebras III

Structure of Lie Groups and Lie Algebras

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李群与李代数的结构

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Structure of Lie Groups and Lie Algebras

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Translated from the Russian
by V. Minachin

Contents

Introduction

This article builds on Vinberg and Onishchik [1988] and is devoted to an exposition of the main results on the structure of Lie groups and finite-dimensional Lie algebras. The greater part of the article is concerned with theorems on the structure and classification of semisimple Lie groups (algebras) and their subgroups (subalgebras). The tables given at the end of the article can be used as reference material in any work on Lie groups.

We consider only the results of the classical theory of Lie groups. Some classes of infinite-dimensional Lie groups and Lie algebras, as well as Lie supergroups and superalgebras, will be dealt with in special articles of one of the following volumes of this series. The same applies to the theory of Lie algebras over fields of finite characteristic. However, the results on Lie algebras given in the present article can be extended to more general fields of characteristic 0 (e.g., the field $\mathbb{C}$ of complex numbers can be replaced by any algebraically closed field of characteristic 0).

For the theory of linear representations of Lie groups and algebras, the reader is referred to the volumes especially devoted to this theory, although we had to include in this article some classical theorems on finite-dimensional representations, which form an inseparable part of the structural theory. We also use some results from the theory of algebraic groups. Almost all of them can be found in Springer [1989], and some in Chap. 1, Sect. 6. On the other hand, the results on complex and real algebraic groups contained in Springer [1989] can be treated as results on Lie groups. Some of them (e.g. the Bruhat decomposition) are not dealt with in this volume.

The authors have tried, whenever possible, to give the reader the ideas of the proofs.

The terminology and notation of the article follow that of Vinberg and Onishchik [1988]. In particular, Lie groups are denoted by upper-case Roman letters, and their tangent algebras by lower-case Gothic.

Chapter 1
General Theorems

All vector spaces and Lie algebras considered in this chapter are assumed to be finite-dimensional. The ground field is denoted by K, which is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

§ 1. Lie's and Engel's Theorems

1.1. Lie's Theorem. Denote by $T_n(K)$ the subgroup of $\mathrm{GL}_n(K)$ consisting of all nondegenerate upper triangular matrices, and by $\mathfrak{t}_n(K)$ the subalgebra of the Lie algebra $\mathfrak{gl}_n(K)$ consisting of all triangular matrices. The group $T_n(K)$ (respectively, Lie algebra $\mathfrak{t}_n(K)$) can be interpreted as a subgroup of the full linear group $\mathrm{GL}(V)$ (respectively, subalgebra of the full linear algebra $\mathfrak{gl}(V)$), where V is an n-dimensional vector space over K consisting of operators preserving some full flag, i.e. a set of subspaces $V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V$, where $\dim V_i = i$. The group $T_n(K)$ and the Lie algebra $\mathfrak{t}_n(K)$ are solvable (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.5). The following theorem, first proved by Sophus Lie, shows that the subgroup $T_n(\mathbb{C})$ (subalgebra $\mathfrak{t}_n(\mathbb{C})$) is, up to conjugation, the only maximal connected solvable Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ (respectively, maximal solvable subalgebra of $\mathfrak{gl}_n(\mathbb{C})$).

Theorem 1.1 (see Bourbaki [1975], Jacobson [1962]). (1) *Let $R: G \to \mathrm{GL}(V)$ be a complex linear representation of a connected solvable Lie group G. Then there is a full flag in V invariant under $R(G)$.*

(2) *Let $\mathfrak{g}$ be a solvable Lie algebra, and $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ a complex linear representation of it. Then there is a full flag in V invariant under $\rho(\mathfrak{g})$.*

Because of the correspondence between solvable Lie groups and Lie algebras (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.5), statements (1) and (2) of the theorem are equivalent. We now give an outline of the proof of statement (1).

We start with some definitions and simple auxiliary statements.

Let $R: G \to \mathrm{GL}(V)$ be a linear representation of a group G over an arbitrary field K. For any character χ of the group G, i.e. a homomorphism $\chi: G \to K^\times$, where $K^\times$ is the multiplicative group of the field K, we set

$$V_\chi = V_\chi(G) = \{v \in V | R(g)v = \chi(g)v \quad \text{for all} \quad g \in G\}.$$

If $V_\chi \neq 0$, then the character χ is said to be a *weight* of the representation R, the subspace V_χ is called the *weight subspace*, and its nonzero vectors the *weight vectors* corresponding to the weight χ. Similarly, for any linear representation ρ of the Lie algebra $\mathfrak{g}$ over the field K and any linear form

$\lambda \in \mathfrak{g}^*$ let

$$V_\lambda(\mathfrak{g}) = \{v \in V | \rho(x)v = \lambda(x)v \quad \text{for all} \quad x \in \mathfrak{g}\}.$$

If $V_\lambda(\mathfrak{g}) \neq 0$, then the form λ is said to be a *weight* of the representation ρ, the subspace $V_\lambda(\mathfrak{g})$ is called the *weight subspace*, and its nonzero vectors the *weight vectors* corresponding to the weight λ.

Weight subspaces corresponding to different weights are linearly independent. Thus a finite-dimensional linear representation may have only finitely many weights.

The proof of Lie's theorem is based on the following property of weight subspaces.

Lemma 1.1. *Let H be a normal subgroup of the group G, χ the character of H, and $R: G \to \mathrm{GL}(V)$ a linear representation. Then for any $g \in G$ we have*

$$R(g)V_\chi(H) = V_{\chi^g}(H),$$

where $\chi^g(h) = \chi(g^{-1}hg)$ $(h \in H)$.

Outline of the proof of Theorem 1.1. First, one shows by induction on $\dim G$ that R has at least one weight in V. For $\dim G = 1$ the statement is evident. In the general case, the definition of a solvable Lie group implies that there is a virtual normal Lie subgroup H of G of codimension 1. Clearly, $G = CH$, where C is a connected virtual one-dimensional Lie subgroup. By the inductive hypothesis, $V_\chi(H) \neq 0$ for some character χ of the group H. In view of Lemma 1.1, the operators $R(g)$, $g \in G$, permute the weight subspaces of the group H. Since G is connected, $V_\chi(H)$ is invariant under $R(G)$.

The one-dimensional subgroup C has a one-dimensional invariant subspace in $V_\chi(H)$, which is evidently invariant under the action of the entire group G.

Thus, there is a one-dimensional subspace in V invariant under G. The existence of a full flag in V invariant under G is then proved by induction on $\dim V$. □

Corollary 1. *Any irreducible complex linear representation of a connected solvable Lie group or a solvable Lie algebra is one-dimensional.*

Corollary 2. *Let $G \subset \mathrm{GL}(V)$ be a connected irreducible complex linear Lie group. Then either G is semisimple, or* $\mathrm{Rad}\, G = \{cE | c \in \mathbb{C}^\times\}$.

Proof. Suppose that G is not semisimple. Consider the vector subspace $W = V_\chi(\mathrm{rad}\, G) \neq 0$. Lemma 1.1 implies that it is invariant under G. Hence $W = V$, i.e. $\mathrm{Rad}\, G$ contains scalar operators only. □

Corollary 3. *A Lie algebra $\mathfrak{g}$ over $K = \mathbb{C}$ or $\mathbb{R}$ is solvable if and only if the Lie algebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}_n(K)$ is nilpotent.*

Proof. If $\mathfrak{g} = \mathfrak{t}_n(K)$, then $[\mathfrak{g}, \mathfrak{g}]$ is the nilpotent Lie algebra of all upper diagonal matrices with zeros on the diagonal. In the general case one can assume, using the complexification procedure if necessary, that $K = \mathbb{C}$. We now see, by Lie's theorem, that if $\mathfrak{g}$ is solvable, then the Lie algebra ad $[\mathfrak{g}, \mathfrak{g}] = [\operatorname{ad} \mathfrak{g}, \operatorname{ad} \mathfrak{g}]$ is nilpotent and therefore $\mathfrak{g}$ is also nilpotent. □

1.2. Generalizations of Lie's Theorem. First we consider the possibilities of generalizing Lie's theorem to Lie algebras over an arbitrary field K. If a representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ over K has an invariant full flag, then the characteristic numbers of all operators $\rho(x)$, $x \in \mathfrak{g}$, must belong to the field K, which is far from being always true if K is not algebraically closed. If $\operatorname{char} K = 0$, then the above mentioned property of the operators $\rho(x)$, $x \in \mathfrak{g}$, turns out to be also sufficient for the existence of an invariant flag.

Theorem 1.2. *Let $\mathfrak{g}$ be a solvable Lie algebra over a field K of characteristic 0 and $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ a linear representation of it over K. If all characteristic numbers of all operators $\rho(x)$, $x \in \mathfrak{g}$, belong to K, then there is a full flag in V invariant under $\rho(\mathfrak{g})$.*

The proof is similar to that of Theorem 1.1, and makes use of the following analogue of Lemma 1.1.

Lemma 1.2. *Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ over a field K of characteristic 0, $\mathfrak{h}$ an ideal in $\mathfrak{g}$, and $V_\lambda(\mathfrak{h})$ a weight subspace of the representation $\rho|\mathfrak{h}$. Then the following two equivalent statements hold: (1) $V_\lambda(\mathfrak{h})$ is invariant under $\rho(\mathfrak{g})$; (2) $\lambda(z) = 0$ for any $z \in [\mathfrak{g}, \mathfrak{h}]$.*

Corollary 3 to Theorem 1.1 is extended to the case of an arbitrary field of characteristic 0. If a field of characteristic 0 is algebraically closed, then the analogues of Corollaries 1 and 2 hold.

The condition imposed by Theorem 1.2 on the characteristic is essential, as the following example shows.

Example. If $\operatorname{char} K = 2$, then the Lie algebra $\mathfrak{gl}_2(K)$ is solvable, but its identity representation in K^2 has no weight vectors.

Without going into details, we note that Lie's theorem can be extended to connected solvable linear algebraic groups over an algebraically closed field of arbitrary characteristic. This follows from Borel's fixed point theorem (see Springer [1989], Chap. 1, Sect. 3.5). We also state the following simple theorem on representations of abstract solvable groups.

Theorem 1.3 (see Merzlyakov [1987]). *Let G be a solvable group, and $R: G \to \mathrm{GL}(V)$ a complex linear representation of it. Then there is a full flag in V invariant under a subgroup of finite index $G_1 \subset G$.*

Proof. Consider the algebraic closure $H = {}^aR(G)$ of the subgroup $R(G)$ of $\mathrm{GL}(V)$. The solvable linear algebraic group H has a finite number of connected components. According to Theorem 1.1, there is a full flag in V invariant under H^0. But then it is also invariant under the subgroup $G_1 = R^{-1}(H^0)$, which is of finite index in G. □

In addition to the main statement of Theorem 1.3 one can also show that the subgroup G_1 can be chosen in such a way that its index does not exceed a number depending on $\dim V$ only (see Merzlyakov [1987]).

1.3. Engel's Theorem and Corollaries to It. The cornerstone in the theory of nilpotent Lie algebras and Lie groups is the following theorem first proved by F. Engel.

Theorem 1.4 (see Bourbaki [1975], Jacobson [1955]). *Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ over an arbitrary field K. Suppose that for each $x \in \mathfrak{g}$ the linear operator $\rho(x)$ is nilpotent. Then there is a basis in V with respect to which the operators $\rho(x)$, $x \in \mathfrak{g}$, are represented by upper triangular matrices with zeros on the diagonal. In particular, the Lie algebra $\rho(\mathfrak{g})$ is nilpotent.*

Proof. As for Lie's theorem, induction on $\dim V$ reduces the theorem to the proof of the existence of a weight vector (with the weight 0). The latter is achieved by induction on $\dim \mathfrak{g}$. For $\dim \mathfrak{g} = 1$ the statement is evident. Suppose that the statement holds for all Lie algebras of dimension less than m, and let $\dim \mathfrak{g} = m$. It follows from the statement of the theorem and the inductive hypothesis that there is an ideal $\mathfrak{h}$ of codimension 1 in $\mathfrak{g}$ (one can take for $\mathfrak{h}$ any maximal subalgebra of $\mathfrak{g}$). Then $\mathfrak{g} = \mathfrak{h} + \langle y \rangle$, where $y \in \mathfrak{g}$. Consider the weight subspace $V_0(\mathfrak{h}) \neq 0$. Since $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, Lemma 1.2 implies that $V_0(\mathfrak{h})$ is invariant under $\mathfrak{g}$. The operator $\rho(y)$ is nilpotent, whence there is a vector $v_0 \in V_0(\mathfrak{h})$, $v_0 \neq 0$, such that $\rho(y)v_0 = 0$. Evidently, v_0 is the desired weight vector with respect to $\mathfrak{g}$. □

Corollary 1. *If under the conditions of Theorem 1.4 the representation ρ is irreducible, then it is trivial and one-dimensional.*

An application of Engel's theorem to the adjoint representation easily yields the following corollary.

Corollary 2. *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if either of the following two conditions is satisfied:*

(1) *For any $x \in \mathfrak{g}$ the operator $\operatorname{ad} x$ is nilpotent.*
(2) *There is a basis $\{e_i\}$ in $\mathfrak{g}$ such that $[e_i, e_j]$ is a linear combination of the elements $e_k, e_{k+1}, \ldots, e_m$, where $k = \max(i, j) + 1$.*

A Lie algebra $\mathfrak{g}$ is said to be *engelian* if all the operators $\operatorname{ad} x$, $x \in \mathfrak{g}$, are nilpotent. Corollary 2 implies that a finite-dimensional Lie algebra is engelian if and only if it is nilpotent. For an infinite-dimensional Lie algebra this statement does not hold, in general. If, however, $\mathfrak{g}$ is finitely generated and $(\operatorname{ad} x)^k = 0$ for some $k \in \mathbb{N}$ and all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

We also note that a stronger version of Engel's theorem is also valid, namely its conclusion holds for linear representations ρ of a Lie algebra $\mathfrak{g}$ such that $\rho(\mathfrak{g})$ is generated (as a Lie algebra) by a set of nilpotent operators closed under the commutator.

The next theorem lists other important properties of nilpotent Lie algebras proved with the use of Engel's theorem.

Theorem 1.5 (see Bourbaki [1975], Jacobson [1955], Serre [1987]). *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra. Then*

(i) $\operatorname{codim}[\mathfrak{g},\mathfrak{g}] \geq 2$.

(ii) *If* $\mathfrak{a}$ *is a subspace in* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g},\mathfrak{g}]$, *then* $\mathfrak{a}$ *generates* $\mathfrak{g}$ *as a Lie algebra.*

(iii) *If* $\mathfrak{h}$ *is an ideal in* $\mathfrak{g}$, *then* $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}) \neq 0$.

(iv) *If* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$, *then its normalizer* $\mathfrak{n}(\mathfrak{h})$ *strictly contains* $\mathfrak{h}$.

Finally, we note the following application of Engel's theorem to the theory of nilpotent Lie groups.

Theorem 1.6. *A connected Lie group* G *is nilpotent if and only if all operators* $\operatorname{Ad} g$ $(g \in G)$ *are unipotent. Any compact subgroup of a connected nilpotent Lie group* G *is contained in* $Z(G)$.

Proof. The first statement follows from Corollary 2 to Theorem 1.4 and the correspondence between nilpotent Lie groups and Lie algebras (see Vinberg and Onishchik [1988], Chap. 2, Theorem 5.13). To prove the second statement, consider the restriction R of the representation Ad to a compact subgroup $L \subset G$. Since R is completely reducible (see below Chap. 4, Corollary to Proposition 2.1), Corollary 1 to Theorem 1.4 implies that R is trivial. Hence $L \subset \operatorname{Ker} \operatorname{Ad} = Z(G)$.

1.4. An Analogue of Engel's Theorem in Group Theory. The following theorem can be considered as a group-theoretical analogue of Engel's theorem. It is not a formal consequence of Engel's theorem because it applies to groups that are not necessarily Lie groups.

Theorem 1.7 (Kolchin, see Merzlyakov [1987], Serre [1987]). *Let* G *be a group, and* $R\colon G \to \mathrm{GL}(V)$ *a linear representation of it over a field* K. *Suppose that* $V \neq 0$ *and all operators* $R(g)$, $g \in G$, *are unipotent. Then* $\chi \equiv 1$ *is a weight of the representation* R.

Proof. Consider the system of linear equations $(R(g) - E)v = 0$, where g runs over the entire group G. Since we are looking for nontrivial solutions of the system, the field K can be assumed to be algebraically closed. Replacing V by its minimal nonzero invariant subspace, one can also assume that R is irreducible. The Burnside theorem (see Kirillov [1987]) implies that the operators $R(g)$, $g \in G$, generate $\mathfrak{gl}(V)$ as a vector space.

On the other hand, let $Z = R(g) - E$. Then $\operatorname{tr} R(g) = \operatorname{tr} E + \operatorname{tr} Z = \dim \mathfrak{g}$ does not depend on $g \in G$. If $g,\ g' \in G$, then

$$\operatorname{tr}(ZR(g')) = \operatorname{tr}((R(g) - E)R(g')) = \operatorname{tr} R(gg') - \operatorname{tr} R(g') = 0.$$

Hence $\operatorname{tr}(ZX) = 0$ for any $X \in \mathfrak{gl}(V)$, whence $Z = 0$, i.e. $\rho(g) = E$. □

§ 2. The Cartan Criterion

2.1. Invariant Bilinear Forms. Let G be a Lie group over a field K. A bilinear form b on the tangent algebra $\mathfrak{g}$ of the group G is said to be *invariant* if

$$b((\operatorname{Ad} g)x, (\operatorname{Ad} g)y) = b(x, y) \tag{1}$$

for all $g \in G$, $x, y \in \mathfrak{g}$. It follows from formula (18) in Vinberg and Onishchik [1988], Chap. 2 that the invariant form b satisfies the relation

$$b([x, y], z) + b(y, [x, z]) = 0 \tag{2}$$

for all $x, y, z \in \mathfrak{g}$. Conversely, relation (2) implies (1) if G is connected. A bilinear form b on an arbitrary Lie algebra $\mathfrak{g}$ is said to be *invariant* if it satisfies property (2).

Example 1. Let E be a three-dimensional Euclidean space with the scalar product (,). Fix an orientation in E and consider the vector product in E. Then E becomes a Lie algebra over $\mathbb{R}$ such that the form (,) is invariant.

Example 2. In the Lie algebra $\mathfrak{gl}(V)$ of linear transformations of a vector space V over K there is an invariant bilinear form

$$b(X, Y) = \operatorname{tr}(XY). \tag{3}$$

Example 3. Let $\mathfrak{g}$ be a Lie algebra over K, and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a linear representation of it. Then the symmetric bilinear form

$$b_\rho(x, y) = \operatorname{tr}(\rho(x)\rho(y))$$

is invariant on $\mathfrak{g}$. In particular, there is an invariant bilinear form

$$k_\mathfrak{g}(x, y) = b_{\mathrm{ad}}(x, y) = \operatorname{tr}((\operatorname{ad} x)(\operatorname{ad} y)),$$

called the *Killing form* of the algebra $\mathfrak{g}$.

In what follows we always assume that an invariant bilinear form b on a Lie algebra $\mathfrak{g}$ is symmetric. The following assertions are proved without difficulty.

Proposition 2.1. *Let b be an invariant bilinear form on a Lie algebra $\mathfrak{g}$ and $\mathfrak{a}$ an ideal in $\mathfrak{g}$. Then $\mathfrak{a}^\perp = \{x \in \mathfrak{g} | b(x, y) = 0 \quad \forall y \in \mathfrak{a}\}$ is also an ideal in $\mathfrak{g}$. If $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{a}^\perp \supset \mathfrak{z}(\mathfrak{g})$, and if b is nondegenerate, then $\mathfrak{a}^\perp = \mathfrak{z}(\mathfrak{g})$.*

Proposition 2.2 *The Killing form $k = k_\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$ satisfies the relation*

$$k(a(x), a(y)) = k(x, y)$$

for all $x, y \in \mathfrak{g}$ and any $a \in \operatorname{Aut} \mathfrak{g}$. If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then the restriction of the form $k_\mathfrak{g}$ to $\mathfrak{a}$ coincides with $k_\mathfrak{a}$.

2.2. Criteria of Solvability and Semisimplicity. In this section we denote by b the invariant bilinear form in $\mathfrak{gl}(V)$ defined by formula (3).

Theorem 2.1. *A subalgebra* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *is solvable if and only if* $b([X,Y],Z) = 0$ *for all* $X, Y, Z \in \mathfrak{g}$.

Proof. For the proof of Theorem 2.1 one can assume that $K = \mathbb{C}$ (the real case is reduced to the complex one by considering the complexification, i.e. the Lie algebra $\mathfrak{g}(\mathbb{C}) = \mathfrak{g} + i\mathfrak{g} \subset \mathfrak{gl}(V(\mathbb{C}))$ (see Sect. 7)). For any $X \in \mathfrak{gl}(V)$ denote by X_s and X_n the semisimple and nilpotent components respectively in the additive Jordan decomposition $X = X_s + X_n$, (see Springer [1989], Sect. 3.1.1). Denote by $\overline{X}_s$ the semisimple operator having the same eigenvectors as X_s but with the complex conjugate eigenvalues. Let $b([X,Y],Z) = 0$ for all $X, Y, Z \in \mathfrak{g}$. By virtue of Engel's theorem, it is sufficient to prove that $X_s = 0$ for any $X \in [\mathfrak{g}, \mathfrak{g}]$. Write $X = \sum_{i=1}^{p} [X_i, Y_i]$, where $X_i, Y_i \in \mathfrak{g}$. Then

$$b(X, \overline{X}_s) = \sum_{i=1}^{p} b([X_i, Y_i], \overline{X}_s) = \sum_{i=1}^{p} b(Y_i, [\overline{X}_s, X_i]).$$

The relation $(\operatorname{ad} X)(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ and the equality $\operatorname{ad} \overline{X}_s = (\overline{\operatorname{ad} X})_s$ imply that $(\operatorname{ad} \overline{X}_s)(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. Hence $b(X, \overline{X}_s) = \operatorname{tr}(X\overline{X}_s) = 0$, whence $X_s = 0$. The converse statement easily follows from Lie's theorem. □

Corollary. *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $k_{\mathfrak{g}}([x,y],z) = 0$ *for all* $x, y, z \in \mathfrak{g}$ *or if* $k_{\mathfrak{g}}(x,y) = 0$ *for all* $x, y \in [\mathfrak{g}, \mathfrak{g}]$.

Theorem 2.2. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if the Killing form* $k_{\mathfrak{g}}$ *is nondegenerate.*

Proof. Let $\mathfrak{u} = \{x \in \mathfrak{g} | k_{\mathfrak{g}}(x,y) = 0 \quad \forall y \in \mathfrak{g}\}$. By virtue of Proposition 2.1, $\mathfrak{u}$ is an ideal in $\mathfrak{g}$, while Theorem 2.1 implies that the Lie algebra $\operatorname{ad}\mathfrak{u}$ is solvable. Since $\operatorname{ad}\mathfrak{u} \simeq \mathfrak{u}$, we have $\mathfrak{u} = 0$ if $\mathfrak{g}$ is semisimple. Conversely, if $\mathfrak{g}$ is not semisimple, and $\mathfrak{a}$ is its nonzero abelian ideal, then $\mathfrak{a} \subset \mathfrak{u}$ because $((\operatorname{ad} x)(\operatorname{ad} y))^2 = 0$ for all $x \in \mathfrak{a}$, $y \in \mathfrak{g}$. □

Remark. A similar proof yields the following assertion: if ρ is a faithful linear representation of a semisimple Lie algebra $\mathfrak{g}$, then the form b_ρ (see Example 3) is nondegenerate on $\mathfrak{g}$.

Corollary. *If* $\mathfrak{g}$ *is semisimple, then* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

2.3. Factorization into Simple Factors

Proposition 2.3. *If* $\mathfrak{g}$ *is semisimple, and* $\mathfrak{a}$ *is an ideal in* $\mathfrak{g}$, *then* $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{a}^\perp$. *Any ideal* $\mathfrak{a} \subset \mathfrak{g}$ *and the quotient algebra* $\mathfrak{g}/\mathfrak{a}$ *are semisimple.*

Proof. As in the proof of Theorem 2.2, one can verify that $\mathfrak{a} \cap \mathfrak{a}^\perp$ is a solvable ideal in $\mathfrak{g}$. □

The following theorem is now derived without difficulty.

Theorem 2.3. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g}$ *can be decomposed into the direct sum*