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Yu. G. Reshetnyak (Ed.)

Geometry IV

Non-regular Riemannian Geometry

几 何 IV

非正规黎曼几何

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Preface

The book contains a survey of research on non-regular Riemannian geometry, carried out mainly by Soviet authors. The beginning of this direction occurred in the works of A.D. Aleksandrov on the intrinsic geometry of convex surfaces. For an arbitrary surface F, as is known, all those concepts that can be defined and facts that can be established by measuring the lengths of curves on the surface relate to intrinsic geometry. In the case considered in differential geometry the intrinsic geometry of a surface is defined by specifying its first fundamental form. If the surface F is non-regular, then instead of this form it is convenient to use the metric ρ_F, defined as follows. For arbitrary points $X, Y \in F$, $\rho_F(X, Y)$ is the greatest lower bound of the lengths of curves on the surface F joining the points X and Y. Specification of the metric ρ_F uniquely determines the lengths of curves on the surface, and hence its intrinsic geometry. According to what we have said, the main object of research then appears as a metric space such that any two points of it can be joined by a curve of finite length, and the distance between them is equal to the greatest lower bound of the lengths of such curves. Spaces satisfying this condition are called spaces with intrinsic metric. Next we introduce metric spaces with intrinsic metric satisfying in one form or another the condition that the curvature is bounded. This condition is introduced by comparing triangles in space with triangles on a surface of constant curvature having the same lengths of sides.

The book contains two articles. The first is devoted to the theory of two-dimensional manifolds of bounded curvature. This theory at present has a complete character. It is a generalization of two-dimensional Riemannian geometry. For a manifold of bounded curvature there are defined the concepts of area and integral curvature of a set, the length and turn (integral curvature) of a curve.

One of the main results of the theory is the closure of the class of two-dimensional manifolds with respect to the passage to the limit under certain natural restrictions. In particular, this enables us to define two-dimensional manifolds of bounded curvature by means of approximation by polyhedra. The proof of the possibility of such an approximation is one of the main results of the theory. In the account given here it is essential to use the analytic representation of two-dimensional manifolds of bounded curvature by means of a line element of the form $ds^2 = \lambda(z)(dx^2 + dy^2)$. The function $\lambda(z)$ is such that its logarithm is the difference of two subharmonic functions. In contrast to the case of Riemannian manifolds the function λ here may vanish and have points of discontinuity. Some results in the theory of manifolds of bounded curvature do not have a complete analogue in two-dimensional Riemannian geometry. Here we should refer to some estimates and solutions of extremal problems, the theorem on pasting, and so on.

The second article is devoted to the theory of metric spaces whose curvature is contained between certain constants K_1 and K_2, where $K_1 < K_2$. The main result of this theory is that these spaces are actually Riemannian. In each such space we can locally introduce a coordinate system in which its metric is defined by a line element $ds^2 = g_{ij}\, dx^i\, dx^j$, where the functions g_{ij} satisfy almost the same regularity conditions as in ordinary Riemannian geometry. (We say "almost the same" because the functions g_{ij} only have second derivatives, generalized in the sense of Sobolev, that are summable in any degree $p > 0$; this implies that the coefficients g_{ij} are continuous.) The theory of curvature in Riemannian geometry can be transferred to the case of such spaces. Some relations here are satisfied only almost everywhere (for example, the formula for representing the sectional curvature of a manifold). In this article the authors also consider some questions of Riemannian geometry. Applications are given of the theorem on the Riemann property of spaces of two-sided bounded curvature to global Riemannian geometry.

In particular, an axiomatic definition of a Riemannian space is obtained here, based on representations in the spirit of synthetic geometry. A priori it is not required that the spaces under consideration should be manifolds. This fact follows from other axioms.

Yu.G. Reshetnyak

I. Two-Dimensional Manifolds of Bounded Curvature

Yu.G. Reshetnyak

Translated from the Russian
by E. Primrose

Contents

Chapter 1
Preliminary Information

§ 1. Introduction

1.1. General Information about the Subject of Research and a Survey of Results. The theory of two-dimensional manifolds of bounded curvature is a generalization of two-dimensional Riemannian geometry. Formally a two-dimensional manifold of bounded curvature is a two-dimensional manifold in which there are defined the concepts of the length of a curve, the angle between curves starting from one point, the area of a set, and also the integral curvature of a curve and the integral curvature of a set. For the case when the given manifold is Riemannian, the integral curvature of a curve is equal to the integral of the geodesic curvature along the length of the curve, and the integral curvature of a set is equal to the integral of the Gaussian curvature of the manifold with respect to the area. The remaining concepts in this case have the meaning that is usual in Riemannian geometry. For an arbitrary two-dimensional manifold of bounded curvature the integral curvature is a completely additive set function, which may not admit representations in the form of an integral with respect to area.

Another particular case of two-dimensional manifolds of bounded curvature consists of surfaces of polyhedra (not necessarily convex) in three-dimensional Euclidean space. For them the integral curvature is an additive set function concentrated on some discrete set, namely the set of vertices of the polyhedron. If the set consists of a unique point, a vertex of the polyhedron, then its integral curvature is equal to $2\pi - \theta$, where θ is the total angle of the polyhedron at this vertex, that is, the sum of the angles of all its faces that meet at this vertex.

Three methods are known for introducing two-dimensional manifolds of bounded curvature. The first of them is *axiomatic*. A two-dimensional manifold of bounded curvature is defined as a metric space satisfying some special axioms. The second method is based on *approximation* by two-dimensional Riemannian manifolds or manifolds with polyhedral metric. It turns out that under certain natural assumptions the limit of the sequence of two-dimensional manifolds of bounded curvature is also a manifold of bounded curvature. Exact formulations are given later; here we just mention that a certain condition of boundedness of the curvature is fundamental in these assumptions. In particular, the limit of a sequence of manifolds with polyhedral metric is a manifold of bounded curvature. This fact can be used for the definition of the class of two-dimensional manifolds of bounded curvature.

A *two-dimensional Riemannian manifold* is a smooth manifold such that for each local coordinate system there is defined in it a differential quadratic form

$$ds^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(x_1, x_2)\, dx_i\, dx_j.$$

The main concept of two-dimensional Riemannian geometry is the *Gaussian curvature*. In order that it can be defined it is necessary to require that the coefficients g_{ij} $(i, j = 1, 2)$ have partial derivatives of the first and second orders. There naturally arises the idea of considering "generalized" Riemannian geometries obtainable if we weaken the requirements of regularity imposed on the coefficients of the quadratic form ds^2. It turns out that two-dimensional manifolds of bounded curvature can be defined in such a way. We shall show later how to do this. Here we consider the case when the line element of the manifold has a certain special structure, namely such that $g_{11} = g_{22}$, $g_{12} \equiv 0$. For the case of Riemannian manifolds the line element can always be reduced to such a form by a transformation of the coordinates. The system of coordinates for which $g_{11} \equiv g_{22}$, $g_{12} \equiv 0$ is called *isothermal*. Using such a form of the coordinate system, we obtain a third *analytic* method of introducing two-dimensional manifolds of bounded curvature.

The general plan of the theory of two-dimensional manifolds of bounded curvature is due to A.D. Aleksandrov, who developed the geometrical aspects of this theory (see Aleksandrov (1948b), (1948c), (1949b), (1950), (1954), (1957a), (1957b), Aleksandrov and Burago (1965), Aleksandrov and Strel'tsov (1953), (1965), Aleksandrov, Borisov and Rusieshvili (1975). An account of the theory constructed by Aleksandrov is given in a monograph of Aleksandrov and Zalgaller (1962). An analytic approach to the introduction and study of two-dimensional manifolds of bounded curvature is due to Yu.G. Reshetnyak (Reshetnyak (1954), (1959), (1960), (1961b), (1962), (1963a), (1963b)). Other authors also took part in the development of individual aspects of the theory (the corresponding references are given later in the main text).

The concept of a two-dimensional manifold of bounded curvature was introduced as a development of the research of Aleksandrov on the intrinsic geometry of convex surfaces (Aleksandrov (1944), (1945a), (1945b), (1947), (1948a)), and presented completely in his monograph Aleksandrov (1948a).

Chapter I of this article has an auxiliary character. Two-dimensional manifolds of bounded curvature are defined as metric spaces satisfying certain special conditions. One of these conditions is that the metric of the space must be intrinsic. In § 2 we give necessary conditions and a summary of the basic facts relating to the theory of metric spaces with intrinsic metric. In § 3 we consider two-dimensional manifolds with intrinsic metric. Here we go into details on the definition of the operations of cutting up and pasting such manifolds. In addition, the concept of a side of a simple arc in a two-dimensional manifold has important significance for what follows.

In § 4 we give a summary of the basic results of two-dimensional Riemannian geometry. The main information concerning two-dimensional manifolds with polyhedral metric is contained in § 5. In particular, for such manifolds we define the concepts of integral curvature (or the turn) of a curve and the curvature of a set, and we study the structure of a shortest curve on a two-dimensional polyhedron. Polyhedra play a special role in the theory of manifolds of bounded curvature. By approximating an arbitrary manifold by polyhedra, in many cases it turns out to be possible to reduce the solution of this or that problem to the case of polyhedra, for which it becomes a problem with respect to the formulation belonging to elementary geometry. This enables us to use for the solution of such problems arguments based on intuitive geometric representations. For example, some extremal problems are related to a number of problems for which such a way of action leads to success.

The definition of two-dimensional manifolds of bounded curvature is given in § 6. We regard the axiomatic definition as fundamental. The following fact relating to classical Riemannian geometry is well known. Let T be a *triangle* in a two-dimensional Riemannian manifold, that is, a domain homeomorphic to a disc whose boundary is formed by three geodesics. We denote by α, β and γ the angles of this domain at the vertices of the triangle T and let $\omega(T)$ be the integral over T of the Gaussian curvature with respect to area. We put $\delta(T) = \alpha + \beta + \gamma - \pi$. The quantity $\delta(T)$ is called the *excess* of the triangle T. As we know, $\delta(T) = \omega(T)$. (This statement is a special case of the Gauss-Bonnet theorem; see § 4.) If the Gaussian curvature is non-negative, then it follows that $\delta(T) \geqslant 0$ for any triangle.

Let U be an arbitrary domain in a two-dimensional Riemannian manifold. For any system of pairwise non-overlapping geodesic triangles $T_i \subset U$, $i = 1, 2, \ldots, m$, we have the inequality

$$\sum_{i=1}^{m} \omega(T_i) \leqslant \int_U [\mathscr{K}(x)]^+ \, d\sigma(x),$$

where $\mathscr{K}$ is the Gaussian curvature, $d\sigma$ is the element of area, $a^+ = \max\{a, 0\}$. This property is taken as the basis for constructing the axiomatics of a two-dimensional manifold of bounded curvature. A two-dimensional manifold of bounded curvature is defined as a certain metric space. A geodesic is a curve, any sufficiently small arc of which is a shortest curve, that is, such that its length is equal to the distance between the ends. The concept of a shortest curve is naturally defined for the case of metric spaces. It is also clear what we need to call a triangle. In order to define the concept of the excess for a triangle in an arbitrary metric space we need to know what the angle between two curves starting from one point is, in the given case the angle between the sides of the triangle. The corresponding definition is given in § 6. A manifold of bounded curvature can be defined as a metric space that is a two-dimensional manifold and is such that for any point of it there is a neighbourhood U for which the sum of the excesses of pairwise non-overlapping geodesic triangles contained in U does not exceed some constant $C(U) < \infty$, however these triangles are chosen. The exact formulations are given in § 6. The final version of the axiomatics of two-dimensional manifolds of bounded curvature is defined by the argument that of the different equivalent forms of the axiomatics we must choose the weakest.

One of the main results of the theory of two-dimensional manifolds of bounded curvature is the characterization of such manifolds by means of approximation by two-dimensional polyhedra, or, which reduces to the same thing, by two-dimensional Riemannian manifolds. The difficulties that must be overcome here are connected with the fact that starting from the axioms of a manifold of bounded curvature it is required to establish some very deep properties of it. In § 6 of this article we give an outline of the proof of theorems on the approximation of a two-dimensional manifold of bounded curvature by Riemannian manifolds. The reader can find complete proofs in the monograph Aleksandrov and Zalgaller (1962). The proof of the necessary conditions is based on arguments that are a development of the ideas worked out by Aleksandrov in the study of the intrinsic geometry of convex surfaces. The proof of the sufficient conditions outlined in § 6 is based on arguments different from those given in Aleksandrov and Zalgaller (1962). (For a complete account of this proof, see Reshetnyak (1962).)

The analytic characteristic of two-dimensional manifolds of bounded curvature is given in § 7. We dwell on it in more detail, bearing in mind the fact that for specialists thinking in terms of categories of mathematical analysis it is the shortest path towards determining what is a two-dimensional manifold of bounded curvature.

We first consider Riemannian manifolds. In a neighbourhood of any point of such a manifold we can introduce a coordinate system in which the line element of the manifold is expressed by the formula

$$ds^2 = \lambda(x, y)(dx^2 + dy^2).$$

(As we said above, such a coordinate system is called isothermal.) The Gaussian curvature $\mathscr{K}$ of a given manifold in this coordinate system admits the representation

$$\mathscr{K}(x, y) = -\frac{1}{2\lambda(x, y)} \Delta \ln \lambda(x, y).$$

Using known results of potential theory, we thus obtain

$$\ln \lambda(z) = \frac{1}{\pi} \iint_G \ln \frac{1}{|z - \zeta|} \mathscr{K}(\zeta)\lambda(\zeta)\, d\xi\, d\eta + h(z).$$

Here $z = (x, y)$, $\zeta = (\xi, \eta)$, G is a domain on the plane, and $h(z)$ is a harmonic function. We now observe that for an arbitrary set $E \subset G$

$$\omega(E) = \iint_E \mathscr{K}(\zeta)\lambda(\zeta)\, d\xi\, d\eta$$

is the integral curvature of the corresponding set in the Riemannian manifold. By virtue of this the integral representation for $\ln \lambda(z)$ given above can be written in the form

$$\ln \lambda(z) = \frac{1}{\pi} \iint_G \ln \frac{1}{|z - \zeta|}\, d\omega(\zeta) + h(z). \tag{1}$$

The last relation naturally suggests that if we wish to have generalized Riemannian manifolds in some sense, for which the integral curvature is an arbitrary completely additive set function, then it is sufficient in (1) to substitute such a function for ω, and then to consider the geometry defined by the corresponding line element $ds^2 = \lambda(z)(dx^2 + dy^2)$. Such a path leads to a *two-dimensional manifold of bounded curvature.*

In § 7 we give only drafts of the necessary proofs. A complete account of them can be found in Reshetnyak (1954), (1960), (1961a).

For an arbitrary two-dimensional manifold of bounded curvature there are defined the concepts of integral curvature and area of a set, and the integral curvature (or the turn) of a curve. In § 8 we show how all these concepts can be defined. We rely on the analytic representation of two-dimensional manifolds of bounded curvature described in § 7.

In § 8 we give a survey of the main results of the theory of two-dimensional manifolds of bounded curvature. Here we are concerned first of all with a theorem on pasting of two-dimensional manifolds of bounded curvature and theorems on passage to the limit. The class of two-dimensional manifolds of bounded curvature turns out to be closed with respect to passages to the limit under significantly weaker assumptions than for the class of Riemannian manifolds.

Among the main results of the theory of manifolds of bounded curvature there are, in particular, those that concern extremal problems for manifolds of bounded curvature. One of the main instruments for research is the method of cutting and pasting created by Aleksandrov. This method uses essentially the specific character of two-dimensional manifolds of bounded curvature. The totality of all such manifolds is invariant with respect to operations connected with the method indicated, which we cannot say, for example, about the class of Riemannian manifolds.

In § 9 of this chapter we give a survey of further research into the theory of two-dimensional manifolds of bounded curvature. The author has tried to express everything that is most essential in this topic.

1.2. Some Notation and Terminology. Later we assume that the concepts of topological and metric spaces are known, like all the basic facts of general topology. In particular, we assume that the reader knows what is a neighbourhood of a point in a topological space, an open or closed set, a connected component, and so on.

Let us recall some standard notation, used in what follows.

Let A be a set in a topological space $\mathfrak{R}$. Then $\overline{A}$ denotes the closure of A, A^0 denotes the totality of all interior points of A, and $\partial A = \overline{A} \setminus A^0$ denotes the boundary of A.

The symbol $\mathbb{R}^n$ denotes the n-dimensional arithmetic Euclidean space of points $x = (x_1, x_2, \dots, x_n)$, where $x_1, x_2, \dots, x_n$ are arbitrary real numbers. For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ we put

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

For arbitrary points $x, y \in \mathbb{R}^n$ the distance between x and y is assumed to be equal to $|x - y|$. The function $\rho: (x, y) \to |x - y|$ is the *metric*. In a well-known way a given metric defines some topology in $\mathbb{R}^n$. Speaking of $\mathbb{R}^n$ as a topological space, we shall always have this topology in mind.

The space $\mathbb{R}^2$ will be called an *arithmetic Euclidean plane*. The symbol $\mathbb{C}$ denotes the set of complex numbers. Later we shall often identify $\mathbb{R}^2$ and $\mathbb{C}$, regarding the point $(x, y) \in \mathbb{R}^2$ and the complex number $z = x + iy$ as one and the same object.

The usual Euclidean plane is denoted later by the symbol $\mathbb{E}^2$. As a metric space $\mathbb{E}^2$ is isometric to $\mathbb{R}^2$.

Let $B(0, 1)$ be the open disc $\{(x, y) | x^2 + y^2 < 1\}$ in the plane $\mathbb{R}^2$, and $\bar{B}(0, 1)$ the closed disc $\{(x, y) | x^2 + y^2 \leqslant 1\}$.

Henceforth the statement that some set in a topological space is homeomorphic to a disc (a closed disc) always means that this set is homeomorphic to the disc $B(0, 1)$ (respectively, the disc $\bar{B}(0, 1)$.

§ 2. The Concept of a Space with Intrinsic Metric

2.1. The Concept of the Length of a Parametrized Curve. We assume that the concept of a metric space and some of the simplest information relating to it are known.

Let M be a set in which a metric ρ is specified. We shall denote the metric space obtained in this way by the symbol (M, ρ). This notation is appropriate in that later there will often arise the necessity of considering different metrics on the same set. When no misunderstanding is possible we shall simply talk about a metric space M.

Let M be a metric space and ρ its metric. A *parametrized curve* or *path* in the space M is any continuous map $x: [a, b] \to M$ of the interval $[a, b]$ of the set of real numbers $\mathbb{R}$ into M. We shall say that the path $x: [a, b] \to M$ joins the points $X, Y \in M$ if $x(a) = X$, $x(b) = Y$.

A metric space M with metric ρ is called *linearly connected* if for any two points X, Y of it there is a path joining these points.

A path $x: [a, b] \to M$ is called *simple* if it is a one-to-one map of the interval $[a, b]$. A set L in the space M is called a *simple arc* if there is a simple path $x: [a, b] \to M$ such that $L = x([a, b])$. Any simple path $x: [a, b] \to M$ satisfying this condition is called a *parametrization of the simple arc L*.

A set Γ in a metric space (M, ρ) is called a *simple closed curve* in M if it is a topological image of the circle $S(0, 1)$ on the plane $\mathbb{R}^2$. If Γ is a simple closed curve in the metric space M, then there is a path $x: [a, b] \to M$ such that $x(a) = x(b)$, $x([a, b]) = \Gamma$ and for any $t_1, t_2 \in [a, b]$ such that $t_1 \neq t_2$ and at least one of the points t_1 and t_2 is not an end of the interval $[a, b]$ the points $x(t_1)$ and $x(t_2)$ are distinct.

We shall call a set $L \subset M$ a *simple curve* if L is closed and is either a simple closed curve in M or a topological image of an arbitrary interval of the number line $\mathbb{R}$ (which, generally speaking, may not be closed).

Suppose we are given a path $x\colon [a, b] \to M$ in a metric space M. We specify arbitrarily a finite sequence $\alpha = \{t_0, t_1, \ldots, t_m\}$ of points of the interval $[a, b]$ such that $t_0 = a \leqslant t_1 \leqslant \cdots \leqslant t_m = b$ and put

$$s(x, \alpha) = \sum_{i=1}^{m} \rho[x(t_{i-1}), x(t_i)].$$

The least upper bound of $s(x, \alpha)$ on the totality of all sequences α satisfying the conditions mentioned above is called the *length of the path* x and denoted by the symbol $s_\rho(x; a, b)$ or simply $s(x; a, b)$ when no misunderstanding is possible. (The notation $s_\rho(x; a, b)$ is necessary for those cases when we consider different metrics in M and compare the lengths of the path $x\colon [a, b] \to M$ in these metrics.)

We mention the following properties of length that follow immediately from the definition.

I. Any path $x\colon [a, b] \to M$ in the space (M, ρ) satisfies the inequality

$$\rho[x(a), x(b)] \leqslant s_\rho(x; a, b).$$

II. Suppose we are given a path $x\colon [a, b] \to M$. Then for any c such that $a < c < b$ we have

$$s_\rho(x; a, b) = s_\rho(x; a, c) + s_\rho(x; c, b).$$

III. Suppose we are given a sequence of paths $(x_\nu\colon [a, b] \to M)$, $\nu = 1, 2, \ldots$ and a path $x_0\colon [a, b] \to M$. We assume that $x_0(t) = \lim\limits_{t \to 0} x_\nu(t)$ for any $t \in [a, b]$. Then

$$s_\rho(x; a, b) \leqslant \varliminf_{\nu \to \infty} s_\rho(x_\nu; a, b).$$

Let L be a simple arc in the metric space (M, ρ). We specify arbitrarily a parametrization $x\colon [a, b] \to M$ of the arc L. Then it is easy to establish that $s_\rho(x; a, b)$ does not depend on the choice of parametrization x of the arc L. In this case we shall call $s_\rho(x; a, b)$ the *length of the simple arc* L and denote it by $s_\rho(L)$ or simply $s(L)$.

Similarly, if Γ is a simple closed curve and $x\colon [a, b] \to M$ is an arbitrary parametrization of it, then $s_\rho(x; a, b)$ does not depend on the choice of this parametrization and is denoted henceforth by $s_\rho(\Gamma)$ or simply $s(\Gamma)$.

Let L be a simple arc in the metric space M, and $x\colon [a, b] \to L$ a parametrization of L. The points $A = x(a)$ and $B = x(b)$ are called the *end-points* of L. All the remaining points of L are called *interior points* of it. Let $X = x(t_1)$ and $Y = x(t_2)$, $t_1 < t_2$, be two arbitrary points of the simple arc. The set of all points $Z = x(t)$, where $t_1 \leqslant t \leqslant t_2$, is obviously a simple arc. We shall denote it by $[XY]$. From property I of the length of a parametrized curve it follows that for any simple arc L with end-points A and B we have

$$\rho(A, B) \leqslant s(L).$$

From property II it follows that for any interior point C of the simple arc

$$s([AB]) = s([AC]) + s([CB])$$

(A and B are the end-points of L).

Let Γ be a simple closed curve and A an arbitrary point of it. Then there is a parametrization $x\colon [a, b] \to M$ of Γ such that $x(a) = x(b) = A$. Any two distinct points X, Y of the simple closed curve Γ split it into two simple arcs, which we denote by Γ_1 and Γ_2. From Property II of the length of a parametrized curve it follows that

$$s_\rho(\Gamma) = s_\rho(\Gamma_1) + s_\rho(\Gamma_2).$$

In the given definitions, in principle, there can be an infinite value of the length. If the length of the path $x\colon [a, b] \to M$ in the metric space (M, ρ) is finite, then the given path is called *rectifiable*. Similarly, a simple arc (simple closed curve) is called rectifiable if its length is finite.

Let L be a simple arc in the space (M, ρ). We assume that L is rectifiable. Then it admits a parametrization $x\colon [0, l] \to M$ such that s is equal to the length of the arc $[x(0)x(s)]$ for each $s \in [0, l]$.

2.2. A Space with Intrinsic Metric. The Induced Metric. Suppose we are given a metric space M and a set $A \subset M$. We shall say that the path $x\colon [a, b] \to M$ lies in the set A (or goes into A) if $x(t) \in A$ for all $t \in [a, b]$.

A set A in a metric space (M, ρ) is said to be *metrically connected* if for any two of its points there is a rectifiable path joining these points and lying in the set A. In particular, the space (M, ρ) itself is said to be metrically connected if for any two of its points X, Y there is a rectifiable path joining these points.

A metric space (M, ρ) is called a *space with intrinsic metric* if it is linearly connected and for any two of its points X, Y the quantity $\rho(X, Y)$ is equal to the greatest lower bound of lengths of arcs joining these points.

If (M, ρ) is a space with intrinsic metric, then M is metrically connected.

Suppose, for example, that M is the usual plane $\mathbb{E}^2$. For arbitrary points X, $Y \in \mathbb{E}^2$ suppose that $\rho(X, Y) = 0$ if $X = Y$ and that $\rho(X, Y)$ is equal to the length of the interval with end-points X and Y if $X \neq Y$. The metric defined in this way on the plane $\mathbb{E}^2$ is obviously intrinsic.

Similarly, if M is a sphere Σ_K of radius $r = 1/\sqrt{K}$ in the space $\mathbb{E}^3$, then taking for $\rho(X, Y)$ the length of the shortest arc of the great circle passing through the points X and Y, we obtain an intrinsic metric on the sphere Σ_K. At the same time, the metric $\rho_0(X, Y)$, where $\rho_0(X, Y)$ is the length of the interval in $\mathbb{E}^3$ joining the points X and Y on the sphere Σ_K, is not intrinsic.

The metric spaces known from analysis, namely Hilbert space and, more generally, any normed vector space, are spaces with intrinsic metric.

Let (M, ρ) be a metric space and $a \in M$ an arbitrary point of M. Let us specify arbitrarily a number $r > 0$. We denote the set of all points $x \in M$ such that $\rho(x, a) < r$ by the symbol $B(a, r)$ and call it the *open ball* with centre a and radius r. In certain cases considered later, instead of the word "ball" we shall say "disc".

The totality of all points $x \in M$ for which $\rho(x, a) = r$ is denoted by the symbol $S(a, r)$ and called the *sphere* with centre a and radius r. In those cases when $B(a, r)$ is called a *disc* we shall call the set $S(a, r)$ a *circle*. We put $\bar{B}(a, r) = B(a, r) \cup S(a, r)$. The set $\bar{B}(a, r)$ is called the *closed ball* with centre a and radius r.

We mention the following properties of spaces with intrinsic metric.

Theorem 2.2.1 (Aleksandrov and Zalgaller (1962)). *Let (M, ρ) be a space with intrinsic metric. If M is locally compact (that is, any point $X \in M$ has a neighbourhood whose closure is compact), then for any $r > 0$ the closed ball $\bar{B}(X, r)$ is a compact set.*

The metric space (M, ρ) is called *complete* if any sequence (x_ν), $\nu = 1, 2, \ldots$, of points of this space for which $\lim\limits_{\nu\to\infty,\mu\to\infty} \rho(x_\nu, x_\mu) = 0$ is convergent. According to a well-known theorem of Hausdorff, for any metric space (M, ρ) there is a complete metric space $(\bar{M}, \bar{\rho})$ such that $M \subset \bar{M}$, $\rho(x, y) = \bar{\rho}(x, y)$ for any $x, y \in M$, and the set M is everywhere dense in $\bar{M}$. The space $(\bar{M}, \bar{\rho})$ is unique in the following sense. If (M', ρ') is another metric space connected with (M, ρ) like $(\bar{M}, \bar{\rho})$, then there is a map $j: \bar{M} \to M'$ such that $j(\bar{M}) = M'$, $j(x) = x$ for any $x \in M$, and $\rho'[j(x), j(y)] = \bar{\rho}(x, y)$ for any $x, y \in \bar{M}$. We shall call $(\bar{M}, \bar{\rho})$ the *Hausdorff completion of the space* (M, ρ). Henceforth the metric of the Hausdorff completion will be denoted like the metric of the original space.

Theorem 2.2.2 (Aleksandrov and Zalgaller (1962)). *The Hausdorff completion of a metric space with intrinsic metric is also a space with intrinsic metric.*

We mention here a general scheme for constructing the metric. Suppose we are given a metric space (M, ρ), and let A be a connected set of this space. The set A with metric ρ is itself a metric space – a subspace of (M, ρ). Even if (M, ρ) is a space with intrinsic metric, the metric space (A, ρ) may not be of this kind. Let us define a metric in the set A, which we denote by ρ_A. Namely, for arbitrary points $X, Y \in A$ we denote by $\rho_A(X, Y)$ the greatest lower bound of lengths of paths in the space (M, ρ) joining the points X and Y and lying in the set A.

Theorem 2.2.3. *If $A \subset M$ is a metrically connected set of the space (M, ρ), then the function $(X, Y) \to \rho_A(X, Y)$ of a pair of points of A, defined in the way indicated above, is a metric on the set A. This metric is intrinsic and for any path $x: [a, b] \to M$ lying in the set A we have $s_\rho(x; a, b) = s_{\rho_A}(x; a, b)$.*

The metric ρ_A is called the *induced intrinsic metric* on the set A of the metric space (M, ρ).

Suppose, for example, that M is the three-dimensional Euclidean space $\mathbb{E}^3$, and that the set A is the sphere $S(a, R)$ in this space. It is easy to show that in the given case the quantity $\rho_A(X, Y)$ is equal to the length of the shortest arc of the great circle passing through the points X and Y, that is, the induced intrinsic metric on the sphere $S(a, R)$ coincides with the metric defined above.

Let L be a rectifiable simple arc in the metric space (M, ρ). Then for arbitrary points $X, Y \in L$ the quantity $\rho_L(X, Y)$ is equal to the length of the arc $[XY]$ of the curve L. We assume that Γ is a rectifiable simple closed curve in the metric