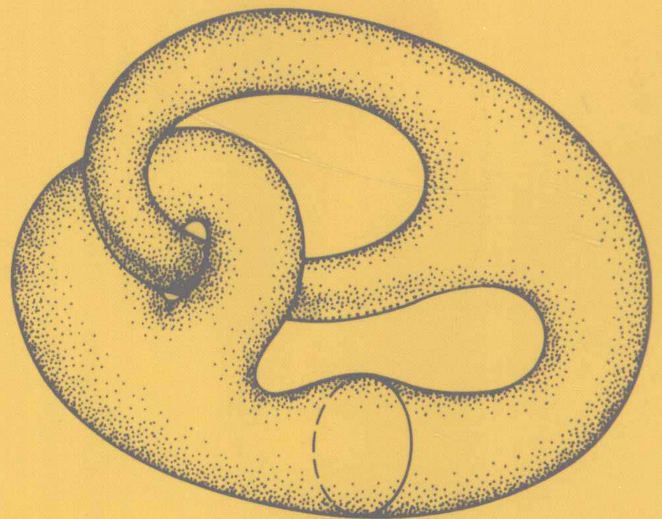
Undergraduate Texts in Mathematics

M. A. Armstrong

Basic Topology

基本拓扑学



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Basic Topology

With 132 Illustrations

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(continued after index)

Dedicated to the memory of

PAUL DUFÉTELLE

Preface to the Springer Edition

This printing is unchanged, though the opportunity has been taken to correct one or two misdemeanours. In particular Problems 2.13, 3.13 and 3.19 are now correctly stated, and Tietze has regained his final "e". My thanks go to Professor P. R. Halmos and to Springer-Verlag for the privilege of appearing in this series.

M.A.A.
Durham, January 1983.

Preface

This is a topology book for undergraduates, and in writing it I have had two aims in mind. Firstly, to make sure the student sees a variety of different techniques and applications involving point set, geometric, and algebraic topology, without delving too deeply into any particular area. Secondly, to develop the reader's geometrical insight; topology is after all a branch of geometry.

The prerequisites for reading the book are few, a sound first course in real analysis (as usual!), together with a knowledge of elementary group theory and linear algebra. A reasonable degree of 'mathematical maturity' is much more important than any previous knowledge of topology.

The layout is as follows. There are ten chapters, the first of which is a short essay intended as motivation. Each of the other chapters is devoted to a single important topic, so that identification spaces, the fundamental group, the idea of a triangulation, surfaces, simplicial homology, knots and covering spaces, all have a chapter to themselves.

Some motivation is surely necessary. A topology book at this level which begins with a set of axioms for a topological space, as if these were an integral part of nature, is in my opinion doomed to failure. On the other hand, topology should not be presented as a collection of party tricks (colouring knots and maps, joining houses to public utilities, or watching a fly escape from a Klein bottle). These things all have their place, but they must be shown to fit into a unified mathematical theory, and not remain dead ends in themselves. For this reason, knots appear at the end of the book, and not at the beginning. It is not the knots which are so interesting, but rather the variety of techniques needed to deal with them.

Chapter 1 begins with Euler's theorem for polyhedra, and the theme of the book is the search for topological invariants of spaces, together with techniques for calculating them. Topology is complicated by the fact that something which is, by its very nature, topologically invariant is usually hard to calculate, and vice versa the invariance of a simple number like the Euler characteristic can involve a great deal of work.

The balance of material was influenced by the maxim that a theory and its payoff in terms of applications should, wherever possible, be given equal weight. For example, since homology theory is a good deal of trouble to set up (a whole chapter), it must be shown to be worth the effort (a whole chapter of applications). Moving away from a topic is always difficult, and the temptation to include more and more is hard to resist. But to produce a book of reasonable length some topics just have to go; I mention particularly in this respect the omission of any systematic method for calculating homology groups. In formulating definitions, and choosing proofs, I have not always taken the shortest path. Very often the version of a definition or result which is most convenient to work with, is not at all natural at first sight, and this is above all else a book for beginners.

Most of the material can be covered in a one-year course at third-year (English) undergraduate level. But there is plenty of scope for shorter courses involving a selection of topics, and much of the first half of the book can be taught to second-year students. Problems are included at the end of just about every section, and a short bibliography is provided with suggestions for parallel reading and as to where to go next.

The material presented here is all basic and has for the most part appeared elsewhere. If I have made any contribution it is one of selection and presentation.

Two topics deserve special mention. I first learned about the Alexander polynomial from J. F. P. Hudson, and it was E. C. Zeeman who showed me how to do surgery on surfaces. To both of them, and particularly to Christopher Zeeman for his patience in teaching me topology, I offer my best thanks.

I would also like to thank R. S. Roberts and L. M. Woodward for many useful conversations, Mrs J. Gibson for her speed and skill in producing the manuscript, and Cambridge University Press for permission to reproduce the quotation from Hardy's 'A Mathematician's Apology' which appears at the beginning of Chapter 1. Finally, a special word of thanks to my wife Anne Marie for her constant encouragement.

M.A.A.
Durham, July 1978.

Contents

1. Introduction

Beauty is the first test: there is no permanent place in the world for ugly mathematics.

G. H. HARDY

1.1 Euler's theorem

We begin by proving a beautiful theorem of Euler concerning polyhedra. As we shall see, the statement and proof of the theorem motivate many of the ideas of topology.

Figure 1.1 shows four polyhedra. They look very different from one another,

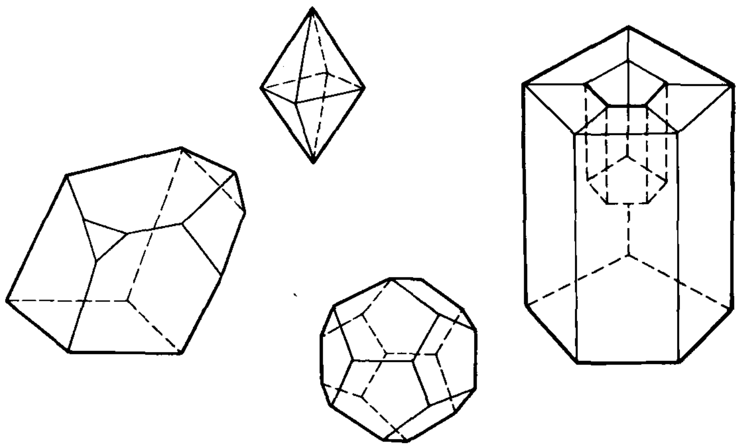


Figure 1.1

yet if for each one we take the number of vertices (v), subtract from this the number of edges (e), then add on the number of faces (f), this simple calculation always gives 2. Could the formula $v - e + f = 2$ be valid for all polyhedra? The answer is no, but the result is true for a large and interesting class.

We may be tempted at first to work only with regular, or maybe convex, polyhedra, and $v - e + f$ is indeed equal to 2 for these. However, one of the examples in our illustration is not convex, yet it satisfies our formula and we would be unhappy to have to ignore it. In order to find a counterexample we need to be a little more ingenious. If we do our calculation for the polyhedra shown in Figs 1.2 and 1.3 we obtain $v - e + f = 4$ and $v - e + f = 0$ respectively. What has gone wrong? In the first case we seem to have cheated a little by constructing a polyhedron whose surface consists of two distinct pieces; in

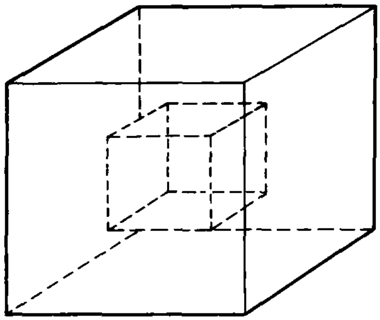


Figure 1.2

A cube with a smaller cube removed from its interior

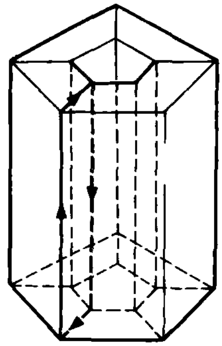


Figure 1.3

A prism with a hole straight through the centre

technical language its surface is not connected. We suspect (quite correctly) that we should not allow this, since each of the pieces of surface contributes 2 to $v - e + f$. Unfortunately, this objection does not hold for Fig. 1.3, as the surface of the polyhedron shown there is certainly all one piece. However, this surface differs from those shown earlier in one very important respect. We can find a loop on the surface which does not separate it into two distinct parts; that is to say, if we imagine cutting round the loop with a pair of scissors then the surface does not fall into two pieces. A specific loop with this property is labelled with arrows in Fig. 1.3. We shall show that $v - e + f = 2$ for polyhedra which do not exhibit the defects illustrated in Figs 1.2 and 1.3.

Before proceeding any further, we need to be a little more precise. In our discussion so far we have only made use of the surfaces of the solids illustrated (except, that is, when we have mentioned convexity). So let us agree to use the word 'polyhedron' for such a surface, rather than for the solid which it bounds. A *polyhedron* is therefore a finite collection of plane polygons which fit together nicely in the following sense. If two polygons meet they do so in a common edge, and each edge of a polygon lies in precisely one other polygon. In addition, we ask that if we consider the polygons which contain a particular vertex, then we can label them $Q_1, Q_2, \ldots, Q_k$ in such a way that Q_i has an edge in common with Q_{i+1} for $1 \leqslant i < k$, and Q_k has an edge in common with Q_1. In other words, the polygons fit together to form a piece of surface around the given vertex. (The number k may vary from one vertex to another.) This last condition rules out, for example, two cubes joined together at a single vertex.

(1.1) Euler's theorem. *Let* P *be a polyhedron which satisfies:*

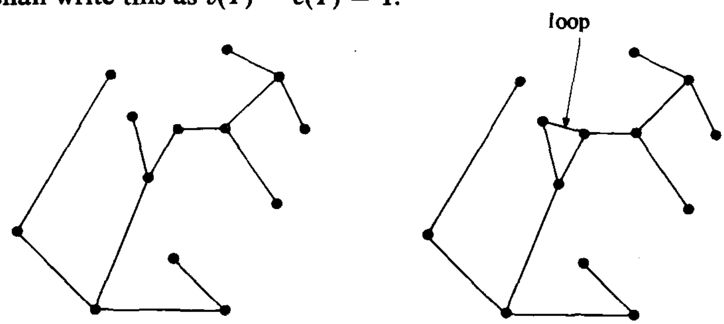
(a) *Any two vertices of* P *can be connected by a chain of edges.*

(b) *Any loop on* P *which is made up of straight line segments (not necessarily edges) separates* P *into two pieces.*

Then $\mathrm{v} - \mathrm{e} + \mathrm{f} = 2$ *for* P.

The formula $v - e + f = 2$ has a long and complicated history. It first appears in a letter from Euler to Goldbach dated 1750. However, Euler placed no restrictions on his polyhedra and his reasoning can only be applied in the convex case. It took sixty years before Lhuilier drew attention (in 1813) to the problems raised by polyhedra such as those shown in our Figs 1.2 and 1.3. The precise statement of theorem (1.1), and the proof outlined below, are due to von Staudt and were published in 1847.

Outline proof. A connected set of vertices and edges of P will be called a *graph*: connected simply means that any two vertices can be joined by a chain of edges in the graph. More generally, we shall use the word graph for any finite connected set of line segments in 3-space which fit together nicely as in Fig. 1.4. (If two segments intersect they are required to do so in a common vertex.) A graph which does not contain any loops is called a *tree*. Notice that for a tree, the number of vertices minus the number of edges is equal to 1. If the tree is denoted by T we shall write this as $v(T) - e(T) = 1$.



(*a*) Tree (*b*) Graph – not a tree

Figure 1.4

By hypothesis (a), the set of all vertices and edges of P is a graph. It is easy to show that in any graph one can find a subgraph which is a tree and which contains all the vertices of the original. So choose a tree T which consists of some of the edges and *all* of the vertices of P (Fig. 1.4a shows such a tree for one of the polyhedra of Fig. 1.1).

Now form a sort of 'dual' to T. This dual is a graph Γ defined as follows. For each face A of P we give Γ a vertex $\hat{A}$. Two vertices $\hat{A}$ and $\hat{B}$ of Γ are joined by an edge if and only if the corresponding faces A and B of P are adjacent with intersection an edge that is not in T. One can even represent Γ on P in such a way that it misses T (the vertex $\hat{A}$ corresponding to an interior point of A) though to do this we have to allow its edges to be bent. Figure 1.5 illustrates the procedure.

It is not too hard to believe that this dual Γ is connected and is therefore a graph. Intuitively, if two vertices of Γ cannot be connected by a chain of edges of Γ, then they must be separated from one another by a loop of T. (This does

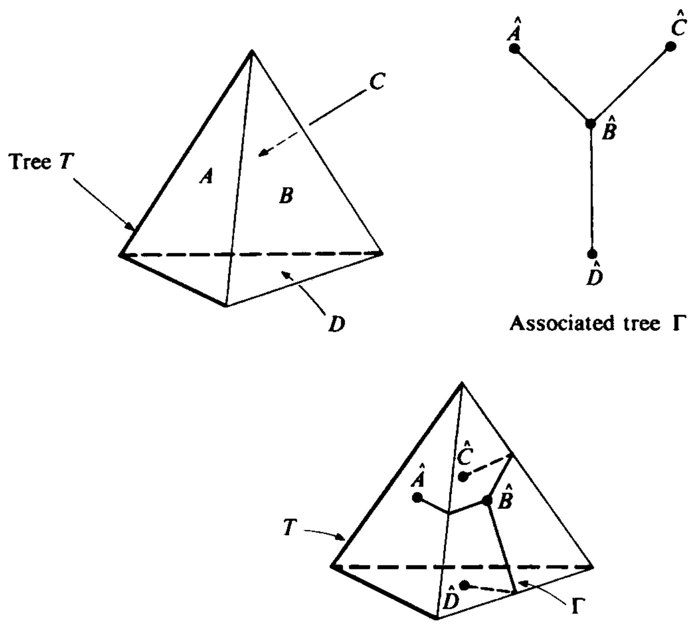


Figure 1.5

Γ represented on P

need some proof and we shall work out the details in Chapter 7.) Since T does not contain any loops we deduce that Γ must be connected.

In fact Γ is a tree. For if there were a loop in Γ it would separate P into two distinct pieces by hypothesis (b), and each of these pieces must contain at least one vertex of T. Any attempt to connect two vertices of T which lie in different pieces by a chain of edges results in a chain which meets this separating loop, and therefore in a chain which cannot lie entirely in T. This contradicts the fact that T is connected. Therefore Γ is a tree. (The proof breaks down here for a polyhedron such as that shown in Fig. 1.3, because the dual graph Γ will contain loops.)

Since the number of vertices of any tree exceeds the number of edges by 1 we have $v(T) - e(T) = 1$ and $v(\Gamma) - e(\Gamma) = 1$. Therefore

$$v(T) - [e(T) + e(\Gamma)] + v(\Gamma) = 2.$$

But by construction $v(T) = v$, $e(T) + e(\Gamma) = e$ and $v(\Gamma) = f$. This completes the argument.

1.2 Topological equivalence

There are several proofs of Euler's theorem. We have chosen the one above for two reasons. Firstly, its elegance; most other proofs use induction on the number of faces of P. Secondly, because it contains much more information than Euler's formula. With very little extra effort it actually tells us that P is made up of two discs which are identified along their boundaries. To see this, simply thicken each of T and Γ a little on P (Fig. 1.6) to obtain two disjoint discs. (Thickening a tree always gives a disc, though thickening a graph with loops will give a space with holes in it.) Enlarge these discs little by little until their boundaries coincide. The polyhedron P is now made up of two discs which have a common boundary. Granted these discs may have a rather odd shape, but they can be deformed into ordinary, round flat discs. Now remember that the sphere consists of two discs, the north and south hemispheres, sewn along their common boundary the equator (Fig. 1.7). In other words, the hypotheses of Euler's theorem tell us that P looks in some sense like a rather deformed sphere.

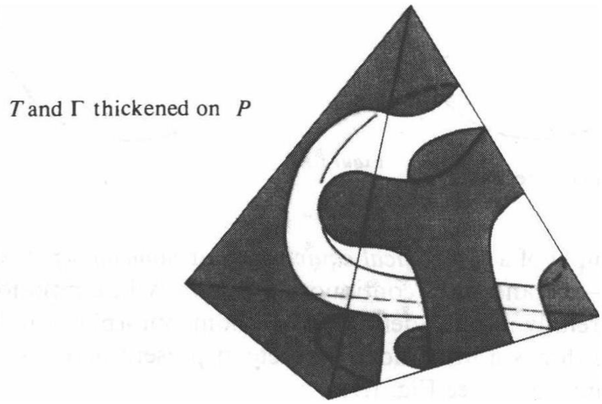


Figure 1.6

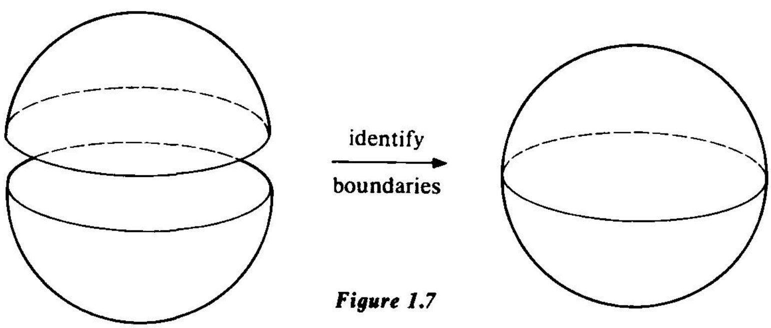


Figure 1.7

Of course, for a specific polyhedron it may be very easy to set up a decent correspondence between its points and those of the sphere. For example, in the case of the regular tetrahedron T we can use radial projection from the centre of gravity $\hat{T}$ of T to project T onto a sphere with centre $\hat{T}$. The faces of T project to curvilinear triangles on the sphere as shown in Fig. 1.8. In fact Legendre used exactly this procedure (in 1794) to prove Euler's theorem for convex polyhedra; we shall describe Legendre's argument later.