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地图的色和理论

Theory of Chromatic Sums of Maps

李赵祥 著

In this book, a new method of studying chromatic sums and dichromatic sums of maps on surfaces have been found. It will apply algebra, combibation, topology and asymptotics analysis of complex analysis. Applying this new method, some new result of chromatic sums and dichromatic sums of maps on some surfaces can be obtained. The general enumerating problem of different sort maps can be studied by chromatic sum theory, a new method of studying general map enumeration can be obtained.

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Preface

A map, always denoted by M, may be regarded as an embedding of an 1-complex(*i.e.*, graph)into a compact closed 2-manifold (*i.e.*, surface) in a view of topology. By means of algebras shown in [37], it may be defined as a basic permutation J (*i.e.*, x and αx are on different orbits) on a disjoint union $\chi_{\alpha,\beta}$ of quadricells with Axiom 1 and Axiom 2 bellow.

Let $M=(\chi_{\alpha,\beta}, J)$, $\chi_{\alpha,\beta} \sum_{x\in X} Kx$, X be a finite set, $Kx=\{x, \alpha x, \beta x, \alpha\beta x\}$, where K is Klein group of four elements denoted by 1, $\alpha, \beta, \alpha\beta$.

Axiom 1 $\alpha J=J^{-1}\alpha$.

Axiom 2 The group Ψ_g which is generated by $g=\{\alpha, \beta, J\}$ is transitive on $X_{\alpha,\beta}$.

Further, if a map $M=(\chi_{\alpha\beta}(X), J)$ satisfies the following Axiom 3, then it is said to be *nonorientable*; otherwise, *orientable*.

Axiom 3 The group Ψ_L which is generated by $L=\{\alpha\beta, J\}$ is transitive on $\chi_{\alpha,\beta}(X)$.

A is a compact 2-manifold. An(A) *orientable*(*nonorientable*) surface of genus g is homeomorphic to the sphere with g *handle* (*crosscaps*) and is denoted by $S_g(N_g)$. A map M on(or embedded on) S_g(*or* N_g) is a graph drawn on the surface so that each vertex is a point on the surface, each edge $(x, y\}$, $x \neq y$, is a simple open curve whose endpoints are x and y, each loop incident to a vertex x is a simple closed curve containing x, no edge contains a vertex to which it is not incident, and each connected region of the complement of the graph in the surface is homeomorphic to a disc and is called a *face*. A map is *rooted*, if an edge, a direction along the edge, and a side of the edge are all distinguished. If the root is the oriented edge from u to v, then u is the root-vertex while the face on the oriented side of the edge is defined as the root-face. The theory of chromatic sums of maps, and its special case $\lambda=\infty$, *i.e.*, the enumerative theory of maps were, respectively, found by W. T. Tutte in 1970's and 1960's. One of his aim of finding the chromatic sums is to attack the colouring average problem which is harder than the four colour problem. A series of Tutte's census papers[70-74] in 1960's has laid a ground work on the enumerative theory of rooted planar maps. Since then, the theory has been simplified and developed by W. G. Brown [10-12], Liu[37-50], R. C. Mullin[52] and Tutte himself. E. A. Bender, E. R. Canfield, N. C. Wormald, R. W. Robinson, L. B. Richmond and Z. C. Gao et al. set up many equations for rooted nonplanar maps.

In most cases(nonplanar maps), they failed to find the whole solutions of the equations they presented. What they got were partial solutions or an asymptotic evaluation and limited to the general maps with nearly no restrictions containing only one parameter. However, the book[37] by Liu presented a new way to investigate maps on surfaces especially those with smaller genera in orientable and nonorientable cases based on the sphere. Since finding the number of nonplanar maps is very difficult, people such as E. A. Bender et al [3-5] developed an *asymtotic* evaluation to obtain the statistic behaviors of the maps given.

On chromatic sum, the first paper which was published in 1973 by W. T. Tutte [62] is for rooted planar triangulations. Since 1973, Tutte has published a series of paper[62-66] on chromatic sums for rooted planar triangulations to tackle the coloring average problem. And then, Y. P. Liu[38,39,41,45] has a numbers of papers for rooted outerplanar maps, rooted cubic planar maps, and general planar maps. All results of chromatic sums having been published are on the plane. Based on the initial works of Tutte and Y. P. Liu et al., the authour has studied chromatic sums of some maps on the sphere, projective plane, torus and and Klein bottle, and their special case $\lambda=\infty$, *i.e.*, the enumerative problem of maps.

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Contents

Chapter 1 General Maps

§ 1.1 Chromatic Sums of General Maps on the Sphere and the Projective Plane

§ 1.1.1 Introduction

Let C be a circuit(or curve) on a surface Σ. If $\Sigma - C$ has a connected region homeomorphic to a disc, then C is called trivial (or contractible as some scholars defined it); otherwise, it is essential(or noncontractible).

Let μ and P be, respectively, the set of all rooted general maps on the sphere and the projective plane. Their chromatic sum functions are, respectively,

$$\begin{aligned} f &= f(x, y, z, t, \omega; \lambda) \\ &= \sum_{M \in \mu} P(M; \lambda) x^{m(M)} y^{n(M)} z^{l(M)} t^{s(M)} \omega^{d(M)} \\ f_P &= f_p(x, y, z, t, \omega; \lambda) \\ &= \sum_{M \in P} P(M; \lambda) x^{m(M)} y^{n(M)} z^{l(M)} t^{s(M)} \omega^{d(M)} \end{aligned}$$

where $m(M)$, $n(M)$, $l(M)$, $s(M)$ and $d(M)$ be, respective-ly, the valency of the root-vertex of M, the valency of the root-face of M, the number of edges of M, the number of nonroot-vertices of M and the number of nonroot-faces of M. $P(M;\lambda))$ is the chromatic polynomial of M.

Now two well-known formula on chromatic polynomials of maps should be mentioned for the further use. The first one is

$$P(M;\ \lambda)=P(M-e;\lambda)-P(M\bullet e;\lambda) \qquad (1.1.1)$$

For any map M, where e is an edge of M, $M-e$ and $M \bullet e$ stand for the maps obtained by deleting and contracting e from M, respectively. The second is

$$P(M_1 \cup M_2;\lambda)=\frac{P(M_1;\ \lambda)P(M_2;\lambda)}{\lambda(\lambda-1)\ldots(\lambda-i+1)} \qquad (1.1.2)$$

provided that $M_1 \cap M_2=K_i$, the complete graph of order $i \geqslant 1$.

In this section, for any map M, $e_r(M)$ stand for the root-edge of M.

§ 1.1.2 Maps on the Sphere

In this section, we will set up equations satisfied by the chromatic sum functions of general maps on the sphere. An edge is called double edge(a double edge on the plane is also called isthmus by some authors) if each side of it is on the boundary of the same face.

The set μ may be divided into four parts as

$$\mu=\mu_1+\mu_2+\mu_3+\mu_4 \tag{1.1.3}$$

where μ_1 consists of only the vertex map,

$$\mu_2=\{M|M\in\mu-\mu_1,\ e_r(M)\ is\ a\ loop\};$$

$$\mu_3=\{M|M\in\mu-\mu_1,\ e_r(M) is\ a\ double\ edge\};$$

$$\mu_4=\{M|M\in\mu-\mu_1-\mu_2-\mu_3\}.$$

Let $f_i(i=1,2,3,4)$ be the chromatic sum function of μ_i. Let $f|_{x=1}=f(1,y,z,t,\omega;\lambda)$, $f|_{y=1}=f(x,1,z,t,\omega;\lambda)$. For chromatic sum function f, we denote $\delta_\varphi f=\frac{\varphi f-f|_{\varphi=1}}{\varphi-1}$, here $\varphi=x$, or $\varphi=y$. Then

$$f_1=\lambda;f_2=0;\ f_3=\lambda^{-1}(\lambda-1)xy^2ztff|_{x=1} \tag{1.1.4}$$

It is easy to see that the following Lemmas hold.

Lemma1. 1. 1 Let $\mu_{<4>}=\{M-e_r(M)|M\in\mu_4\}$. Then

$$\mu_{<4>}\subseteq\mu.$$

For a map M, let $\Delta_i(M)$ be the map obtained by adding a new edge from the root-vertex to the *ith* vertex on the root-face boundary, $i=0,1,2,\ldots,\ n(M)$ ($n(M)$ *is the valency of root — face of* M). Let $\mu^*=\sum_{M\in\mu}\{\Delta_i(M)|n(M)\geqslant i\geqslant 0\}$. It is easily seen that $\mu_4=\subseteq\mu^+$.

Lemma 1. 1. 2 For μ_4, we have $\mu_4=\mu^*-\mu_2$.

Lemma 1. 1. 3 Let $\mu_4=\{M\bullet e_r(M)|M\in\mu_4\}$. Then

$$\mu_{(4)}\subseteq\mu.$$

For a map $M\in\mu$, let $\nabla(M)$ be the map by map obtained by splitting the root-vertex of M into vertices o_1 and o_2 and adding a new edge $R=(o_1, o_2)$ as the root-edge of the resultant map.

Let $\mu^+=\{(M)\mid M\in\mu\}$. It is easily seen that $\mu_4\subseteq\mu^+$.

Lemma 1.1.4 $\mu^+=\mu-\mu_1-\mu_2$.

Applying (1.1.1), for chromatic sum functions, the contribution of μ_4 to f is

$$f_4=\sum_{M\in\mu_4}P(M-e_r(M);\lambda)x^{m(M)}y^{n(M)}z^{l(M)}t^{s(M)}\omega^{d(M)}$$
$$-\sum_{M\in\mu_4}P(M\bullet e_r(M);\lambda)x^{m(M)}y^{n(M)}z^{l(M)}t^{s(M)}\omega^{d(M)} \quad (1.1.5)$$

The first and the last summations in (1.1.5) are denoted by f_{4A} and f_{4B}, respectively.

Applying Lemmas 1.1.1-1.1.2 and (1.1.2), we have

$$f_{4A}=xyz\omega\{(\sum_{M\in\mu}P(M;\lambda)x^{m(M)}z^{l(M)}t^{s(M)}\omega^{d(M)}\sum_{i=0}^{n(M)}y^i)-\lambda^{-1}ff|_{y=1}\}$$
$$=xyz\omega\delta_y f-\lambda^{-1}xyz\omega ff|_{y=1} \quad (1.1.6)$$

Applying Lemma 1.1.3-1.1.4 and 1.1.2, we have that

$$f_{4B}=xyzt\{(\sum_{M\in\mu}P(M;\lambda)y^{n(M)}z^{l(M)}t^{s(M)}\omega^{d(M)}\sum_{i=0}^{m(M)}x^i)-\lambda^{-1}ff|_{x=1}\}$$
$$=xyzt\delta_x f-\lambda^{-1}xyztff|_{y=1} \quad (1.1.7)$$

Applying (1.1.3)-(1.1.7), we have that

Theorem 1.1.1 f satisfies the following functional equation:

$$f=\lambda+\lambda^{-1}(\lambda-1)xy^2ztff|_{x=1}+xyz\omega\delta_y f-\lambda^{-1}xyz\omega ff|_{y=1}-xyzt\delta_x f+\lambda^{-1}xyztff|_{x=1} \quad (1.1.8)$$

§ 1. 1. 3 Maps on the Projective Plane

In this section, we will set up the equations satisfied by the chromatic sum functions of rooted general maps on the projective plane.

The set P may be divided into two parts as

$$P=P_1+P_2 \tag{1.1.9}$$

where $P_1=\{M|M\in P,\ e_r\ \textit{is a loop}\}$;

$P_2=\{M|M\in P,\ M\notin P_1\}$;.

The set P_2 may be divided into two parts as

$$P_2=P_{21}+P_{22} \tag{1,1,10}$$

where $P_{21}=\{M|M\in P_2,\ e_r(M)\ \textit{is a double edge}\}$;

$P_{22}=\{M|M\in P_{22},\ e_r(M)\ \textit{is not a double edge}\}$.

The P_{21} set may be divided into two parts as

$$P_{21}=P_{211}+P_{212} \tag{1.1.11}$$

where $P_{211}=\{M|M\in P_2,\ e_r(M)\ \textit{is not on an essential circuit}\}$;

$P_{212}=\{M|M\in P_{22},\ e_r(M)\ \textit{is on an essential circuit}\}$.

Lemma 1. 1. 5 Let $P_{<212>}=\{M-e_r(M)\,|M\in P_{<212>}\}$. Then $P_{<212>}\subset\mu$.

Proof For any $M'\in P_{<212>}$, $M'=M-e_r(M)$, where $M\in P_{212}$. Since $e_r(M)$ is a double edge and $e_r(M)$ is on an essential circuit, M' is a map on the sphere.

Lemma 1.1.6 Let $P_{<22>}=\{M-e_r(M)\,|\,M\in P_{22}\}$. Then

$$P_{<22>}\subset P.$$

Proof For any $M'\in P_{<22>}$, $M'=M-e_r(M)$, where $M\in P_{22}$. Since $e_r(M)$ is not a double edge, we know $M'\in P$.

Similarly, we have that

Lemma 1.1.7 Let $P_{(2)}=\{M\bullet e_r(M)\,|\,M\in P_2\}$. Then

$$P_{(2)}=P.$$

Let $f_{Pi}(i=1,2)$ be the chromatic sum function of P_i. Then

$$f_P=f_{P_1}+f_{P_2};\ f_{P_1}=0 \tag{1.1.12}$$

Applying(1.1.1), for chromatic sun functions, the contribution of P_2 to f_P is

$$f_{P_2}=\sum_{M\in P_2}P(M-e_r(M);\lambda)x^{m(M)}y^{n(M)}z^{l(M)}t^{s(M)}\omega^{d(M)}$$
$$-\sum_{M\in P_2}P(M\bullet e_r(M);\lambda)x^{m(M)}y^{n(M)}z^{l(M)}t^{s(M)}\omega^{d(M)} \tag{1.1.13}$$

The first and the last summations in(1.1.13) are denoted by $f_{p_{2A}}$ and $f_{p_{2B}}$, respectively.

By(1.1.10) and(1.1.11), we obtain that

$$f_{P_{2A}}=f_{P_{211A}}+f_{P_{212A}}+f_{P_{22A}} \tag{1.1.14}$$

For any $M\in P_{211}$, $M-e_r(M)$ are two maps, one map is on the sphere, another map is on the projective plane. We obtain

$$f_{P_{211A}}=xy^2ztff_P|_{x=1}+xy^2ztf|_{x=1}f_P \tag{1.1.15}$$

here $f|_{x=1}=f(1,y,z,t,\omega;\lambda)$; $f_P|_{x=1}=f_P(1,y,z,t,\omega;\lambda)$.