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V. I. Arnol'd S. P. Novikov (Eds.)

Dynamical Systems VII
Integrable Systems, Nonholonomic Dynamical Systems

动力系统 VII
可积系统，不完整动力系统

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I. Nonholonomic Dynamical Systems, Geometry of Distributions and Variational Problems

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Translated from the Russian
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Contents

Introduction

0. A nonholonomic manifold is a smooth manifold equipped with a smooth distribution. This distribution is in general nonintegrable. The term 'holonomic' is due to Hertz and means 'universal', 'integral', 'integrable' (literally, ὁλος – entire, νομος – law). 'Nonholonomic' is therefore a synonym of 'nonintegrable'.

A nonholonomic manifold is the geometric (or, more precisely, kinematic) counterpart of a nonholonomic dynamical system with linear constraints. As we shall see later, there are two main ways to construct dynamics on nonholonomic manifolds. They will be referred to (somewhat conventionally) as mechanical and variational, respectively.

The aim of the present survey is to give a possibly self-consistent account of the geometry of distributions and to lay down a foundation for a systematic study of nonholonomic dynamical systems. Our review is somewhat different in its character from other reviews in the present series. The reasons for this are rooted in the peculiar history of the subject as will be explained below.

Due to lack of space, the present volume includes only the first part of our survey which comprises geometry of distributions and variational dynamics. Geometry and dynamics of nonholonomic mechanical systems, nonholonomic connections, etc., will be described separately. However, in this introduction we will discuss all these topics, at least in historical aspect. Several problems of nonholonomic mechanics have already been considered in Volume 3 of this Encyclopaedia (Arnol'd, Kozlov and Nejshtadt [1985]). In this survey we give a modern exposition of some earlier results as well as new results which were not published previously.

1. Nonholonomic geometry and the theory of nonholonomic systems are the subject of numerous papers. A few of them are due to prominent geometers of the beginning of our century, others and still more numerous date back to the early past-war period. Nevertheless, this theory has not gained popularity in a broader mathematical audience. For instance, in most textbooks on Riemannian geometry and the calculus of variations there are hardly any facts on these subjects, save perhaps for the classical Frobenius theorem. Even the term 'nonholonomic' is scarcely mentioned. The exceptions are rather rare (e.g. the recent book of Griffiths [1983] where such results as the Chow-Rashewsky theorem are exposed).

There are many reasons to feel dissatisfied with this state of affairs. First of all, nonholonomic systems always held a sufficiently important place in mechanics. Mathematicians have always been thoroughly studying classical dynamics and the mathematical structures that are inherent to it. In view of this long-standing tradition the neglect of nonholonomic problems is almost striking. It is even not commonly known that nonholonomic mechanical problems cannot be stated as variational problems (cf. below).

Secondly, nonholonomic variational problems have much in common with optimal control problems which have been the subject of so many papers in recent years. This similarity has not been noticed until recently. Although the statements of nonholonomic variational problems are classical, some features of their solutions (e.g. the structure of the accessibility set, cf. Chapter 2) makes them quite close to non-classical problems. However, in textbooks on variational calculus even existence theorems for the simplest nonholonomic problems are lacking. The corresponding theorem in Chapter 2 of our survey is based on a recent observation made in connection with non-classical problems.

Thirdly, problems of thermodynamics (Gibbs, Carathéodory) and of quantum theory (Dirac) also lead to nonholonomic variational problems. A modern mathematical treatment of this kind of problems is still to be given.

Fourthly, nonholonomic problems are closely connected with the general theory of partial differential equatons. The best known results that display this connection are the theorems of L. Hörmander and A.D. Aleksandrov on hypoellipticity and hypoharmonicity. The study of these questions from a nonholonomic point of view has been actively pursued in recent years (we included some additional references to make this translation more up-to-date).

Finally, the general mathematical theory of dynamical systems (now in its maturity, as confirmed by the present edition) may well find in nonholonomic dynamics a vast source of new problems, examples, and paradoxes. We hope to support this view by the present survey.

2. We shall now give a brief review of the history of our subject. We believe that it will also explain the isolation of nonholonomic theory from the rest of mathematics which has continued up to the present time. A systematic development of the nonholonomic theory was started in the twenties and thirties, following a pattern which dates back to the turn of the century. This pattern was shaped in a series of classical papers on nonholonomic mechanics, due to many mathematicians and physicists: Hertz, Voss, Hölder, Chaplygin, Appell, Routh, Woronets, Korteweg, Carathéodory, Horac, Volterra, to mention a few. See Aleksandrov [1947], and Synge [1936] for a review. As mentioned by Grigoryan and Fradlin [1982], nonholonomic mechanical problems were already treated by Euler. However, it was not until the turn of the century that a clear understanding of their special features was gained. Hertz's name (Prinzipien der Mechanik, 1894) should be ranked first in this respect. A less known source of the theory comes from physics, namely, from the works of Gibbs and Carathéodory on the foundations of thermodynamics. They deal with contact structure which is the simplest example of nonholonomic structure. As for pure mathematics itself, in particular, the theory of distributions, which is an indispensable part of nonholonomic theories, we should begin with the theory of Pfaffian systems and subsequent works on the general theory of differential equations. The contribution of E. Cartan to this domain was of particular importance. He was the first to introduce differential forms and codistributions. Unfortunately, these tools were not widely used in nonholonomic problems.

Finally, we mention one trend of research in geometry going back to Issaly (1880), or, perhaps, even earlier. We mean the studies of nonholonomic surfaces, i.e. of nonholonomic 2-dimensional distributions in 3-dimensional space generalizing the ordinary theory of surfaces. This trend has been developed further by D.M. Sintsov and his school and by some other geometers in the Soviet Union. However, these works were not sufficiently known even to the experts and did not play a major role.

3. In the twenties when Levi-Civita and H. Weyl defined the notions of Riemannian and affine connections and discovered deep relations between mechanics and geometry, it became clear that nonholonomic mechanics should also serve as a source of new geometrical structures which, in turn, provide mechanics and physics with a convenient and concise language. This mutual interaction was started by Vranceanu and Synge. In the Soviet Union nonholonomic problems were activey advertized for by V. Kagan. In 1937 he proposed the following theme for the Lobachevsky prize competition (Vagner, [1940]): "To lay down the foundation of a general theory of nonholonomic manifolds. ⟨...⟩ Applications to mechanics, physics, or integration of Pfaffian systems are desirable".

The most important results on nonholonomic geometry and its connections with mechanics were obtained in the pre-war years and are due to Vranceanu and Synge, and also to Schouten and V. Vagner. In two short notes and an article a Romanian mathematician Vranceanu [1931] gave the first precise definition of a nonholonomic structure on a Riemannian manifold and outlined its relation to the dynamics of nonholonomic systems. Synge [1927, 1928] has studied the stability of the free motion of nonholonomic systems. In his work he anticipated the notion of the curvature of a nonholonomic manifold. It was formally introduced somewhat later and in two stages. First, Schouten defined what was later to be called truncated connection, i.e. parallel transport of certain vectors along certain vector fields. The geodesics of this connection are precisely the trajectories studied by Synge. Finally, a Soviet geometer V. Vagner made the next important step in a series of papers which won him the Lobachevsky prize of Kazan University for young Soviet mathematicians in 1937. He defined (in a very complicated way) the general curvature tensor which extends the Schouten tensor and satisfies all the natural conditions (e.g. it is zero if and only if the Schouten-Vranceanu connection is flat). In his subsequent papers Vagner extended and generalized his results. (Notably, Schouten was in the jury when Vagner defended his thesis.)[1]

4. The geometry of the straightest lines (i.e. classical mechanics of nonholonomic systems) is the subject of quite a few papers written mainly between the wars. By contrast, much less was done on the geometry of the shortest lines, i.e. on the variational theory of nonholonomic systems. The main

[1] Recently, a more modern exposition of Vagner's main work has appeared (Gorbatenko [1985].)

contributions bearing on this subject may be listed quite easily. The starting point for the theory was a paper of Carathéodory [1909] in which he proves that any two points on a contact manifold may be connected by an admissible curve. (This statement was already mentioned without proof in earlier papers, e.g. by Hertz.) It is interesting to notice that Carathéodory needed this theorem in connection with his work on foundations of thermodynamics, namely in order to justify the definition of thermodynamical entropy. Although this theorem has a kinematic nature, it may be used to define a variational, or nonholonomic, metric sometimes referred to as the Carnot-Carathéodory metric (see Chapter 3). An extension of this theorem to arbitrary totally nonholonomic manifolds was proved independently by Chow [1939] and by Rashevsky [1938]. Several results of the classical calculus of variations were extended to the nonholonomic case by Schoenberg who was specially studying variational problems. A comparison of mechanical and variational problems for nonholonomic manifolds was given by Franklin and Moore [1931]. An interpretation of nonholonomic variational problems in mechanical or optical terms has been proposed quite recently (Arnol'd, Kozlov and Nejshtadt [1985], Karapetyan [1981], Kozlov [1982a, b, 1983]).

5. Before we come to describe the contents of this paper it is worth commenting on the reasons which possibly account for the contrast between the importance of these subjects and their modest position in "main-stream" mathematics. The point is that most papers which bear on the subject were written extremely vaguely, even if one allows for the usual difficulties of "coordinate language". To put it in a better way, at that time it was practically impossible to give a clear exposition of the theory (which is far from being simple by any standards). The key concept that was badly needed was that of a connection on a principal bundle in its almost full generality. However, nothing of the kind was used at that time. The absence of an adequate language was really painful and resulted in enormous and completely unmanageable texts. It was difficult to extract from them even the main notions, to say nothing of theorems. As a consequence, these papers were not duly understood.[2] One needed such tools as jets, germs, groups of germs of diffeomorphisms, etc. This was already clear to E. Cartan. Although he never studied this subject specially, he frequently stresses that connections on principal fibre bundles should be used in nonholonomic problems (ironically, he quotes this idea in the same volume dedicated to the Lobachevsky prize competition that we already mentioned (Vagner [1940])).

All this may be the reason why the fundamental reshaping of differential geometry in a coordinate-free manner has left aside the nonholonomic theory. In the fifties and sixties this theory was already out of fashion and was to remain

[2] V. Vagner wrote in 1948: "The lack of rigour which is typical for differential geometry is reflected also in the absence of precise definitions of such notions as spaces, multi-dimensional surfaces, etc. Differential geometry is certainly dropping behind and this became even more dangerous when it lost its direct contacts with theoretical physics".

in obscurity for many years to come. Although many of the more recent papers on nonholonomic theory already used modern language, they were isolated from the fertilizing applications which had served as the starting point for geometers of the prewar time.

After a complete renewal of its language in the fifties and sixties modern differential geometry became one of the central parts of contemporary mathematics. Along with topology, the theory of Lie groups, the theory of singularities, etc., it has created a genuine mathematical foundation of mechanics and theoretical physics in general. An invariant formulation of dynamics permitted to apply various powerful tools in this domain. Gradually, this process has brought to bear on nonholonomic theory. Quite recently, the Schouten-Vranceanu connection was rediscovered by Vershik and Faddeev [1975]. (See also Godbillon [1969], Vershik [1984].) In this paper nonholonomic mechanics was exposed systematically in terms of differential geometry. In particular, it was shown that the local d'Alembert principle regarded as a precise geometric axiom implies the above mentioned theorem on geodesics in nonholonomic theory.

A good deal of the authors' efforts was aimed to extract geometrical ideas and constructions from the papers of the past years and to present them in a modern form. This goal has not been fully achieved, but it seems indispensable in order to develop systematically the qualitative and geometric theory of nonholonomic dynamical systems, in analogy with other theories of dynamical systems (e.g. Hamiltonian, smooth, ergodic, etc.). We tried to give the basic definitions and to describe the simplest (3-dimensional) examples. Many mathematicians and physicists have helped us by pointing out various scattered papers on the subject. We are particularly indebted to A.D. Aleksandrov, V.I. Arnol'd, A.M. Vassil'ev, A.M. Vinogradov, A.V. Nakhmann, V.V. Kozlov, N.V. Ivanov, Yu.G. Lumiste, Yu.I. Lyubich, N.N. Petrov, A.G. Chernyakov, V.N. Shcherbakov, and Ya.M. Eliashberg.

6. Let us now turn to a brief description of the general structure of the survey. Nonholonomic dynamics is based on the geometry of distributions which is the subject of Chapter 1. The simplest and best known example of a nonholonomic manifold is the contact structure, i.e. a maximally nonholonomic distribution of codimension 1. Since in the existing literature very little is available on distributions of larger codimension, we present the main definitions and the most important examples of distributions in Section 1 of Chapter 1. In Section 2 we study generic distributions and the classification problem. In particular, we present results on the existence of functional moduli of distributions for almost all growth vectors. We also briefly mention the notion of nilpotentization which is of particular importance, especially in the recent advances of the theory. (For more information consult the list of references which was enlarged to make this translation more up-to-date.) As already mentioned, there are two completely different dynamics associated with a nonholonomic Riemannian manifold: the dynamics of the 'straightests', or mechanical, and the dynamics of the 'shortests', or variational. The terms 'straightest' and 'shortest' were first intro-

duced in connection with mechanics by Hertz. The difference between them is briefly as follows. Using the distribution we may introduce the so called truncated connection (Schouten [1930]). The study of its geodesics and of the corresponding flow is connected with mechanics of systems with linear constraints, e.g., with the problem of rolling, etc. (The general theory comprises also nonlinear restrictions, cf. for instance Vershik and Faddeev [1975], Vershik and Gershkovich [1984].) These questions will be considered separately. If we restrict the metric to the distribution, we get a new metric on the manifold. Its geodesics (the shortests) are the subject of variational theory discussed in detail in Chapter 2. The phase space of a nonholonomic variational problem is the so called mixed bundle, i.e. the direct sum of the distribution regarded as a subbundle of the tangent bundle and of its annihilator regarded as a subbundle of the cotangent bundle. As already mentioned, variational problems also admit a proper mechanical interpretation.

In Section 1 of Chapter 2 we present the main notions and constructions related to nonholonomic variational problems, such as nonholonomic geodesic flow, nonholonomic metrics, the nonholonomic exponential mapping, wave fronts, etc. In Section 2 we compute the accessibility sets for nonholonomic problems (or control sets, in the language of control theory).

In Chapter 3 we consider nonholonomic variational problems on Lie groups and homogeneous spaces. As usual, problems on Lie groups offer the most important class of examples, as well as a training field to develop constructions and methods which may be then extended to the general setting. In Section 1 we discuss local questions: the wave front and the ε-sphere of a nonholonomic Riemannian metric. In Section 2 we present a complete description of the dynamics of systems associated with the nonholonomic geodesic flow on homogeneous spaces of 3-dimensional Lie groups. Our approach is based on the wide use of geometry (more precisely, nonholonomic Riemannian geometry) and of the theory of nilpotent Lie groups. These two sources provide a better understanding of various domains connected with the study of distributions, such as nonholonomic mechanics, the theory of hypoelliptic operators, etc. At the same time this approach leads to new problems in geometry and in the theory of Lie groups.

Chapter 1
Geometry of Distributions

§ 1. Distributions and Related Objects

1.1. Distributions and Differential Systems. In the sequel without further notice all objects, such as manifolds, functions, mappings, distributions, vector fields, forms, etc., are supposed to be infinitely differentiable.

Definition 1.1. Let X be a real smooth manifold without boundary, and let TX be its tangent bundle. A subbundle $V \subset TX$, i.e. a family $\{V_x\}_{x \in X}$ of linear subspaces $V_x \subset T_x$ of the tangent spaces which depend smoothly on the point $x \in X$, is called a *distribution* on X. If X is connected, the number $\dim V_x \equiv \dim V$ is called the *dimension* of the distribution.

In the simplest case a distribution has the following structure: there is a decomposition of X into submanifolds (leaves), and V_x is the tangent space to the leaf passing through x. In this case the distribution is said to be *integrable* and determines a foliation. Its leaves are called maximal integral submanifolds of the distribution; their dimension is equal to that of V. If $\dim V = 1$, V is always integrable and its integral submanifolds are (locally) integral curves of a vector field that generates V.

In this survey we shall be mainly interested in the opposite case of nonintegrable, or nonholonomic distributions. The simplest example of a nonholonomic distribution is provided by two-dimensional distributions in $\mathbb{R}^3$, for instance, given by $V_x = \mathrm{Lin}\{\partial/\partial x_1, -\partial/\partial x_2 + x_1 \partial/\partial x_3\}$, $x = (x_1, x_2, x_3)$. As is frequently done, the distribution is defined here as the linear span of vector fields. Another way to define the same distribution is as follows: V is the null-space of a 1-form $x_1\, dx_2 + dx_3$ which defines a contact structure on $\mathbb{R}^3$. The description of a distribution as the set of null-spaces of a system of differential forms will also be frequently used.

A k-dimensional distribution on X may be regarded as a section of the *Grassmann bundle* associated with TX, i.e. of the bundle of k-dimensional subspaces of the tangent spaces. This construction equips the space $\mathscr{V}_k(X)$ of such distributions with the natural topology of C^∞-sections. If X is an open ball, then, for $k \geqslant 2$, nonintegrable distributions (and even maximally nonintegrable distributions, see Section 2) form an open dense subset in $\mathscr{V}_k(X)$. On the contrary, the integrability of a distribution, i.e. the existence of foliation, is an extremely rare (nowhere dense) event.

Definition 1.2. We shall say that a vector field ξ on X is *subordinate* (or *belongs*) *to* $V = \{V_x\}$ if $\xi_x \in V_x$ for all $x \in X$. If $V_x = \mathrm{Lin}\{\xi^i_x, i = 1, \ldots, n\}$, the vector fields ξ^i are said to *generate* V. An integral curve γ of a vector field belonging to V is called *admissible* (with respect to V): $\dot\gamma_x \in V_x$, $x \in X$.

Recall that the linear space (over $\mathbb{R}$) of smooth vector fields Vect X is a Lie algebra with respect to the Lie bracket $[\xi, \eta] = \xi\eta - \eta\xi$. (Vector fields may be regarded as derivations, and their product means composition of derivations.) Moreover, Vect X is a $C^\infty(X)$-module, since each $\xi \in$ Vect X may be multiplied by $f \in C^\infty(X)$.

If V is a distribution, the set of all vector fields belonging to V (i.e. the set of its sections) is a C^∞-submodule in Vect X. We introduce the following

Definition 1.3. A *differential system* on X is a linear space of vector fields on X which is a $C^\infty(X)$-module[1].

As we have explained above each distribution V gives rise to a differential system $N(V)$. However, there exist other differential systems that correspond to distributions with singularities, i.e. to fields of linear subspaces in TX of non-constant dimension. Such distributions appear quite naturally. For instance, the Lie bracket of two distributions may already have singularities, which motivates the necessity of the above definition.

Let F be a differential system. The set of all vectors $v \in T_xX$ for which there is a vector field $\xi \in F$ such that $\xi_x = v$ is a linear subspace $V_x \subset T_x$. If dim $V_x =$ const, F is generated by a distribution $\{V_x\} = V$, $F = N(V)$; otherwise such a distribution does not exist.

Proposition. *A differential system on a smooth manifold X is the space of sections of a distribution if and only if it is a projective $C^\infty(X)$-module.*

Recall that if X is an open ball, any projective module is free, hence, in local problems, differential systems that are free $C^\infty(X)$-modules are the same as distributions. (A free module is the direct sum of several copies of $C^\infty(X)$, a projective module is a direct summand of a free module).

Definition 1.4. A distribution V (a differential system N) is *involutive* if $N(V)$ (respectively, N) is a Lie algebra; in other words, the Lie bracket of two vector fields that are subordinate to V (respectively, belong to N) also belongs to V (respectively, to N).

In the sequel we shall maninly deal with local problems, and so it is useful to introduce local versions of the main definitions using the language of germs and jets. (Concerning the notions of germs, jets, etc, see Bröcker and Lander [1975], Golubitsky and Guillemin [1973].) Let W_n^r, $1 < r \leqslant \infty$, $n = 1, \ldots,$ be the space of *r-jets of vector fields* at $0 \in \mathbb{R}^n$, let ω_n be the space of germs of vector fields in a neighborhood of $0 \in \mathbb{R}^n$. The spaces $W_n^\infty \equiv W_n$ and ω_n are Lie algebras with respect to the Lie bracket of vector fields. Moreover, these spaces are modules over the ring of jets J_n^∞ and the ring of germs of functions E_n, respectively.

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[1] Some authors use the term 'differential systems' for distributions. Since the latter term is well established in Russian literature we shall use the term 'differential system' for a different notion. Recall that the notion of $C^\infty(X)$-module means that vector fields from F may be multiplied by an arbitrary element of $C^\infty(X)$: $\forall \xi \in F \, \forall f \in C^\infty(X) \, f\xi \in F$.

The ring E_n of germs of C^∞-functions is a local algebra, i.e. it contains a unique nontrivial maximal ideal $\mathfrak{M}_n$ generated by the germs of functions that vanish at $0 \in \mathbb{R}^n$. The ideal of flat relations $\mathfrak{M}_n^\infty = \bigcap_k \mathfrak{M}_n^k$ has a very complicated structure. The quotient ring $E_n/\mathfrak{M}_n^\infty$ is called the ring of jets and is described by the following theorem (cf. Bröcker and Lander [1975]).

Borel Theorem.

$$J_n^\infty = E_n/\mathfrak{M}_n^\infty \cong \mathbb{R}[[y_1, \dots, y_n]].$$

Hence this ring, as a ring of power series, is Noetherian, i.e. it contains only finite sequences of strictly increasing ideals.

We shall keep the name "differential system" for submodules both in W_n and in ω_n. Thus in these terms a *jet* (*a germ*) *of a distribution* means a free submodule in W_n (or in ω_n).

In complete analogy with the global statement above, a differential system F is generated by the germ of a distribution (and is a free module over E_n) if and only if dim V_x is the germ of a constant function (here $V_x = \{\xi(x); \xi \in F\}$).

1.2. Frobenius Theorem and the Flag of a Distribution. The classical Frobenius theorem (cf. Sternberg [1964]) asserts that each involutive distribution is integrable. Integrability of a distribution means that there exists a foliation such that the distribution consists of tangent spaces to its leaves. We shall give a proof of this theorem that focuses on the algebraic side of the question (cf. also Treves [1980]). Since the statement of the Frobenius theorem is local, we shall consider only the local case.

Frobenius Theorem. *The germ* (*or the jet*) *of an involutive distribution is generated by commuting vector fields. More precisely, let N be an involutive differential system* (*either in germs, or in jets*) *which is a subalgebra and a free E_n-module of dimension m. Then N is generated, as a module, by m commuting vector fields.*

Corollary. *All jets* (*germs*) *of involutive distributions of a given dimension constitute a single orbit of the group of jets* (*germs*) *of diffeomorphisms of $\mathbb{R}^n$. In other words, the jets* (*germs*) *of two involutive distributions of the same dimension may be transformed one into another by a jet* (*germ*) *of a diffeomorphism.*

Hence the orbit is defined by the 1-jet of a distribution, since involutivity is a 1-jet condition.

The final part of Frobenius theorem in its usual form (construction of leaves) is now reduced to the standard existence theorems for ordinary differential equatons. Indeed, the integral curves for commuting vector fields may be determined successively, which yields the desired leaf.

Proof. Let V be the germ of an involutive distribution of dimension m in $\mathbb{R}^n$ generated by $\xi_1, \dots, \xi_m \in W_n$. We shall prove by induction over m that V is generated by commuting vector fields $\Psi_1, \dots, \Psi_m$. Choose a coordinate system

$\{x_i\}_{i=1}^n$ in $\mathbb{R}^n$ in such a way that $\xi_1 = \frac{\partial}{\partial x_1}$. Let us decompose ξ_i with respect to $\frac{\partial}{\partial x_i}$: $\xi_i = \sum_{j=1}^n c_{ij} \frac{\partial}{\partial x_j}$, $c_{ij} \in E_n$, where E_n is the ring of germs of smooth functions in $\mathbb{R}^n$.

The vector fields $\tilde{\xi}_i = \xi_i - c_{i,1}\xi_1$, $i = 2, \ldots, m$, form an involutive system. By the induction hypothesis, there exist commuting vector fields $\Psi_2, \ldots, \Psi_m$ such that $V(\tilde{\xi}_2, \ldots, \tilde{\xi}_m) = V(\Psi_2, \ldots, \Psi_m)$. Choose a coordinate system $\{y_i\}$ in $\mathbb{R}^n$ in such a way that $\Psi_i = \frac{\partial}{\partial y_i}$, $i = 2, \ldots, m$, and decompose ξ_1, with respect to $\frac{\partial}{\partial y_i}$: $\xi_1 = \sum_{i=1}^n b_i \frac{\partial}{\partial y_i}$. Put $\tilde{\xi}_1 = \xi_1 - \sum_{i=2}^m b_i \xi_i$ $(b_i \in E_n)$. Then $\tilde{\xi}_1 = b_1 \frac{\partial}{\partial y_1} + \sum_{j=m+1}^n b_j \frac{\partial}{\partial y_j}$. There exists $j \in \{1, m+1, m+2, \ldots, n\}$ such that $b_j(0) \neq 0$. After a suitable reordering of coordinates we may assume that $j = 1$. Put $\Psi_1 = b_1^{-1}\tilde{\xi}_1$. Then $V(\Psi_1, \ldots, \Psi_m) = V(\xi_1, \ldots, \xi_m)$ and the fields Ψ_i commute with each other. The only claim that requires some comment is that $[\Psi_i, \Psi_1] = 0$ for $i \geqslant 2$. Indeed,

$$[\Psi_i, \Psi_1] = \sum_{j=m+1}^n \frac{\partial}{\partial y_i}\left(\frac{b_j}{b_1}\right)\frac{\partial}{\partial y_j}. \qquad (*)$$

On the other hand, there exist functions c_i^k, $k = 1, \ldots, m$, such that

$$[\Psi_i, \Psi_1] = \sum_{k=1}^m c_i^k \Psi_k = \sum_{k=1}^m c_i^k \frac{\partial}{\partial y_k} + \sum_{k=m+1}^n c_i^1 b_1^{-1} b_k \frac{\partial}{\partial y_k} \qquad (**)$$

Comparing the coefficients in (*) and (**) we get $c_i^k = 0$ for all k, which concludes the proof. □

A submanifold $Y \subset X$ is called an *integral submanifold* of the distribution V if $T_x Y \subset V_x$, $x \in Y$. If the distribution is not involutive, and hence nonintegrable, there are no integral submanifolds of dimension equal to the dimension of the distribution. However, integral submanifolds exist (e.g., the integral curves of a vector field which is subordinate to V). One can show that in the case of generic distributions of codimension $\geqslant 3$ there are no integral submanifolds of dimension greater than one at any point (we shall discuss this question later in more detail).

Let N be a differential system in W_n (respectively, in ω_n).

Definition 1.5. The *flag of a differential system* N (or simply the flag of N) is the sequence of differential systems $N_1 = N$, $N_2 = [N, N]$, $\ldots$, $N_l = [N_{l-1}, N]$. Here $[A, B]$ is the C^∞-submodule generated by $[\alpha, \beta]$, $\alpha \in A$, $\beta \in B$.

Proposition. *The sequence $N_1, N_2, \cdots \subset W_n$ stabilizes, i.e. there exists an integer r such that $N_{r-1} \neq N_r = N_{r+1} = \cdots$, moreover, N_r is a Lie subalgebra of* Vect X.

Proof. The set W_n of jets of vector fields is a finitely generated J_n-module. According to the Borel theorem (see the end of Section 1.1.), J_n is a Noetherian ring, hence W_n is a Noetherian module, so that an increasing sequence of its submodules must stabilize.

If $N_r = W_n$, the differential system N is called *totally nonholonomic*, and the smallest r such that $N_r = W_n$ is called the *nonholonomicity degree* of N (this degree depends on the point of the manifold).

Let V be a distribution, let $N = N(V)$ be the differential system associated with V, and let $N(V) = N_1 \subset N_2 \subset \cdots$ be its flag. In general, the Pfaffian systems N_i are not generated by distributions (see Section 1.1). If this is the case for all i, we may define the flag of the distribution $V = V_1 \subset V_2 \subset \cdots$.

Definition 1.6. A distribution V is called *regular* if its flag is well defined.

Let $n_i = \dim V_i$. The vector $n_1 < n_2 < \cdots$ is called the *growth vector* of the distribution V. Clearly, the sequence V_i stabilizes. If $V_r = TX$ for some r, the distribution V is *totally nonholonomic* and r is called its *nonholonomicity degree*. If the sequence V_i stabilizes for $i = 1$, so that $V_1 = V_2$, then V is a subalgebra and we are dealing with the integrable case.

1.3. Codistributions and Pfaffian Systems. All the notions introduced above admit a dualization: thus, distributions correspond to codistributions and differential systems to Pfaffian systems. This correspondence may be established both globally and locally, for jets and germs. Classics (E. Cartan, for instance) used the language of forms and codistributions much more frequently than the language of vector fields. For instance, in mechanics and geometry a distribution is usually defined as the annihilator of a codistribution. However, some authors, e.g. Vessiot, wrote in the same years (1926) that many concepts of Cartan's theory might be expressed more naturally (at least from the algebraic point of view) by using the language of vector fields. The advantage of differential forms is the existence of exterior derivative and wedge product while the advantage of vector fields is the existence of the Lie algebra structure. In the sequel, $\Omega^1(X)$ denotes the space of 1-forms on X, i.e. of smooth sections of T^*X.

Definition 1.7. A *codistribution* on X is a subbundle of the cotangent bundle T^*X or, in other words, a field of subspaces of the cotangent spaces which depend smoothly on the point of X. A *Pfaffian system* is a submodule of the $C^\infty(X)$-module $\Omega^1(X)$.

The annihilator of a distribution V, i.e. $V^\perp = \{\omega : \langle \omega, \xi \rangle = 0 \ \forall \xi \in V\}$, is a codistribution. In a similar fashion, the annihilator of a codistribution is a distribution. In the sequel we shall use both the language of vector fields and the language of differential forms. Below we give a table of parallel notions. The transition from a notion to its dual is based on the duality between the space of vector fields (or the jets (germs) of vector fields) and the space of 1-forms (respectively, of their jets, etc.) regarded as modules over the ring of functions (respectively, the ring of jets (germs) of functions). Let W_n^* denote the space of ∞-jets of 1-forms at $0 \in \mathbb{R}^n$, and ω_n^* the space of their germs. Let $\langle \xi, \omega \rangle$ be the pairing