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I. R. Shafarevich (Ed.)

Algebraic Geometry I

Algebraic Curves, Algebraic Manifolds and Schemes

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代数曲线，代数流形与概型

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I. Riemann Surfaces and Algebraic Curves

V. V. Shokurov

Translated from the Russian
by V. N. Shokurov

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Introduction[1]

The name 'Riemann surface' is a rare case of a designation which is fully justified historically: all fundamental ideas connected with this notion belong to Riemann. Central among them is the idea that an analytic function of a complex variable defines some natural set on which it has to be studied. This need not coincide with the domain of the complex plane where the function was initially given. Usually, this natural set of definition does not fit into the complex plane $\mathbb{C}$, but is a more complicated surface, which must be specially constructed from the function: this is what we call the Riemann surface of the function. One can get a complete picture of the function only by considering it on the whole of its Riemann surface. This surface has a nontrivial geometry, which determines some of the essential characters of the function.

The extended complex plane, obtained by adjoining a point at infinity, can be perceived as an embryonic form of this approach. Topologically, the extended plane is a two-dimensional sphere, also known as the Riemann sphere. This example already displays some features which are characteristic of the general notion of a Riemann surface:

1) The Riemann sphere $\mathbb{CP}^1$ can be defined by gluing together two disks (i.e., circles) of the complex plane; for instance, the disks $|z| < 2$ and $|w| < 2$, in which the annuli $\frac{1}{2} < |z| < 2$ and $\frac{1}{2} < |w| < 2$ are identified by means of the correspondence $w = z^{-1}$. (This yields the shaded area in Fig. 1.)

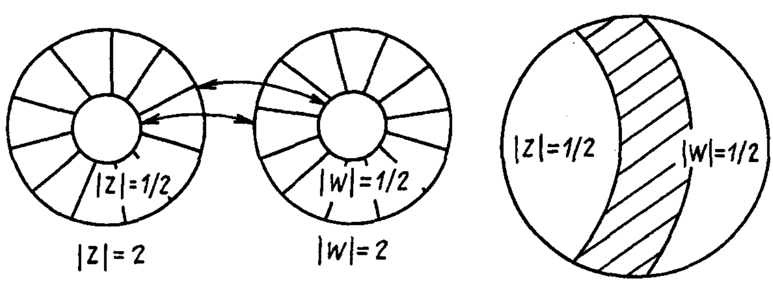

Fig. 1

2) The relation $w = z^{-1}$, which defines the gluing, is a one-to-one and analytic (conformal) correspondence of the domains it identifies. For that reason the property of being analytic at some point agrees in both circles, $|z| < 2$ and $|w| < 2$, on the identified regions. This leads to a unified notion

[1] The author expresses his profound gratitude to Professor I. R. Shafarevich for numerous remarks and suggestions, which have contributed to the improvement of the text, and for writing this introduction, which provides a fascinating bird's-eye view of the charming world of algebraic geometry.

of analytic function on the Riemann sphere glued from them. It is therefore possible to state and prove such theorems as: 'a function which is holomorphic on the whole Riemann sphere is constant', or: 'a function on the Riemann sphere which has only poles for singularities, is a rational function'.

The same principles underlie the general notion of a Riemann surface. We shall deal only with compact Riemann surfaces. By definition, this is a closed (compact) surface S glued from a finite number of disks $U_1, \dots, U_m$ in the complex plane: for any two disks, U_i and U_j, some domains, $V_{ij} \subset U_i$ and $V_{ji} \subset U_j$, are identified by means of a correspondence $\varphi_{ij}\colon V_{ij} \to V_{ji}$, which is one-to-one and analytic.

In other words, a Riemann surface is a union of sets $U_1, \dots, U_N$, each of which is endowed with a coordinate function z_i $(i = 1, \dots, N)$. This is a one-to-one mapping of U_i onto a disk in the complex plane. Further, in an intersection $V_{ij} = U_i \cap U_j$, the coordinate z_j is expressed in terms of z_i as an analytic function, and similarly z_i in terms of z_j.

Thus, just as in the case of the Riemann sphere, there is a well-defined notion of analyticity for a continuous complex-valued function, given in a neighbourhood of some point $p \in S$. Further, we can carry over to functions given on the surface S such notions as a pole, the property of being meromorphic, and so forth. Hence a Riemann surface is a set on which it makes sense to say that a function is analytic, and locally *(in a sufficiently small domain)* this amounts to the ordinary concept of analyticity in some domain of the complex plane. This definition is explained in detail in § 1 of Chapter 1.

So, with the notion of a Riemann surface, we run into an entity of a new mathematical nature. It must be rated on a par with such notions as a Riemannian manifold in geometry, or a field in algebra. Just as some metric concepts are defined in a Riemannian manifold, and algebraic operations in a field, so is the notion of analytic function on a Riemann surface. In particular, it is now possible to formulate and prove the theorem stating that a function which is holomorphic on an entire (compact) Riemann surface is constant.

That the concept of Riemann surface is nontrivial, is manifest from its connection with the theory of multivalued analytic functions. In fact, for every such function one can construct a Riemann surface on which it becomes single-valued. We restrict ourselves to algebraic functions, so the corresponding Riemann surfaces are compact.

The simplest case, represented by the function $w = \sqrt[n]{z}$, does not yet necessitate any new type of surface. Indeed we have $z = w^n$; so, even though w is a multivalued function of z, the function $z(w)$ is single-valued. Therefore we can regard w as an independent variable, running over the Riemann sphere S, which is just the Riemann surface of the function w. The relation $z = w^n$ defines a mapping of the w-sphere S onto the z-sphere $\mathbb{CP}^1$. One can think of the sphere S as lying 'above' $\mathbb{CP}^1$ (in some larger space), in such a way that above each point $z = z_0$ we find the points which are mapped into it. Then for $z_0 \neq 0, \infty$ the inverse image on S of a disk $U\colon |z - z_0| < \varepsilon$, for sufficiently small ε, is made up of n disjoint domains W_i, $i = 1, \dots, n$:

$$w = w_i g(t), \quad |t| < \frac{\varepsilon}{|z_0|}, \quad g(t) = \sqrt[n]{1+t}, \quad g(0) = 1, \quad t = \frac{z}{z_0} - 1,$$

where the w_i are the distinct values of $\sqrt[n]{z_0}$ (Fig. 2a). But, in a neighbourhood of the point 0 (respectively, of ∞), the inverse image of a disk $|z| < \varepsilon$ (respectively, $|t| < \varepsilon$, with $t = z^{-1}$) is constituted by a single circle W: $|w| < \sqrt[n]{\varepsilon}$, which lies above the disk in the form of a 'helix' (see Fig. 2b, where $n = 2$).

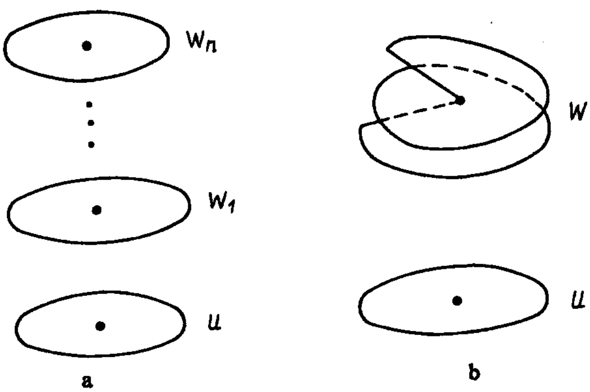

Fig. 2

In the general case, an algebraic function is defined by an equation $f(z, w) = 0$, where $f(z, w)$ is a polynomial $f(z, w) = a_0(z)\, w^n + \ldots + a_n(z)$, and the $a_i(z)$ are polynomials in z. As a first, rough approximation to the Riemann surface of the function w, we shall look at the set $\widetilde{S}$ of all solutions (z, w) of $f(z, w) = 0$. On this set, w is tautologically the function that takes on the value w_0 at (z_0, w_0). However, this definition must be made more precise. We shall assume that $\widetilde{S} \subset \mathbb{C}^2$, where $\mathbb{C}^2$ is the plane of the two complex variables z, w, and where the topology of $\widetilde{S}$ is inherited from $\mathbb{C}^2$. In other words, $\widetilde{S}$ is a complex algebraic curve lying in the plane $\mathbb{C}^2$.

To start with, suppose z_0 is such that $f(z_0, w) = 0$ has n distinct roots $w_1, \ldots, w_n$. This means that $a_0(z_0) \neq 0$ and $f'_w(z_0, w_i) \neq 0$. Then, by the implicit function theorem, w is an analytic function $g_i(z)$ of z in some neighbourhood $|z - z_0| < \varepsilon$ of z_0. More precisely, all solutions of $f(z, w) = 0$ close to (z_0, w_i) can be represented in the form $(z, g_i(z))$, $i = 1, \ldots, n$. That is to say, the solutions with $|z - z_0| < \varepsilon$ fall into n disks W_i, $i = 1, \ldots, n$:

$$|z - z_0| < \varepsilon, \quad w = g_i(z),$$

exactly as in Fig. 2a. We call them disks because the function z maps them in a one-to-one manner onto the disk U: $|z - z_0| < \varepsilon$.

It remains to consider the cases we have omitted, in which the number of solutions of $f(z_0, w) = 0$ is less than n, and also the case where $z_0 = \infty$ on the Riemann sphere $\mathbb{CP}^1$. In all these cases there exists a disk $U: |z - z_0| < \varepsilon$ (respectively, $|t| < \varepsilon$, $t = z^{-1}$, if $z_0 = \infty$) with the property that, for all points $z \in U$, $z \neq z_0$, we are in the case previously considered. We denote by $\widetilde{U}$ the associated punctured disk: $|z - z_0| < \varepsilon$, $z \neq z_0$, and by $\widetilde{W}$ its inverse image in $\widetilde{S}$. The set $\widetilde{W}$ may turn out to be disconnected.

Trivially, if $f(z_0, w) = 0$ has two distinct solutions, w_i and w_j, then two small neighbourhoods in $\widetilde{S}$ do not meet and give rise to different connected components of $\widetilde{W}$, like the sets $W_1, \dots, W_n$ in Fig. 2*a*. But there are less trivial cases in which various connected components of $\widetilde{W}$ converge to the same point of $\widetilde{S}$. The idea is that in reality these components must define distinct points of the Riemann surface S of w: they must be 'separated' in S. If, for instance, $w^2 = z^2 + z^3$ then $w = z\sqrt{1+z}$. Now the function $\sqrt{1+z}$ has two branches, $g_1(z)$ and $g_2(z) = -g_1(z)$, in a neighbourhood of $z_0 = 0$. So $\widetilde{W}$ consists of two components: $\widetilde{W}_1 = \{|z| < \varepsilon,\ z \neq 0,\ w = zg_1(z)\}$ and $\widetilde{W}_2 = \{|z| < \varepsilon,\ z \neq 0,\ w = zg_2(z)\}$, which merge as $z \to 0$ (Fig. 3*a*).

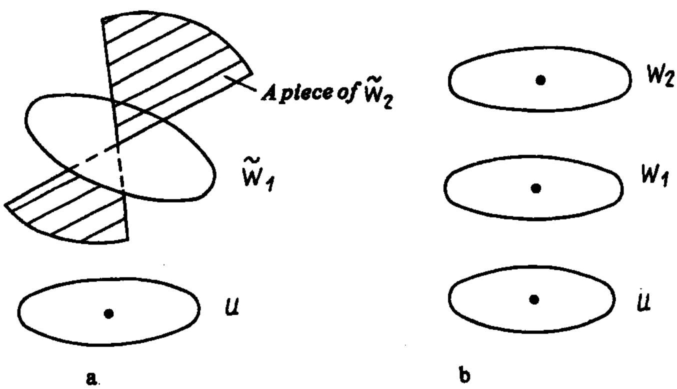

Fig. 3

In the general case, we denote by $\widetilde{W}_1, \dots, \widetilde{W}_r$ the connected components of $\widetilde{W}$. The Riemann surface S is defined in such a way that in it the $\widetilde{W}_i$ are, so to speak, 'isolated' from each other: their closures do not meet as $z \to z_0$. Set-theoretically, S differs from $\widetilde{S}$ in that now there are r distinct points above z_0, each corresponding to its own component $\widetilde{W}_i$. More precisely, each $\widetilde{W}_i$ is a connected unramified covering of the punctured disk $\widetilde{U}$: above every point $z \in \widetilde{U}$, we find the same number n_i (say) of points in $\widetilde{W}_i$, and $n_1 + \dots + n_r = n$. It is easy to prove that a function w_i can be defined on each $\widetilde{W}_i$ in such a way that $\widetilde{W}_i$ is given as the punctured disk $|w_i| < \varepsilon^{1/n_i}$, $w_i \neq 0$, and the mapping $\widetilde{W}_i \to \widetilde{U}$ is defined as $z - z_0 = w_i^{n_i}$. We can then look at the unpunctured disk W_i: $|w_i| < \varepsilon^{1/n_i}$. The various disks W_i are regarded as disjoint sets in the Riemann surface S (cf. Fig. 3*b*). Each of them is mapped by the function w_i onto a disk of the complex plane, and they lie above the Riemann z-sphere as in Fig. 2b.

From all the disks W_i we have constructed, above the various points $z_0 \in \mathbb{CP}^1$ (including $z_0 = \infty$), we can select a finite number, $W_1, \dots, W_N$, whose union already contains all the others. From the analyticity of all the mappings we have encountered, it is easy to deduce that the variety obtained by gluing the disks $W_1, \dots, W_N$ verifies the condition occurring in the definition of a Riemann surface. Thus, S is indeed a Riemann surface. For a detailed justification of this construction, see Chapter 1, § 2.

An arbitrary Riemann surface carries with it a large amount of geometric information. In particular, the Riemann surface of an algebraic function reveals some important characteristics of that function. Since the gluings φ_{ij} are conformal, and hence orientation-preserving, transformations, any Riemann surface is orientable. So, from a topological point of view it has a unique invariant: the genus. In Fig. 4 are depicted surfaces of genus $g = 0, 1, 2, 3, 4$.

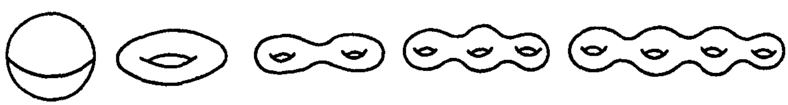

Fig. 4

If, for example, a polynomial $f(z)$ (of degree $2n$ or $2n-1$, say) has no multiple roots, then the Riemann surface of the function $w = \sqrt{f(z)}$ is of genus $n-1$. But, in addition, one can define on a Riemann surface all the notions which are invariant under conformal transformations: it has a 'conformal geometry'. Among such notions are the Laplace operator and harmonic functions. In particular, the real and imaginary parts of a function which is analytic in some domain of a Riemann surface, are harmonic. This enables us to study functions on a Riemann surface by applying the apparatus of elliptic differential operators and even some physical intuition. A harmonic function on a Riemann surface can be conceived as a description of a stationary state of some physical system: a distribution of temperatures, for instance, in case the Riemann surface is a homogeneous heat conductor. Klein (following Riemann) had a very concrete picture in his mind:

"This is easily done by covering the Riemann surface with tin foil ... Suppose the poles of a galvanic battery of a given voltage are placed at the points A_1 and A_2. A current arises, whose potential u is single-valued, continuous, and satisfies the equation $\Delta u = 0$ across the entire surface, except for the points A_1 and A_2, which are discontinuity points of the function."

[*Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert,* p. 260]

The existence of functions, which is suggested by such physical considerations, is established on the basis of the theory of elliptic partial differential equations. This provides an absolutely new method of constructing analytic functions on a Riemann surface: once a harmonic function u has been constructed, we select its conjugate function v, so that $u + iv$ is analytic.

In particular, this enables one to describe the stock of all meromorphic functions on any Riemann surface S. If S is the Riemann surface of an algebraic function w given by $f(z, w) = 0$, *then both* w *and* z *are meromorphic* functions on S. Therefore any rational function of w and z is meromorphic. It can easily be proved that this is the way all meromorphic functions on S are obtained. This is a generalization of the theorem saying that a meromorphic function on the Riemann sphere is a rational function of z. For an arbitrary Riemann surface, however, it is by no means obvious that there is even one nonconstant meromorphic function. Such a function is constructed, as we have just said, by using methods from the theory of elliptic partial differential equations. Furthermore, one can construct along the same lines two meromorphic functions w and z on S, connected by a relation of the form $f(z, w) = 0$, where f is a polynomial, and with the property that S is just the Riemann surface of the algebraic function w defined by the equation $f = 0$. This result is known as 'Riemann's existence theorem'.

Hence the abstract notion of a (compact) Riemann surface reduces to that of Riemann surface for an algebraic function. This is a highly nontrivial result, with powerful applications. Indeed, in a number of particular situations, what arises is an 'abstract' Riemann surface. Then the preceding theorem provides a very explicit realization of such a surface. The simplest example of such a situation is when S is the quotient group $\mathbb{C}/\Lambda$ of the complex plane $\mathbb{C}$ modulo a lattice $\Lambda = \{\omega_1 n_1 + \omega_2 n_2 \mid n_1, n_2 \in \mathbb{Z}\}$, spanned by two complex numbers ω_1 and ω_2. Let U be any sufficiently small disk, so that no two of its points differ by a vector from Λ. Then the coordinate z on $\mathbb{C}$ is a one-to-one mapping of U onto a domain in $S = \mathbb{C}/\Lambda$ (Fig. 5). Further, these disks form a covering of S. Topologically S is a torus: it is of genus 1. In this situation, Riemann's existence theorem shows that S is the Riemann surface of an algebraic function $w = \sqrt{z^3 + az + b}$, where a and b are some complex numbers and the polynomial $z^3 + az + b$ has no multiple roots. It can be shown that every Riemann surface of genus 1 can be obtained in this way. The meromorphic functions on S are interpreted as being all meromorphic functions of z which are invariant under translations by vectors of the lattice Λ, that is, elliptic functions. In this case, Riemann's existence theorem furnishes a very explicit description of an elliptic function field.

Such a description is possible for Riemann surfaces of genus $g > 1$ as well. One has to consider discrete groups of linear fractional transformations acting in the disk $|z| < 1$. Two points are identified if they are sent to each other by an element of such a group Γ. Thus the Riemann surface is represented as a quotient $\Gamma\backslash\mathbb{D}$, where $\mathbb{D}$ is the unit disk. Just like the plane $\mathbb{C}$ (for surfaces of genus 1), the unit disk, for genus $g > 1$, is the universal covering of the Riemann surface S. For $g = 0$, S is nothing else than the Riemann sphere and is its own universal covering. In the plane $\mathbb{C}$ the Euclidean metric