21世纪高等院校教材

Ordinary Differential Equations and Their Applications
——*Theories and Models*

（常微分方程及其应用——理论与模型）

周宇虹　罗建书　编著

科学出版社

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内 容 简 介

本书是常微分方程课程的英文教材，是作者结合多年的双语教学经验编写而成. 全书共 5 章，包括一阶线性微分方程，高阶线性微分方程，线性微分方程组，Laplace 变换及其在微分方程求解中的应用，以及微分方程的稳定性理论. 书中配有大量的应用实例和用 Matlab 软件绘制的微分方程解的相图，并介绍了绘制相图的程序.

本书可作为高等院校理工科偏理或非数学专业的本科双语教材，也可供相关专业的研究生、教师和广大科技人员参考.

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前　言

“常微分方程”是数学专业的一门专业基础课, 随着常微分方程理论与方法的应用领域不断扩大, 它也被许多工科专业列为本科生和研究生的选修课. 本书是作者根据多年为信息科学与计算专业本科生上的“常微分方程”双语课、公共“常微分方程”选修课, 以及数学专业的“科技英语”课的讲稿整理而成的. 科学技术的进步和发展, 使得常微分方程的理论与方法得到了越来越广泛的应用, 其应用领域涉及自然科学、工程技术、社会、经济、生物、生态与环境、医学、交通、军事等, 硕果累累. 因此, 在内容的编排方面, 作者除了积极吸纳国内外优秀教材在反映最新教学理念和知识结构方面的精彩之处外, 还在每一章中精选了不同领域的应用案例, 以期让学生了解常微分方程是如何用于解决实际问题的. 根据学生的专业特性, 书中介绍了用 Laplace 变换解微分方程和方程组的方法和技巧. 书中还首次介绍了 Iode 软件及其在平衡点附近解的几何特征研究和平衡点稳定性分析中的应用 (这部分内容由罗建书负责编写).

本书是一本双语课教材. 将常微分方程作为双语课, 是因为学生不仅可以学到有关常微分方程、微积分和高等代数方面的专业术语, 还可以通过不同领域的应用案例拓展阅读范围, 从而提高学生的专业英语水平. 为了使学生更好地理解和掌握学科知识, 提高双语教学的教学质量和教学效果, 书中对专业术语、比较难懂的词句作了注释.

华中师范大学的严国政教授和中北大学的靳祯教授对书稿作了认真仔细的审阅, 并提出了宝贵的修改意见. 在此, 作者向他们表示衷心的感谢. 本书的出版得到南京林业大学“精品教材”建设项目和国防科学技术大学专业建设项目的资助. 对两所学校多年来对双语教学给予的鼓励和支持, 作者在此表示衷心的感谢.

对于书中不可避免的各种疏漏及不妥之处, 敬请各位读者和同行批评指教, 作者将不胜感激.

作　者

2010 年 10 月于南京

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Chapter 1 First-order Differential Equations[1]

1.1 Introduction

Example 1.1.1 Dating of art works[2]

On May 29, 1945, H.A.Van Meegeren, *a third rate Dutch painter*[3], *was arrested on the charge of collaborating with the enemy for*[4] the sale to Goering of a painting of famed 17th century Dutch painter. Van Meegeren refused to accept the charge and announced, in his *prison cell*[5], that he had never sold painting to Goering. He stated that all the *questioned paintings*[6] were his own works. To settle the question an international panel of distinguished chemists, physicists and art historians was appointed to investigate the matter. The panel took X-rays of the paintings to determine whether other paintings were underneath those paintings, analyzed the *pigments*[7] (coloring materials) used in the paintings, and examined the paintings for certain *signs of old age*[8].

The panel of experts found traces of the modern pigment *cobalt blue*[9] in some paintings. In addition, they also detected *phenoformaldehyde*[10], which was not discovered until *the turn of the 19th century*[11], in several paintings. On the basis of these evidences Van Meegeren *was convicted, of forgery*[12], on October 12, 1947 and *sentenced to one year in prison*[13]. Two months later, he died of a heart attack.

However, many people refused to believe that the famed "*Disciples at Emmaus*[14]" was a *forgery*[15]. In 1967, almost twenty years later, scientists at Carnegie Mellon University proved that the "Disciples at Emmaus" was indeed a forgery.

The key to the dating of materials lies in *the phenomenon*

[1] 一阶微分方程
[2] 艺术品的断代
[3] 一个三流的德国画家
[4] 因 ··· 被判通敌罪
[5] 单人牢房
[6] 受质疑的绘画作品
[7] 颜料
[8] 旧年代的痕迹
[9] 钴蓝（一种现代绘画颜料）
[10] 苯酚甲醛
[11] 19 世纪末
[12] 因伪造被判有罪
[13] 被判服狱一年
[14] Emmaus 的信徒们
[15] 赝品

[16] 放射性现象
[17] 某些放射性元素的原子是不稳定的
[18] 在一个给定的时间段内一定比例的原子会自然衰减并生成一种新元素的原子
[19] 与···成正比
[20] 其中
[21] 物质的衰变系数
[22] 半衰期
[23] 读乐谱的妇人
[24] 弹曼陀林的妇人
[25] 糖尿病的检测
[26] 糖尿病(一种代谢疾病)
[27] 尿液
[28] 葡萄糖耐量测试
[29] 一整夜的禁食
[30] 公认标准

of radioactivity[16] discovered at the turn of the 20th century by the physicist Rutherford and his colleagues. They showed that *the atoms of certain "radioactive" elements are unstable*[17] and that *within a given time period a fixed proportion of the atoms spontaneously disintegrates to form atoms of a new element*[18]. Rutherford also showed that the radioactivity of a substance *is directly proportional to*[19] the number of atoms of the substance present. Let $N(t)$ denotes the number of atoms present at time t, then $\mathrm{d}N/\mathrm{d}t$, the number of atoms that disintegrate per unit time, is proportional to N, thus we have the following equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N, \tag{1.1.1}$$

where[20] constant λ is positive and is known as *the decay constant of the substance*[21]. Usually, we use ***half-life***[22], the time required for half of a given quantity of radioactive atoms to decay, to measure the rate of disintegration of a substance. Assume that $N(t_0) = N_0$, then we have the mathematical model for computing half-life

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N, \quad N(t_0) = N_0. \tag{1.1.2}$$

By evaluating the present disintegration rates of the radioactive pigments in Van Meegeren's questioned paintings, the experts concluded that the paintings "*Disciples at Emmaus*", "*Woman Reading Music*[23]" and "*Woman Playing Mandolin*[24]" must be modern forgeries. ■

Example 1.1.2 *Detection of diabetes*[25]

Diabetes mellitus[26] is a disease of metabolism which is characterized by too much sugar in the blood and *urine*[27]. *Glucose tolerance test*[28] (GTT) is a commonly used method to diagnose the disease. In this test, the patient is asked to take a large dose of glucose after *an overnight fast*[29]. During the next three to five hours, several measurements of the concentration of glucose are made in the patient's blood, and these measurements are used in the diagnosis of diabetes. Unfortunately, there is no *universally accepted criterion*[30] exist for interpreting the results of a GTT. Different *physicians*[31] interpreting the results of a GTT may *come up with*[32] different diagnoses. Here is a case. A Rhode Island physician, after reviewing the results of a GTT, came up with a diagnosis of diabetes. But another physician declared the patient to be normal after reviewing the results of the same GTT. To settle the question, the results of the GTT were sent to a specialist in Boston. After examining these results, the specialist concluded that the patient was suffering from a *pituitary tumor*[33].

In the mid-1960's, Drs. Rosevear and Molnar of the Mayo Clinic and Drs. Ackerman and Gatewood of the University of Minnesota discovered a fairly reliable criterion for interpreting the results of a GTT. They constructed a model which could accurately describe *the blood glucose regulatory system*[34] during a glucose tolerance test and in which one or two *parameters*[35] would *yield criteria for distinguishing normal individuals from mild diabetics and prediabetics*[36].

The basic model is described *analytically*[37] by following *system of equations*[38]

$$\frac{\mathrm{d}G}{\mathrm{d}t} = F_1(G, H) + J(t), \tag{1.1.3}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = F_2(G, H), \tag{1.1.4}$$

where G denotes the concentration of glucose in the blood and H denotes the concentration of the net *hormonal*[39] concentration. The function $J(t)$ is the external rate at which the blood glucose concentration is being increased. ■

Definition 1.1.1 An equation relating an unknown function, *its derivatives*[40] and *independent variables*[41] is called a ***differential equation***[42].

Eq.(1.1.1)-Eq.(1.1.4) are all examples of differential equation.

Differential equations are frequently used to describe the changing universe.

Example 1.1.3 According to ***Newton's law of cooling***[43]: the rate of change with respect to time t of the temperature $T(t)$

[31] (内科) 医生
[32] 得出
[33] 脑垂体肿瘤
[34] 血糖调节系统
[35] 参数
[36] 产生能区别出正常人与中度糖尿病患者和早期糖尿病患者的标准
[37] 解析地
[38] 方程组
[39] 荷尔蒙、激素
[40] 它的各阶导数
[41] 自变量
[42] 微分方程
[43] 牛顿冷却定律

[44] 数学模型
[45] 种群
[46] 出生率和死亡率
[47] 比例系数
[48] 微分方程的解
[49] 对 · · · 微分
[50] 通解

of a body is proportional to the difference between T and the temperature A of the surrounding medium, then the *mathematical model*[44] describing this law is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - A), \tag{1.1.5}$$

where k is a positive constant. ■

Example 1.1.4 Let $P(t)$ denote a *population*[45] with constant *birth and death rates*[46], then the rate of change of $P(t)$ is, in many simple cases, proportional to the size of the population, thus we have the following differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(t), \tag{1.1.6}$$

where k is *the constant of proportionality*[47]. ■

If a continuous function together with its derivatives satisfy a differential equation, then we call this function a ***solution of the differential equation***[48].

Example 1.1.5 Is $y(t) = c_1\sin 2t + c_2\cos 2t$, where c_1 and c_2 are arbitrary constants, a solution of the differential equation $y'' + 4y = 0$?

Solution *Differentiating*[49] $y(t)$ with respect to t, we have that

$$y' = 2c_1\cos 2t - 2c_2\sin 2t, \quad y'' = -4c_1\sin 2t - 4c_2\cos 2t.$$

Hence,

$$\begin{aligned} y'' + 4y &= (-4c_1\sin 2t - 4c_2\cos 2t) + 4(c_1\sin 2t + c_2\cos 2t) \\ &= (-4c_1 + 4c_1)\sin 2t + (-4c_2 + 4c_2)\cos 2t = 0. \end{aligned}$$

y and its derivative satisfy the differential equation. Moreover, y is obviously a continuous function. So $y(t) = c_1\sin 2t + c_2\cos 2t$ is a solution of the differential equation. ■

From Example 1.1.5 we see that a differential equation may have infinite many solutions. The set of all solutions of a differential equation is called the ***general solution***[50] of the differential equation. For example, function $y(t) = c_1\sin 2t + c_2\cos 2t$, where c_1 and c_2 are arbitrary constants, is the general solution of equation $y'' + 4y = 0$ since every solution of the differential equation is

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of this form. Any one solution of a differential equation is called a ***particular solution***[51] of the differential equation. It is not difficult to *verify*[52] that

(a) $y = \sin 2t + \cos 2t$,

(b) $y = 4\sin 2t$,

(c) $y = -3\cos 2t$, and

(d) $y = 0$

are all solutions of the differential equation $y'' + 4y = 0$. So they are all particular solutions of the differential equation.

Remark[53] **1.1.1** The general solution of a differential equation cannot always be expressed by a single formula.

Example 1.1.6 We can verify that the function $y = cx^4$ is a family of solutions of the differential equation $xy' - 4y = 0$ on the *interval*[54] $(-\infty, +\infty)$. Moreover, we can also verify that the *piecewise-defined function*[55]

$$y = \begin{cases} -x^4, & x < 0, \\ x^4, & x \geqslant 0 \end{cases}$$

is a particular solution of the differential equation. Obviously, this particular solution cannot be obtained from $y = cx^4$ by a choice of the parameter c. We refer such extra solution as ***singular solution***[56] of the differential equation. So the family of solutions $y = cx^4$ is not the general solution of the equation. ■

A solution of a differential equation that *is identically zero*[57] on an interval I is called a ***trivial solution***[58].

A differential equation along with *subsidiary conditions*[59] on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an ***initial-value problem***[60] (or ***Cauchy problem***[61]):

$$\begin{cases} F(t; y, y', y'', \cdots, y^{(n)}) = 0, \\ y(t_0) = y_0, y'(t_0) = y_0', y''(t_0) = y_0'', \cdots, y^{(n)}(t_0) = y_0^{(n)}. \end{cases} \tag{1.1.7}$$

Eq.(1.1.2) is an initial-value problem.

If the subsidiary conditions are given at more than one value of the independent variable, the problem is called a ***boundary-***

[51] 特解
[52] 证明
[53] 注
[54] 区间
[55] 分段定义函数
[56] 奇(异)解
[57] 恒为零
[58] 平凡解
[59] 附加条件
[60] 初值问题
[61] 柯西问题

[62] 边值问题
[63] 边界条件
[64] 类型
[65] 阶
[66] 线性性
[67] 常微分方程
[68] 偏微分方程
[69] 微分方程的阶
[70] 一阶微分方程
[71] 二阶微分方程
[72] n 阶常微分方程
[73] 实值函数
[74] 线性的

value problem[62] and the conditions are called ***boundary conditions***[63].

Example 1.1.7 Problem

$$3y''' + 5y'' - y' + 7y = 0; \quad y(1) = 0, y'(1) = 0, y''(1) = 0 \quad (1.1.8)$$

is an initial-value problem. While the problem

$$y'' + 2y' = e^t; \quad y(0) = 1, y'(1) = 1 \quad (1.1.9)$$

is a boundary-value problem. ■

Differential equations can be classified in three ways: *type*[64], *order*[65], and *linearity*[66].

(i) A differential equation is an ***ordinary differential equation***[67] if the unknown function depends on only one independent variable, such as Eq.(1.1.1)—Eq.(1.1.9). If the unknown function depends on two or more independent variables, then we call it a ***partial differential equation***[68]. For example, equation

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = 1 \quad (1.1.10)$$

is a partial differential equation. We will be concerned solely with ordinary differential equations in this textbook.

(ii) The order of the highest derivative of the unknown function that appears in a differential equation is called the ***order of the differential equation***[69]. Thus Eq.(1.1.1) is a *first-order differential equation*[70], Eq.(1.1.9) and Eq.(1.1.10) are *second-order differential equations*[71]. The general form of an *nth-order ordinary differential equation*[72] is

$$F(t, y, y', \cdots, y^{(n)}) = 0, \quad (1.1.11)$$

where F is a *real-valued function*[73] of $(n+2)$ variables, t, y, y', $\cdots$, $y^{(n)}$.

(iii) nth-order ordinary differential equation (1.1.11) is said to be ***linear***[74] if F is linear about y, y', $\cdots$, $y^{(n)}$. Thus the equations

$$(y - t)\mathrm{d}t + 4t\mathrm{d}y = 0,$$

$$y'' + y = 0,$$

$$4t^2\frac{\mathrm{d}^4y}{\mathrm{d}t^4}-3\frac{\mathrm{d}y}{\mathrm{d}t}+5ty=0$$

[75] 非线性的
[76] 证明

are, in turn, linear first-, second-, and fourth-order ordinary differential equations. A ***nonlinear***[75] ordinary differential equation is simply one that is not linear. Therefore,

$$(y-t)^3\mathrm{d}y+4t\mathrm{d}t=0,$$

$$y''+y^2=0,$$

$$4t^2\frac{\mathrm{d}^3y}{\mathrm{d}t^3}-3\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^3+5ty=0$$

are examples of nonlinear first, second, and third-order ordinary differential equations, respectively.

Exercise 1.1

In Problems 1-5 determine whether the given functions $y(t)$ are solutions of the differential equations on left?

1. $y''+2y'+y=0;\quad y(t)=2\mathrm{e}^{-t}+t\mathrm{e}^{-t}.$

2. $y''+2y'+y=t;\quad y(t)\equiv 1.$

3. $(y')^4+y^2=-1;\quad y=t^2-1.$

4. $y'+y^2=0;\quad y_1=\dfrac{1}{t},\quad y_2\equiv 0.$

5. $y''+4y=0;\quad y(0)=0,\quad y'(0)=1;$

$$y_1(t)=\sin 2t,\quad y_2(t)=t, y_3(t)=\frac{1}{2}\sin 2t.$$

6. *Show that*[76] $y=\ln t$ is a solution of $ty''+y'=0$ on interval $(0, +\infty)$ but is not a solution on $(-\infty, +\infty)$.

7. Show that $y=\dfrac{1}{t^2-1}$ is a solution of $y'+2ty^2=0$ on the interval $I=(-1,1)$ but not on any larger interval containing I.

8. Determine the order and linearity in each of the following differential equations.

(a) $y'''-5ty'=\mathrm{e}^t+1;$

(b) $ty''+t^2y'-(\sin t)\sqrt{y}=t^2-1;$

(c) $y\dfrac{\mathrm{d}^2x}{\mathrm{d}y^2}=y^2+1;$

(d) $y\left(\dfrac{\mathrm{d}x}{\mathrm{d}y}\right)^2=x^2+1;$

(e) $y'=(\sin t)y+\mathrm{e}^t;$

(f) $y'=t\sin y+\mathrm{e}^t.$

1.2 First-order Linear Differential Equations

[77] 可积函数
[78] 对 ··· 求关于 t 的积分
[79] 积分常数
[80] 积分
[81] 齐次的

The standard form for a first-order differential equation in the unknown function $y(t)$ is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \tag{1.2.1}$$

where f is any *integrable function*[77] of t.

Consider first the simple case where $f(t, y)$ is only the function of one variable t,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t). \tag{1.2.2}$$

Integrating both sides of Eq.(1.2.2) *with respect to* t[78] which yields

$$y(t) = \int f(t)\mathrm{d}t + c. \tag{1.2.3}$$

Here c is an arbitrary *constant of integration*[79]. Unfortunately, we will not be able to solve most differential equations simply by *integration*[80] since, in most cases, we cannot integrate the function $f(t)$ directly. For example, the solution of the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \mathrm{e}^{t^2}y = 0; \quad y(1) = 2$$

is

$$y(t) = 2\exp\left(-\int_1^t \mathrm{e}^{s^2}\mathrm{d}s\right).$$

We cannot integrate the function e^{t^2} directly!

We should, therefore, turn our attention to those differential equations that we can solve. The "simplest" equations that we can solve are those which are linear.

1.2.1 First-order Homogeneous[81] Linear Differential Equations

Definition 1.2.1 The general form of the first-order linear differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = b(t), \tag{1.2.4}$$

where $a(t)$ and $b(t)$ are continuous functions of t. If $b(t) \equiv 0$, the corresponding equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = 0 \tag{1.2.5}$$

is called a first-order **homogeneous** linear differential equation. If $b(t) \not\equiv 0$, then Eq.(1.2.4) is called a first-order ***nonhomogeneous***[82] linear differential equation.

[82] 非齐次的
[83] 取指数
[84] 上式
[85] 微积分的介值定理

To solve the homogeneous equation (1.2.5), we divide both sides of the equation by y and rewrite it in the form

$$\frac{y'}{y} = -a(t). \tag{1.2.6}$$

Integrating both sides of Eq.(1.2.6) gives

$$\ln|y(t)| = -\int a(t)\mathrm{d}t + c_1,$$

where c_1 is an arbitrary constant of integration. *Taking exponentials*[83] of both sides of *last equation*[84] yields

$$|y(t)| = \exp\left(-\int a(t)\mathrm{d}t + c_1\right) = c\exp\left(-\int a(t)\mathrm{d}t\right)$$

or

$$\left|y(t)\exp\left(\int a(t)\mathrm{d}t\right)\right| = c. \tag{1.2.7}$$

If there exist two different t_1 and t_2 such that

$$\left(y(t)\exp\left(\int a(t)\mathrm{d}t\right)\right)\Big|_{t=t_1} = c,$$

$$\left(y(t)\exp\left(\int a(t)\mathrm{d}t\right)\right)\Big|_{t=t_2} = -c,$$

then, by *the intermediate value theorem*[85], $y(t)\exp\left(\int a(t)\mathrm{d}t\right)$ must achieve all values between $-c$ and $+c$, this is contrary to (1.2.7). Hence,

$$y(t)\exp\left(\int a(t)\mathrm{d}t\right) = c.$$

Thus,

$$y(t) = c\exp\left(-\int a(t)\mathrm{d}t\right). \tag{1.2.8}$$

[86] 变化趋势
[87] 趋于无穷大
[88] 对 ⋯ 求从 t_0 到 t 的积分

Eq.(1.2.8) is the general solution of the homogeneous equation (1.2.5).

Example 1.2.1 Solve $y' + y\sqrt{t}\sin t = 0$.

Solution Here $a(t) = \sqrt{t}\sin t$, so

$$y(t) = c\exp\left(-\int \sqrt{t}\sin t\mathrm{d}t\right). \qquad \blacksquare$$

Example 1.2.2 Find the general solution of differential equation $y'(t) + ay(t) = 0$, and then determine the *behavior*[86] of all solutions as $t \to \infty$, where a is a constant.

Solution The general solution is

$$y(t) = c\exp\left(-\int a\mathrm{d}t\right) = c\mathrm{e}^{-at}.$$

If $a < 0$, all solutions, with the exception of $y(t) = 0$, *approach infinity*[87] as $t \to \infty$.

If $a > 0$, all solutions approach zero as $t \to \infty$. $\blacksquare$

The initial-value problem corresponding to Eq.(1.2.5) is

$$\frac{\mathrm{d}y}{\mathrm{d}t} + a(t)y = 0; \quad y(t_0) = y_0. \tag{1.2.9}$$

To find the solution of Eq.(1.2.9) we *integrate* both sides of Eq.(1.2.6) *between t_0 and t*[88],

$$\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\ln|y(s)|\mathrm{d}s = -\int_{t_0}^{t} a(s)\mathrm{d}s.$$

Thus,

$$\ln|y(t)| - \ln|y(t_0)| = -\int_{t_0}^{t} a(s)\mathrm{d}s$$

or

$$\ln\left|\frac{y(t)}{y(t_0)}\right| = -\int_{t_0}^{t} a(s)\mathrm{d}s.$$

Taking exponentials of both sides of last equation we obtain that

$$\left|\frac{y(t)}{y(t_0)}\right| = \exp\left(-\int_{t_0}^{t} a(s)\mathrm{d}s\right)$$

or

$$\left|\frac{y(t)}{y(t_0)}\exp\left(\int_{t_0}^{t} a(s)\mathrm{d}s\right)\right| = 1.$$

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