Applications of Variational Methods

变分方法及应用

贺小明／著

By

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X. He dedicate this book to his parents:

YUN HE & GUI-YING WANG

Preface

Since the birth of the Calculus of Variations, it has been understood that they apply, variational methods can obtain better results than other methods. Moreover, they apply a very large number of situations. It was also realized many years ago that the solutions of a large number of problems are in fact critical points of functionals.

This book mainly reflects a significant part of my research activity during recent years. Except for the first chapter, it is constructed based on the results obtained myself or cooperated directly with other mathematicians such as D. Cao, W. Zou, etc.. The main results of this book generalize and improve some important known results involved in elliptic equations, superlinear problems. So, some newest research progresses on these topics are presented.

The material covered in this book is for advanced graduate and Ph.D students or anyone who engages in critical points theory and its applications. The book is organized as follows.

In Chapter 1 we provide some prerequisites for this book. We summarize some knowledge of Sobolev space, differential functionals and so forth. Basically, these theories are essentially known and readily available in many books. Well trained readers may skip this chapter.

In Chapter 2 we study the existence of nontrivial solutions for some superlinear variational problems, including a generalized Kadomtsev-Petviashvili equation in multi-dimensional spaces, two classes of quasilinear problem with nonlinear boundary conditions and a superlinear elliptic BVPs.

In Chapter 3 we study the existence and multiplicity of solutions for Brezis-Nirenberg problems with Hard terms by means of the pseudo-index theory and

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cohomological index theory.

Chapter 4 deals with the existence of infinitely many nontrivial solutions to Kirchhoff-type problems. The main tools are local minima approach and a variant version of the fountain theorem of M. Willem.

In Chapter 5 we study the existence and multiplicity of solutions for several kinds of singular elliptic equation involving critical Sobolev-Hardy exponents by means of the concentration-compactness principle and variational methods.

I have a good fortune to be a post-doctoral at the Institute of Applied Mathematics, Chinese Academy of Science, following Professor D. Cao for the years 2002 to 2004, and at the Department of Mathematics of Tsinghua University during the years 2006 to 2008, following Professor W. Zou. I am very grateful to Professor Z.Q. Wang for inviting me to visit his department during the year 2009, and enlightening discussions with Wang when I visited Utah State University.

This book mainly consists of the results of my recent research. It is not intended and nor is it possible to be complete. The author wishes to thank the NSFC (No. 10971238), for supporting the publication of this book.

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Chapter 1

Preliminaries

In this chapter, we summarize some classical results on nonlinear functional analysis and partial differential equations. Some of them are well known and we shall omit their proofs. For others, although their proofs may be found in many existing books, see, for instances, [144,150,113,46,153,154], etc., we make no apology for repeating them.

1.1 Sobolev Spaces

Let Ω be an open subset of $\mathbb{R}^N, N \in \mathbb{N}$. Define

$$L^p(\Omega) := \{u : \ \Omega \to \mathbb{R} \ \text{ is Lebesgue measurable, } \ ||u||_{L^p(\Omega)} < \infty\},$$

where

$$||u||_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}, \quad 1 \le p < +\infty.$$

If $p = +\infty$,

$$||u||_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\Omega} |u| := \inf_{A \subset \Omega, meas(A)=0} \sup_{\Omega \setminus A} |u|,$$

where $meas(A)$ denotes the Lebesgue measure of A. Let

$$L^p_{loc}(\Omega) := \{u : \ \Omega \to \mathbb{R}, u \in L^p(V) \text{ for each } V \subset\subset \Omega\},$$

where $V \subset\subset \Omega \Longleftrightarrow V \subset \bar{V} \subset \Omega$ and $\bar{V}$ is compact. We shall in the sequel write $||u||_{L^p(\Omega)}$ as $||u||_p$ or $|u|_p$ for short.

Let $C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions $\phi : \Omega \to \mathbb{R}$ with compact support in Ω. For each $\phi \in C_c^\infty(\Omega)$ and a multiindex $\alpha = (\alpha_1, \cdots, \alpha_N)$ with order $|\alpha| = \alpha_1 + \cdots + \alpha_N$, we denote

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \phi.$$

Definition 1.1. Suppose $u, v \in L^1_{loc}(\Omega)$. We say that v is the α^{th}-weak partial derivative of u, written as $D^\alpha u = v$ if

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx$$

for all $\phi \in C_c^\infty(\Omega)$.

It is easy to verify that the α^{th}-weak partial derivatives of u, if it exists, i s uniquely defined up to a set of measure zero.

Let $C^m(\Omega)$ be the set of functions having derivatives of order $\leq m$ being continuous in $\Omega(m = \text{integer} \geq 0$ or $m = \infty)$. Let $C^m(\bar{\Omega})$ be the set of functions in $C^m(\Omega)$ all of whose derivatives of order $\leq m$ have continuous extension to $\bar{\Omega}$.

Definition 1.2. *Fix $p \in [1, +\infty]$ and $k \in \mathbb{N} \cup \{0\}$. The Sobolev space*

$$W^{k,p}(\Omega)$$

consists of all $u : \Omega \to \mathbb{R}$ which have α^{th}-weak partial derivatives $D^\alpha u$ for each multiindex α with $|\alpha| \leq k$ and $D^\alpha u \in L^p(\Omega)$.

When $p = 2$, we usually set

$$H^k(\Omega) = W^{k,2}(\Omega), \quad k = 0, 1, 2, \cdots$$

Note that $H^0(\Omega) = L^2(\Omega)$.

Definition 1.3. *If $u \in W^{k,p}(\Omega)$, we define its norm as*

$$||u||_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p\right)^{\frac{1}{p}}, & p \in [1, +\infty), \\ \sum_{|\alpha|\leq k} ess \sup_\Omega |D^\alpha u|, & p = +\infty. \end{cases}$$

Definition 1.4. *We define $W_0^{k,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ with respect to its norm defined in Definition 1.3. It is customary to write*

$$H_0^k(\Omega) = W_0^{k,p}(\Omega)$$

and denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$.

The following results can be found in L.C. Evans [61].

Proposition 1.5. *For each $k = 1, 2, \cdots$ and $1 \leq p \leq +\infty$, the Sobolev space $(W^{k,p}(\Omega), ||\cdot||_{W^{k,p}(\Omega)})$ is a Banach space and so is $W^{k,p}(\Omega)$. In particular, $H^k(\Omega), H_0^1(\Omega)$ are Hilbert spaces.*

Definition 1.6. *Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be two Banach spaces, $X \subset Y$. We say that X is continuously imbedded in Y (denoted by $X \hookrightarrow Y$) if the identity $id: X \to Y$ is a linear bounded operator, that is, there is a constant $C > 0$ such that $||u||_Y \leq C||u||_X$ for all $u \in X$. In this case, constant $C > 0$ is called the embedding constant. If moreover, each bounded sequence in X is precompact in Y, we say the embedding is compact, written $X \hookrightarrow\hookrightarrow Y$.*

Definition 1.7. *A function $u: \Omega \subset \mathbb{R}^N \to \mathbb{R}$ is said to be Hölder continuous with exponent $\gamma > 0$ if*

$$[u]^{(\gamma)} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\gamma} < \infty.$$

Definition 1.8. *The Hölder space $C^{k,\gamma}(\bar{\Omega})$ consists of all functions $u \in C^k(\bar{\Omega})$ for which the norm*

$$||u||_{C^{k,\gamma}(\bar{\Omega})} := \sum_{|\alpha| \leq k} ||D^\alpha u||_{C(\bar{\Omega})} + \sum_{|\alpha| = k} [D^\alpha u]^{(\gamma)}$$

is finite. It is a Banach space. We set $C^{k,0}(\bar{\Omega}) = C^k(\bar{\Omega})$.

We have the following imbedding results, see Adams [1], L.C. Evans [61] and D. Gilberg-N.S. Trudinger [69].

Proposition 1.9. *If Ω is a bounded domain in $\mathbb{R}^N$, then*

$$W_0^{k,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & kp < N, 1 \leq q \leq Np/(N - kp), \\ C^{m,\alpha}(\bar{\Omega}), & 0 \leq \alpha \leq k - m - N/p \text{ and} \\ & 0 \leq m < k - N/p < m + 1. \end{cases}$$

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Proposition 1.10. *If Ω is a bounded domain in $\mathbb{R}^N$, then*

$$W_0^{k,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q(\Omega), & kp < N, 1 \le q < Np/(N-kp), \\ C^{m,\alpha}(\bar{\Omega}), & 0 \le \alpha < k-m-N/p \text{ and} \\ & 0 \le m < k - N/p < m+1. \end{cases}$$

The following proposition can be found in H. Brezis [29] and M. Willem [144].

Proposition 1.11. *The following imbedding are continuous:*

$$H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad \text{if } 2 \le p < \infty, N = 1, 2,$$

$$H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad \text{if } 2 \le p \le 2^*, N \ge 3,$$

where $2^* = 2N/(N-2)$ if $N \ge 3$ and $2^* = +\infty$ if $N = 1, 2$, is called a critical exponent.

For $N \ge 3$, let

$$S := \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{||\nabla u||_2^2}{||u||_{2^*}^2}$$

be the best Sobolev constant. Then by G. Talenti's results (see [28]),

$$S = \frac{||\nabla U||_2^2}{||U||_2^2},$$

where

$$U(x) = \frac{(N(N-2))^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}.$$

Note that if $\mathbb{R}^N$ is replaced by a bounded domain, S is never achieved. Let

$$v_\varepsilon(x) := \frac{\bar{N}\xi(x)\varepsilon^{(N-2)/2}}{(\varepsilon^2+|x|^2)^{(N-2)/2}},$$

where $\bar{N} = (N(N-2))^{(N-2)/4}, \varepsilon > 0$ and $\xi \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\xi(x) = 1$ if $|x| \le r/2; \xi(x) = 0$ if $|x| \ge r$, where r can be chosen to meet different requirements.

Proposition 1.12. *The following estimates are true (see e.g. M. Willem [144]).*

$$||v_\varepsilon||_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N), \qquad ||v_\varepsilon||_{2^*-1}^{2^*-1} = O(\varepsilon^{(N-2)/2}),$$

$$||\nabla v_\varepsilon||_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \qquad ||\nabla v_\varepsilon||_1 = ||v_\varepsilon||_1 = O(\varepsilon^{(N-2)/2}),$$

$$||v_\varepsilon||_2^2 = \begin{cases} c\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2), & N = 4, \\ c\varepsilon^2 + O(\varepsilon^{N-2}), & N \geq 5, \end{cases}$$

where $c > 0$ is a constant.

We shall frequently use the following Gagliardo-Nirenberg Inequality, see L.C. Evans [61] and J. Chabrowski [46].

Proposition 1.13. *For every $u \in H^1(\mathbb{R}^N), ||u||_p \leq C||\nabla u||_2^\alpha ||u||_r^{1-\alpha}$ with $\frac{N}{p} = \alpha\frac{N-2}{2} + (1-\alpha)\frac{N}{r}, r \geq 1, \alpha \in [0,1]$, where c is a constant depending on p, α, r, N.*

The following concentration-compactness lemma is due to P.L. Lions [105].

Lemma 1.14. *Let $r > 0$ and $q \in [2, 2^*)$. For any bounded sequence $\{u_n\}$ of $H^1(\mathbb{R}^N)$, if*

$$\sup_{x\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^q dx \to 0, \quad \text{as } n \to \infty,$$

where $B(y,r) := \{x \in \mathbb{R}^N : \ |x-y| \leq r\}$, then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $q < p < 2^*$.

Proof. We only consider the case $N \geq 3$. Choose $p_1, p_2, t, t' > 1$, such that $p_1 t = q, p_2 t' = 2^*, 1/t + 1/t' = 1, p_1 + p_2 = p$. By Hölder Inequality and Proposition 1.10, we obtain

$$\int_{B(y,r)} |u_n|^p dx$$
$$\leq \left(\int_{B(y,r)} |u_n|^{p_1 t} dx\right)^{1/t} \left(\int_{B(y,r)} |u_n|^{p_2 t'} dx\right)^{1/t'}$$

$$
\begin{aligned}
&\leq c\left(\int_{B(y,r)} |u_n|^{p_1 t} dx\right)^{1/t} ||u_n||_{2^*}^{p_2} \\
&\leq c\left(\int_{B(y,r)} |u_n|^{p_1 t} dx\right)^{1/t} \left(\int_{B(y,r)} (u_n^2 + |\nabla u_n|^2) dx\right)^{p_2/2} \\
&\leq c\left(\int_{B(y,r)} |u_n|^{p_1 t} dx\right)^{1/t}.
\end{aligned}
$$

Here and in the sequel the letter c, C will be used repeatedly to denote various constants when the exact values are irrelevant. Covering $\mathbb{R}^N$ by balls with radius r in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, then we have

$$
\int_{\mathbb{R}^N} |u_n|^p dx \leq (N+1)c \sup_{y\in\mathbb{R}^N} \left(\int_{B(y,r)} |u_n|^q dx\right)^{1/t},
$$

which implies the conclusion. This completes the proof.

□

1.2 Differential Functionals

Let E be a Banach space with the norm $||\cdot||$ and $U \subset E$ an open set. The dual space of E is denoted by E', i.e., E' denotes the set of bounded linear functionals on E. Consider a functional $I : U \to \mathbb{R}$.

Definition 1.15. *The functional I has a Fréchet derivative $F \in E'$ at $u \in U$ if*

$$
\lim_{h\in E, h\to 0} \frac{I(u+h) - I(u) - F(h)}{||h||} = 0.
$$

Define $I'(u) = F$ or $\nabla I(u) = F$ and sometimes refer to it as the gradient of I at u. Usually, $I'(\cdot)$ is a nonlinear operator. Let $C^1(U, \mathbb{R})$ be the set of all functionals which have continuous Fréchet derivative on U. A point $u \in U$ is called a critical point of functional $I \in C^1(U, \mathbb{R})$, provided $I'(u) = 0$.

Definition 1.16. *The functional I has a Gateaux derivative $G \in E'$ at $u \in U$ if, for every $h \in E$,*

$$\lim_{t \to 0} \frac{I(u+th) - I(u)}{t} = G(h).$$

The Gateaux derivative at point $u \in U$ is denoted by $DI(u)$. Clearly, if I has a Fréchet derivative $F \in E'$ at $u \in U$, then I has a Gateaux derivative and $I'(u) = DI(u)$. But the converse is not true. However, if I has Gateaux derivative at every point of some neighborhood of $u \in U$ such that $DI(u)$ is continuous at u, then I has a Fréchet derivative and $I'(u) = DI(u)$. This is a straightforward consequence of the Mean Value Theorem.

Let $f(x,t)$ be a function on $\Omega \times \mathbb{R}$, where Ω may be bounded or unbounded. We say that f is a Carathéodory function if $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$.

Lemma 1.17. *Assume that $p \geq 1, q \geq 1$. Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ and satisfy*

$$|f(x,t)| \leq a + b|t|^{p/q}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $a, b > 0$ and Ω is either boubed or unbounded. Define a Carathéodory operator by

$$Bu := f(x, u(x)), \quad u \in L^p(\Omega).$$

Let $\{u_k\}_{k=0}^{\infty} \subset L^p(\Omega)$. If $||u_k - u_0||_p \to 0$ as $k \to +\infty$, then $||Bu_k - Bu_0||_q \to 0$ as $k \to \infty$. Particularly, if Ω is bounded, then B is a continuous and bounded mapping from $L^p(\Omega)$ to $L^q(\Omega)$ and the same conclusion is true if Ω is unbounded and $a = 0$.

Proof. It is easy to see that there is a subsequence, still denoted by $\{u_k\}$ such that

$$u_k(x) \to u_0(x), \quad \text{a.e. } x \in \Omega. \tag{1.1}$$

Since f is a Carathéodory function, we have

$$Bu_k(x) \to Bu_0(x), \quad \text{a.e. } x \in \Omega. \tag{1.2}$$

Put

$$v_k(x) := a + b|u_k(x)|^{p/q}, \quad k = 0, 1, 2, \cdots \tag{1.3}$$

Then by (1.1)-(1.3),

$$|Bu_k(x)| \le v_k(x) \ \text{ for all } x \in \Omega; \quad v_k(x) \to v_0(x) \ \text{ a.e. } x \in \Omega. \tag{1.4}$$

In view of $|u_k|^p + |u_0|^p - ||u_k|^p - |u_0|^p| \ge 0$, by Fatou's Theorem, we get

$$\begin{aligned} &\int_\Omega \liminf_{k\to+\infty} (|u_k|^p + |u_0|^p - ||u_k|^p - |u_0|^p|)\,dx \\ &\quad \le \liminf_{k\to+\infty} \int_\Omega (|u_k|^p + |u_0|^p - ||u_k|^p - |u_0|^p|)\,dx. \end{aligned} \tag{1.5}$$

From (1.1)-(1.5), we see that

$$\lim_{k\to+\infty} \int_\Omega ||u_k|^p - |u_0|^p| dx = 0. \tag{1.6}$$

It follows that

$$\int_\Omega |v_k - v_0|^q dx \le b^q \int_\Omega ||u_k|^p - |u_0|^p| dx \to 0 \tag{1.7}$$

as $k \to \infty$. Since there exist constants $C > 0, C_1 > 0$ such that

$$\begin{aligned} |Bu_k - Bu_0|^q &\le C(|Bu_k|^q + |Bu_0|^q) \\ &\le C(|v_k|^p + |v_0|^q) \\ &\le C_1(|v_k - v_0|^q + |v_0|^q) \end{aligned}$$

a.e. $x \in \Omega$, and by Fatou's Lemma, one has

$$\begin{aligned} &\int_\Omega \liminf_{k\to+\infty} (C_1(|v_k - v_0|^q) - |Bu_k - Bu_0|^q)\,dx \\ &\le \liminf_{k\to+\infty} \int_\Omega (C_1(|v_k - v_0|^q) - |Bu_k - Bu_0|^q)\,dx. \end{aligned} \tag{1.8}$$

Combining (1.2),(1.3),(1.7) and (1.8), we have

$$||Bu_k - Bu||_q \to 0.$$

Finally, if Ω is bounded, then for any $u \in L^p(\Omega)$, obviously we have

$$||Bu||_q \le c + c||u||_p^{p/q}, \tag{1.9}$$

where $c > 0$ is a constant. Inequality (1.9) still remains true if Ω is unbounded and $a = 0$. Therefore, B is a continuous and bounded mapping from $L^p(\Omega)$ to $L^q(\Omega)$ and the same conclusion is true if Ω is unbounded and $a = 0$. □

The following lemma can be found M. Willem [144].

Lemma 1.18. *Assume that* $p_1, p_2, q_1, q_2 \geq 1$. *Let* $f(x,t)$ *be a Carathéodory function on* $\Omega \times \mathbb{R}$ *and satisfy*

$$|f(x,t)| \leq a|t|^{p_1/q_1} + b|t|^{p_2/q_2}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $a, b \geq 0$ *and* Ω *is either bounded or unbounded. Define a Carathéodory operator by*

$$Bu := f(x, u(x)), \quad u \in H := L^{p_1}(\Omega) \cap L^{p_2}(\Omega).$$

Define the space $E := L^{q_1}(\Omega) + L^{q_2}(\Omega)$ *with the norm*

$$||u||_E = \inf\{||v||_{L^{q_1}(\Omega)} + ||w||_{L^{q_2}(\Omega)}\},$$

where $u = v + w \in E, v \in L^{q_1}(\Omega), w \in L^{q_2}(\Omega)$. *Then* $B = B_1 + B_2$, *where* B_i *is a bounded and continuous mapping from* $L^{p_i}(\Omega)$ *to* $L^{q_i}(\Omega), i = 1, 2$. *In particular,* B *is bounded continuous mapping from* H *to* E.

Proof. Let $\xi : \mathbb{R} \to [0,1]$ be a smooth function such that $\xi(t) = 1$ for $t \in (-1,1); \xi(t) = 0$ for $t \notin (-2,2)$. Put

$$g(x,t) = \xi(t) f(x,t), \quad h(x,t) = (1 - \xi(t)) f(x,t).$$

Without loss of generality, we may assume that $p_1/q_1 \leq p_2/q_2$. Then there exist two constants $d > 0, m > 0$ such that

$$|g(x,t)| \leq d|t|^{p_1/q_1}, \quad |h(x,t)| \leq m|t|^{p_2/q_2}.$$

Define

$$B_1 u = g(x,u), \quad u \in L^{p_1}(\Omega);$$
$$B_2 u = h(x,u), \quad u \in L^{p_2}(\Omega).$$

Therefore, by Lemma 1.17, B_i is a bounded and continuous mapping from $L^{p_i}(\Omega)$ to $L^{q_i}(\Omega), i = 1, 2$. It is readily seen that $B := B_1 + B_2$ is a bounded continuous mapping from H to E.

□

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