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V. I. Arnol'd (Ed.)

Dynamical Systems VIII

Singularity Theory II: Applications

动力系统 VIII

奇异理论 II：应用

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Singularity Theory II
Classification and Applications

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Translated from the Russian
by J.S. Joel

Contents

Foreword

In the first volume of this survey (Arnol'd et al. (1988), hereafter cited as "EMS 6") we acquainted the reader with the basic concepts and methods of the theory of singularities of smooth mappings and functions. This theory has numerous applications in mathematics and physics; here we begin describing these applications. Nevertheless the present volume is essentially independent of the first one: all of the concepts of singularity theory that we use are introduced in the course of the presentation, and references to EMS 6 are confined to the citation of technical results.

Although our main goal is the presentation of an already formulated theory, the reader will also come upon some comparatively recent results, apparently unknown even to specialists. We point out some of these results.

In the consideration of mappings from $\mathbb{C}^2$ into $\mathbb{C}^3$ in § 3.6 of Chapter 1, we define the bifurcation diagram of such a mapping, formulate a $K(\pi, 1)$-theorem for the complements to the bifurcation diagrams of simple singularities, give the definition of the Mond invariant N in the spirit of "hunting for invariants", and we draw the reader's attention to a method of constructing the image of a mapping from the corresponding function on a manifold with boundary. In § 4.6 of the same chapter we introduce the concept of a versal deformation of a function with a nonisolated singularity in the class of functions whose critical sets are arbitrary complete intersections of fixed dimension. The corresponding $K(\pi, 1)$-theorem for simple functions is also true here.

In Chapter 2, following V.I. Bakhtin, we discuss the topology of four variants of the Maxwell variety of the singularity A_5, we describe generic "perestroikas" of the maximum function under a change of one parameter of the four and apply this analysis to the study of perestroikas of shock waves propagating in a three-dimensional space (following I.A. Bogaevskiĭ); here we give a "sec + tan" formula for the number of components of the space of Morse functions on the line and a formula of A.A. Vakulenko for the number of components of complements to Maxwell strata.

Chapter 3 contains, among other things, classifications of the singularities of the boundary of the set of hyperbolic differential equations (B.Z. Shapiro and A.D. Vaĭnshteĭn) and the singularities of the boundary of the set of fundamental systems of solutions of linear differential equations. (This theory, due to M.È. Kazaryan, is related to the Schubert stratification of a Grassmannian, to the bifurcation of Weierstrass points of algebraic curves, and to the theory of the focal varieties of projective curves.) In the same chapter we discuss the singularities of the boundary of the set of disconjugate systems (i.e., Chebyshev systems)—the connection of this question with the Schubert stratifications of flag varieties and with Bruhat orderings was recently discovered by B.Z. and M.Z. Shapiro.

Historically the first result based on the theory of monodromy is Newton's theorem on the nonintegrability of plane ovals; in § 4.1 we shall prove multidimensional generalizations of this theorem and give some new Picard-Lefschetz formulas that arise naturally in this problem. § 4.2 is devoted to the theory of Petrovskiĭ lacunas, studying the regularity of the fundamental solutions of hyperbolic partial differential equations close to wave fronts. Among other things, here we shall prove a converse to Petrovskiĭ's local criterion for hyperbolic operators in general position.

In Chapter 5 we enumerate the local lacunas (domains of regularity) for many of the singularities of wave fronts that appear in the tables, including all simple singularities and all singularities of corank 2 with Milnor number $\leqslant 11$. A significant part of these lacunas were found using a computer algorithm, which enumerates all the nonsingular morsifications of complicated real singularities; in § 5.3 we describe this algorithm.

The references inside the volume are organized in the following way. If a reference is to some place within the same chapter, then we give the number of the appropriate section or subsection, as in the Table of Contents. If the reference is to another chapter, then the number of the chapter appears before the number of the section or subsection. References to the first volume are kept to a minimum and are indicated as references to "EMS 6".

Chapter 1 was written by V.V. Goryunov, Chapters 2 and 3, except for § 3.5, were written by V.I. Arnol'd, and Chapters 4 and 5 were written by V.A. Vasil'ev. § 3.5 was written by B.Z. Shapiro. The authors offer their sincere thanks to him.

Chapter 1
Classification of Functions and Mappings

In this chapter we consider classifications with respect to the most frequently encountered and naturally occurring equivalence groups. The principal objects of attention here are the simple singularities, and also the topology of the nonsingular fiber of a mapping and the geometry of bifurcation diagrams. Many of the properties that we shall discuss are analogous to the properties of functions with isolated critical points that were presented in EMS 6 (Arnol'd et al. (1988)). In order to make the presentation as independent as possible from EMS 6, we shall, in the appropriate places, recall the definitions and constructions introduced in EMS 6 for isolated singularities of functions and carried over to those singularities that we treat in this chapter.

Important branches of the theory of singularities such as equivariant mappings remain outside of our consideration. We intend to devote a separate paper in one of the later volumes of this series to them.

§ 1. Functions on a Manifold with Boundary

A *manifold with boundary* is a smooth (real or complex) manifold with a fixed hypersurface. Two functions on a manifold with boundary are said to be *equivalent* if they are mapped into each other under a diffeomorphism of the manifold that maps the boundary into itself. The classification of functions on a manifold with boundary is closely connected with the Lie groups B_k, C_k, F_4, and G_2, and the Coxeter groups H_3, H_4 and $I_2(p)$, whose Dynkin diagrams have multiple edges (Bourbaki (1968)). This connection is analogous to the connection that occurs between the groups A_k, D_k, and E_k and the singularities of functions on smooth manifolds without boundary (EMS 6, 2.5).

1.1. Classification of Functions on a Manifold with a Smooth Boundary. Recall that a function or a mapping is said to be *simple* with respect to some equivalence group if, by an arbitrary sufficiently small perturbation of it, we can obtain representatives of only a finite number of equivalence classes. Thus, for the equivalence of germs of functions $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with respect to the group of coordinate changes in the source (so-called $\mathscr{R}$- or right-equivalence) the simple functions are precisely those that, for a suitable choice of coordinates, have the following normal forms:

$A_k, k \geqslant 0$	$D_k, k \geqslant 4$	E_6	E_7	E_8
$\pm y_1^{k+1}$	$y_1^2 y_2 \pm y_2^{k-1}$	$y_1^3 \pm y_2^4$	$y_1^3 + y_1 y_2^3$	$y_1^3 + y_2^5$

This list is given up to stable equivalence (two *functions* of different numbers of variables are said to be *stably equivalent* if they become equivalent after addition of nondegenerate quadratic forms in the extra variables).

Consider a manifold with a smooth boundary. Locally this is the germ at zero of the pair $(\mathbb{R}^n, \mathbb{R}^{n-1})$ or $(\mathbb{C}^n, \mathbb{C}^{n-1})$.

Theorem (Arnol'd (1978)). *The germs of simple functions at a boundary point of a real manifold with a smooth boundary are described completely, up to a diffeomorphism in the source that takes the boundary into itself, by the following list of germs of functions* $f(x, y)$ *at the point* $x = 0$, $y = 0$ *of the boundary* $x = 0$:

$A_k, k \geqslant 0$	$D_k, k \geqslant 4$	E_6	E_7	E_8
$\pm y^{k+1} + x$	$y_1^2 y_2 \pm y_2^{k-1} + x$	$y_1^3 \pm y_2^4 + x$	$y_1^3 + y_1 y_2^3 + x$	$y_1^3 + y_2^5 + x$

$B_k, k \geqslant 2$	$C_k, k \geqslant 3$	F_4
$\pm x^k \pm y^2$	$xy \pm y^k$	$\pm x^2 + y^3$

When we talk about equivalence of functions that do not have the same number of variables here we mean stable equivalence on a manifold with boundary.

Remark. The set of nonsimple functions has codimension $n + 3$ in the space of functions that take the value 0 at $0 \in \mathbb{R}^n$.

Consider the complex situation. We pass from the manifold $\mathbb{C}^n$ with boundary $x = 0$ to a two-sheeted covering of it, branched along the boundary, by setting $x = z^2$ and $y = y$. There is a natural involution $(z, y) \mapsto (-z, y)$ on the covering. A germ of the function $f(x, y)$ on the manifold with boundary corresponds to a germ of $f(z^2, y)$, which is invariant under the involution. In this way we obtain a one-to-one correspondence between functions on a manifold with a smooth boundary and functions that are invariant under an involution of the space $\mathbb{C}^n$ that preserves the subspace $\mathbb{C}^{n-1}$. This one-to-one correspondence is also a one-to-one correspondence of equivalence classes of singularities. Therefore, the preceding theorem also gives a classification of the simple functions that are invariant under the above action of the group $\mathbb{Z}_2$.

Later research has shown that the above list arises either wholly or partially in many other classification problems. These problems include the projections onto the line and mappings of the plane into three-space described in § 3, and also the linear singularities from § 4.

Recall that the *modality* of a point of a manifold on which a Lie group acts is the smallest number m such that any sufficiently small neighborhood of this point intersects only a finite number of at most m-parameter families of orbits of the group action. The points of modality 0 are therefore precisely the simple points. The points of modality 1 and 2 are respectively called *unimodal* and *bimodal.*

The classification of functions on a manifold with a smooth boundary $x = 0$, which do not have critical points on the ambient space, is easily seen to be equivalent to the classification of the restrictions of these functions to the boundary. The normal forms of such functions are obtained by adding the function x to the normal form of the restriction (cf. the singularities A_k, D_k and E_k in the absolute and boundary variants). In comparison with Chapter 1 of EMS 6 the essentially new point in the classification of boundary singularities is therefore just the classification of functions that have a critical point on the ambient manifold. Up to stable equivalence such functions of modality 1 are exhausted by the following two lists (for the definition of the number μ see Subsection 1.2) (Arnol'd (1978), Matov (1981a, 1981b)).

Unimodal boundary singularities of corank 2

Notation	$\mathbb{C}$-normal form	Restrictions	μ
$F_{1,0}$	$x^3 + axy^2 + y^3$	$4a^3 + 27 \neq 0$	6
$F_{1,p}$	$ax^{p+1} + xy^2 + y^3$	$a \neq 0, p \geqslant 1$	$6 + p$
F_8	$x^4 + y^3 + ax^3y$	—	8
F_9	$x^3y + y^3 + ax^2y^2$	—	9
F_{10}	$x^5 + y^3 + ax^4y$	—	10
$K_{4,2}$	$y^4 + axy^2 + x^2$	$a^2 \neq 4$	6
$K_{4,q}$	$y^4 + axy^2 + x^q$	$a \neq 0, q > 2$	$q + 4$
$K_{p,q}$	$y^p + xy^2 + ax^q$	$a \neq 0, p > 4, q \geqslant 2$	$p + q$
$K^{\#}_{1,2p-3}$	$(x + y^2)^2 + ax^py$	$a \neq 0, p > 1$	$2p + 3$
$K^{\#}_{1,2p-4}$	$(x + y^2)^2 + ax^p$	$a \neq 0, p > 2$	$2p + 2$
K^*_8	$y^4 + x^2y + ax^3$	—	8
K^*_9	$y^4 + x^3 + ax^2y^2$	—	9
K^{**}_8	$y^5 + x^2 + axy^3$	—	8

Unimodal boundary singularities of corank 3

Notation	$\mathbb{R}$-normal form	Restrictions	μ
$L_6 = D_{4,1}$	$y_1^2y_2 \pm y_2^3 + xy_1 + axy_2$	$a^2 \pm 1 \neq 0$	6
$D_{k,l}$	$y_1^2y_2 \pm y_2^{k-1} + axy_1^l + xy_2$	$a \neq 0, k \geqslant 4, l \geqslant 1, k + l > 5$	$k + l + 1$
$E_{6,0}$	$y_1^3 \pm y_2^4 + axy_1 + xy_2$	—	8
$E_{7,0}$	$y_1^3 + y_1y_2^3 + axy_1 + xy_2$	—	9
$E_{8,0}$	$y_1^3 + y_2^5 + axy_1 + xy_2$	—	10
D_5^1	$y_1^2y_2 \pm y_2^4 + xy_1 + axy_2^2$	—	8
$E_{6,1}$	$y_1^3 \pm y_2^4 + xy_1 + axy_2^2$	—	9
D_4^2	$y_1^2y_2 \pm y_2^3 \pm x^2 \pm axy_1^2$	—	8

The bimodal boundary singularities have been classified by Matov (1981b).

Definition. A class of singularities X *adjoins* or is *adjacent* to a class of singularities Y ($X \to Y$) if every function of the class X can be deformed into a function of the class Y by an arbitrarily small perturbation.

All the adjunctions of simple boundary singularities are given in Fig. 1, and the most important adjunctions of unimodal boundary singularities are given in Fig. 2.

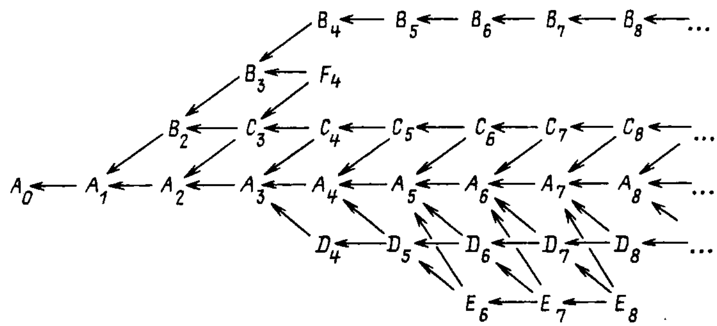

Fig. 1. Adjacent simple boundary singularities

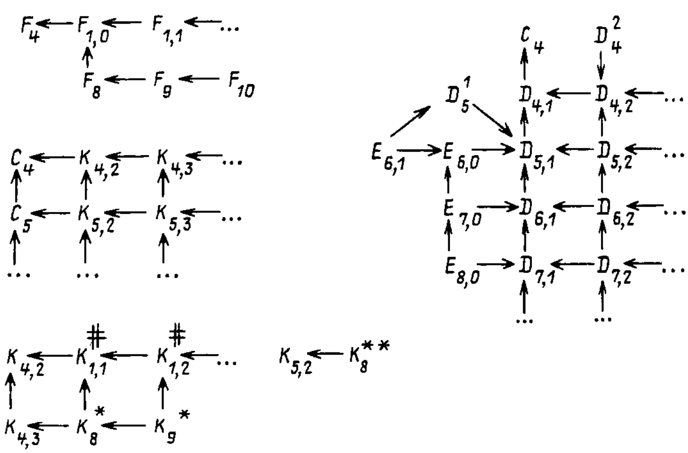

Fig. 2. Main adjunctions of unimodular boundary singularities

1.2. Versal Deformations and Bifurcation Diagrams. We begin by recalling some definitions that relate to deformations of functions and mappings.

A *deformation* of an element f of a manifold M is a germ of a smooth mapping $F: (\Lambda, 0) \to (M, f)$ of a finite-dimensional linear space Λ (the *base* of the deformation). The deformation *induced* from f under a mapping $\theta: (\Lambda', 0) \to (\Lambda, 0)$ is the deformation $\theta^*F = F \circ \theta$.

Suppose that a Lie group G acts on M. Two deformations F_1 and F_2 of the same element f with a common base Λ are said to be *equivalent* if there exists a deformation $g: (\Lambda, 0) \to (G, e)$ of the identity of G such that

$$F_1(\lambda) = g(\lambda)F_2(\lambda).$$

Finally, a deformation F of the point f is said to be *versal* if every other deformation of this point is equivalent to a deformation induced from F. A versal deformation whose base has the smallest possible dimension is termed *miniversal*.

If the manifold M is finite-dimensional, then a versal deformation of a point is a germ of a transversal to the orbit of the point. This result is also valid for points of infinite-dimensional spaces of mappings, having orbits of finite co-dimension, in the case when the group action is sufficiently good (see EMS 6, 3.2).

For complex boundary singularities we can take the space $\mathcal{O}_n$ of germs at the origin of holomorphic functions on $\mathbb{C}^n$ as the manifold. As a Lie group acting on this set we can take the pseudogroup of germs of diffeomorphisms of $\mathbb{C}^n$ that preserve the boundary $x = 0$. In this case a miniversal deformation of the germ $f(x, y_1, \dots, y_{n-1})$ from $\mathcal{O}_n$ with $f(0) = 0$ is given by a transversal to the orbit:

$$F(x, y, \lambda) = f(x, y) + \lambda_1 e_1(x, y) + \cdots + \lambda_\mu e_\mu(x, y),$$

where the λ_i are the parameters of the deformation and $e_1, \dots, e_\mu$ is a basis of the local ring

$$Q_f = \mathcal{O}_n/\mathcal{O}_n\langle xf_0, f_1, \dots, f_{n-1}\rangle, \quad f_0 = \partial f/\partial x, \quad f_j = \partial f/\partial y_j, \quad j > 0.$$

The number $\mu = \dim_{\mathbb{C}} Q_f$ is called the *multiplicity of the critical point* 0. The germs of functions with critical points of infinite multiplicity form a set of infinite codimension in the space of germs.

For simple functions μ is the subscript in the notation of the singularity.

The number μ is related to the Milnor number of the function f on the ambient space and to that of the restriction of f to the boundary:

$$\mu = \mu_1 + \mu_0,$$

$$\mu_1 = \dim_{\mathbb{C}} \mathcal{O}_n/\mathcal{O}_n\langle f_0, f_1, \dots, f_{n-1}\rangle$$

$$\mu_0 = \dim_{\mathbb{C}} \mathcal{O}_{n-1}/\mathcal{O}_{n-1}\langle f_1', \dots, f_{n-1}'\rangle, \quad f' = f|_{x=0}.$$

We fix a representative of the versal deformation F and choose a sufficiently small ball $B_\rho \subset \mathbb{C}^n$ of radius ρ with center at the origin. We choose a δ that is sufficiently small with respect to ρ and for λ from the ball $|\lambda| < \delta$ we consider the local level set $V_\lambda = \{(x, y) \in B_\rho: F(x, y, \lambda) = 0\}$.

Definition. The local *level set* V_λ is said to be *nonsingular* if 1) 0 is not a critical value for $F(\cdot, \lambda)$, and 2) the manifold V_λ is transversal to the boundary.

The germ at zero of the hypersurface $\Sigma \subset \mathbb{C}^\mu$ consisting of those values of λ for which the set V_λ is singular is called the *bifurcation diagram of the zeros (discriminant) of the function f*. The discriminant has two components Σ_1 and Σ_2 corresponding to level manifolds that are not smooth and to those that are not transversal to the boundary. Of course, for functions that are not critical on the ambient space, the first component is empty and the discriminant coincides with the usual discriminant of its restriction to the boundary (see EMS 6, 1.1.10). Figure 3 shows the discriminant of the singularity C_3. The discriminant of B_3 looks the same, except that the components Σ_1 and Σ_0 change places.

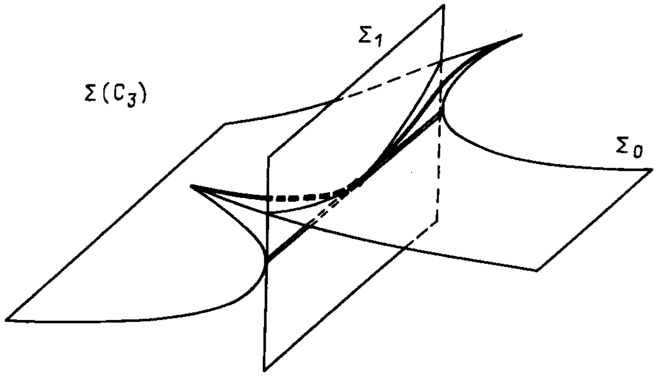

Fig. 3. Discriminant of the singularity C_3

Theorem (Arnol'd (1978)). *The discriminant of a simple critical point B_μ, C_μ or F_4, embedded in $\mathbb{C}^\mu$, is biholomorphically equivalent to the variety of nonregular orbits of the reflection group with the same name* (Bourbaki (1968)), *acting on the complexification of a Euclidean space.*

Thus, the complement to the discriminant of a simple singularity is a $K(\pi, 1)$ space, where π is the braid group of Brieskorn (Brieskorn (1973)) constructed from the corresponding Weyl group.

This result is the extension to the boundary case of the analogous result for simple functions on a smooth manifold (EMS 6, 2.5.6).

One can also extend to boundary singularities the corresponding assertion concerning bifurcation diagrams of functions at simple critical points (EMS 6, 2.5.8).

Let $\mathfrak{m}_n \subset \mathcal{O}_n$ be the ideal of functions that vanish at the origin.

Definition. A *truncated versal deformation* of a boundary singularity is a deformation

$$\Phi(x, y, \lambda) = f(x, y) + \sum_{i=1}^{\mu-1} \lambda_i e_i(x, y),$$

where $e_1, \ldots, e_{\mu-1}$ is a basis of $\mathfrak{m}_n/\mathcal{O}_n\langle xf_0, f_1, \ldots, f_{n-1}\rangle$.

We shall assume the critical points of a function on a manifold with boundary to be its critical points on the ambient manifold and the critical points of its restriction to the boundary. Then for almost every $\lambda \in \mathbb{C}^{\mu-1}$ the function $\Phi(\cdot, \lambda)$ has precisely $\mu = \mu_1 + \mu_0$ distinct critical values in a sufficiently small neighborhood of the point $0 \in \mathbb{C}^n$. The germ at zero of the hypersurface $\Xi \subset \mathbb{C}^{\mu-1}$, which is the complement to the set of above-indicated values of the parameters of the deformation, is called the *bifurcation diagram of functions of the boundary singularity f.*

Theorem (Lyashko (1979, 1983b)). *For a simple function on a manifold with nonsingular boundary the complement to the bifurcation diagram of functions on* $\mathbb{C}^{\mu-1}\setminus\Xi$ *is a* $K(\pi, 1)$ *space. The fundamental group* π *is a subgroup of index* $\mu! h^\mu |W|^{-1}$ *in the braid group on n strands.*

Here h is the Coxeter number and $|W|$ is the order of the Weyl group with the same name as the singularity (Bourbaki (1968)). Recall that the braid group on μ strands is the fundamental group of the space of polynomials $z^\mu + a_1 z^{\mu-2} + \cdots + a_{\mu-1}$ which do not have multiple roots.

1.3. Relative Homology Basis. Let V_λ' be the intersection of the nonsingular local level set V_λ with the boundary $x = 0$ of the manifold $\mathbb{C}^n$.

The nonsingular local level manifold of a function on a smooth manifold is homotopy equivalent to a bouquet of middle-dimensional spheres. The analogue of this fact in our case is the following.

Theorem (Arnol'd (1978)). *The quotient-space* V_λ/V_λ' *has the homotopy type of a bouquet of* μ $n-1$*-dimensional spheres.*

We define a *distinguished basis of vanishing cycles* in the homology group $H_{n-1}(V_\lambda, V_\lambda')$ (Arnol'd (1978)). To do this, in the base of a versal deformation of the function f we consider a line $\mathbb{C}^1$ in general position, that passes through our point λ which does not belong to the discriminant. This line intersects the discriminant Σ in $\mu = \mu_0 + \mu_1$ points. On $\mathbb{C}^1$ we consider the system of non-intersecting paths that go from the original point λ to the points of intersection with the discriminant.

When we move along any of the μ_1 paths leading to the points of Σ_1 corresponding to the nonsmooth local level manifolds, the level V_λ behaves just like the manifold $z_1^2 + \cdots + z_n^2 = \varepsilon$, on which the cycle $\operatorname{Im} z = 0$ contracts for positive $\varepsilon \to 0$ (see EMS 6, 2.1.5). Therefore we obtain μ_1 *vanishing cycles* on V_λ which define elements of the relative homology group $H_{n-1}(V_\lambda, V_\lambda')$.

Over the paths leading to each of the μ_0 points of Σ_0 that correspond to the nontransversality of a local level of the boundary, the same phenomenon arises as on the manifold $x + y_1^2 + \cdots + y_{n-1}^2 = \varepsilon$ for positive $\varepsilon \to 0$. In the latter case the cycle $x = 0$, $\operatorname{Im} y = 0$, $y_1^2 + \cdots + y_{n-1}^2 = \varepsilon$ vanishes. On the manifold this cycle is contracted by the disc $\operatorname{Im} x = 0$, $\operatorname{Im} y = 0$, $x = \varepsilon - y_1^2 - \cdots - y_{n-1}^2 \geqslant 0$. This disc defines a relative cycle in $H_{n-1}(V_\lambda, V_\lambda')$, which one can naturally term a *vanishing semicycle*.

This system of vanishing cycles and vanishing semicycles is a basis of the relative homology group. By numbering the original system of paths on $\mathbb{C}^1$ in the order they leave λ (counterclockwise), we will obtain a numbering of the corresponding elements of the basis (cf. EMS 6, 2.1.5).

1.4. Intersection Form. Although $\dim_{\mathbb{C}} H_{n-1}(V_\lambda, V_\lambda') = \mu$, the most natural results are obtained by considering the action of the fundamental group of the complement to the discriminant on another space of the same dimension, and not on the original one: namely the space of relative cycles with twisted coeffi-